Exercise Sheet 6

Exercise 21 (Topological Vector Spaces) - written

(a) Let $(X; \|\cdot\|)$ be a normed vector space equipped with the norm topology $\mathfrak{T}_{\|\cdot\|}$. Show that this is a topological vector space (i.e. vector addition and scalar multiplication are continuous).

(b) Consider the vector space $X = \mathbb{R}^2$ over $\mathbb{R}$ with the usual vector addition and scalar multiplication, here equipped not with the Euclidean topology but with the finer topology $\mathfrak{T}_{\mathrm{rays}}$, which is generated by the sets

$$\mathcal{O}(x,\varepsilon) = \{\, \rho x \in \mathbb{R}^2 \mid |\rho - 1| < \varepsilon \,\} \qquad \text{(ray segments)}$$

Show that scalar multiplication is continuous but vector addition is not.

Hint: For discontinuity it suffices to find an $\mathcal{O}(x,\varepsilon)$ whose preimage $\mathcal{V} \subset \mathbb{R}^2 \times \mathbb{R}^2$ is not open, i.e. $\exists\, (v,w) \in \mathcal{V}\ \forall\, \delta_v, \delta_w > 0 : \mathcal{O}(v,\delta_v) \times \mathcal{O}(w,\delta_w) \not\subset \mathcal{V}$.

Remark: This topology is induced by the so called "French railway metric" since continuous paths from x to y must go through the origin $0 =$ Paris unless x and y lie on one ray.

Exercise 22 (Smoothing Operator) - oral

Let $\Omega \subseteq \mathbb{R}^d$ be open and consider $C_c^0(\Omega)$ and $C_c^\infty(\Omega)$, each equipped with the supremum norm $\|\cdot\|_\infty$. Choose any function $\Psi \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Psi(y)dy = 1$. For $\varepsilon > 0$, let $\Psi_\varepsilon(x) := \frac{1}{\varepsilon^d}\Psi\left(\frac{1}{\varepsilon}x\right)$ and define the smoothing operator

$$(S_\varepsilon \varphi)(x) = (\Psi_\varepsilon * \varphi)(x) = \int_{\mathbb{R}^d} \Psi_\varepsilon(x-y)\varphi(y)dy$$

(a) Show that for $\varphi \in C_c(\mathbb{R}^d)$ and $\varepsilon > 0$, we always have $S_\varepsilon\varphi \in C_c^\infty(\mathbb{R})$.

(b) Show that $C_c^\infty(\Omega)$ is dense in $C_c^0(\Omega)$.

(c) For the case $\Omega = (0,1) \subseteq \mathbb{R}$, determine the completion of $(C_c^0(\Omega), \|\cdot\|_\infty)$.

Exercise 23 (Distributional Derivatives) - oral

Determine all distributional derivatives up to order 2 of the following functions. Which derivatives can be represented by locally integrable functions?

(a) $f : \mathbb{R} \to \mathbb{R}; t \mapsto |t|^{3/2}$;

(b) $g : \mathbb{R}^2 \to \mathbb{R}; (x_1, x_2) \mapsto \max\{0, x_1 x_2\}$;

Hint: Split the domain of integration into parts where the integrands are smooth.

(please turn over)

Exercise 24 (Local Lebesgue Spaces) - oral

For a domain $\Omega \subset \mathbb{R}^d$ and $p \in [1, \infty]$ we define

$$\mathrm{L}^p_{\mathrm{loc}}(\Omega) := \{\, f : \Omega \to \mathbb{R} \mid f \text{ measurable and } \forall\, \omega \Subset \Omega : f|_\omega \in \mathrm{L}^p(\omega) \,\},$$

where "$\Subset$" means compactly contained.

(a) Show that within the system

$$SN_{\Subset} = \{\, p_\omega : \mathrm{L}^p_{\mathrm{loc}}(\Omega) \to \mathbb{R} \mid \omega \Subset \Omega \,\} \text{ with } p_\omega(f) = \|f\|_{\mathrm{L}^p(\omega)}$$

of semi-norms, there is a countable family $SN_{\mathbb{N}} = \{\, \widetilde{p}_k \mid k \in \mathbb{N} \,\} \subset SN_{\Subset}$ such that for every $\omega \Subset \Omega$ there exist $k \in \mathbb{N}$ and $C_k > 0$ with $p_\omega(f) \le C_k \widetilde{p}_k(f)$ for all f.

(b) Construct a metric on $(\mathrm{L}^p_{\mathrm{loc}}(\Omega), \mathfrak{T}_{SN_{\Subset}})$.

(c) Let $\Omega = (0, 1)$. Construct three examples for a sequence $(f_n)_{n \in \mathbb{N}}$ with $f_n \to f$ in $L^2_{\mathrm{loc}}(\Omega)$ such that, respectively,

1. $f \notin L^2(\Omega)$.
2. $f \in L^2(\Omega)$ but $\|f_n\|_{L^2(\Omega)} \to \infty$.
3. $f \in L^2(\Omega)$, $\|f_n\|_{L^2(\Omega)}$ bounded and $\|f_n - f\|_{L^2(\Omega)} \ge \frac{1}{4}$.