Exercise Sheet 5

Exercise 17 (Lebesgue Spaces) - written

Let $1 \leq s \leq t < \infty$.

(a) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain (i.e. with finite volume). Use Hölder's inequality to prove $L^t(\Omega) \subseteq L^s(\Omega)$. Show that the converse inclusion is, in general, false.

(b) Find an example for an unbounded domain Ω such that $L^t(\Omega) \nsubseteq L^s(\Omega)$.

Exercise 18 (The p-norm for $p \to \infty$) - oral

Let $\Omega \subseteq \mathbb{R}^n$ be a domain with finite nonzero volume $\lambda(\Omega)$. For measurable functions $f : \Omega \to \mathbb{R}$ and $1 \leq p < \infty$, we define

$$\varphi_p(f) := \left(\frac{1}{\lambda(\Omega)} \int_\Omega |f(x)|^p \, dx \right)^{1/p} = \lambda(\Omega)^{-1/p} \|f\|_{L^p} \in [0, \infty]$$

Prove that $p \mapsto \varphi_p(f)$ is monotonically non-decreasing and

$$\|f\|_{L^\infty} = \lim_{p \to \infty} \varphi_p(f) = \lim_{p \to \infty} \|f\|_{L^p}$$

Exercise 19 (Neumann series) - oral

Let X be a Banach space and $A : X \to X$ be a linear map such that $\|Ax\| \leq C\|x\|$ with $C < 1$ for all $x \in X$. We aim at solving the linear equation $x - Ax = f$ for $f \in X$, i.e. we quest for $x = (I - A)^{-1} f$.

(a) Define the sequence $(x_n)_{n \in \mathbb{N}}$ through $x_1 = 0$ and $x_{n+1} = Ax_n + f$. Show that $(x_n)_n$ is a Cauchy sequence. Hint: Show $\|x_{n+1} - x_n\| \leq C\|x_n - x_{n-1}\|$.

(b) Show that the limit x of x_n is the requested solution, and that there is at most one solution.

(c) Prove the formula

$$x = (I - A)^{-1} f = \sum_{k=0}^{\infty} A^k f$$

which generalises the geometric series.

(please turn over)

Exercise 20 (Hermite Polynomials) - oral

The *Hermite polynomials* are defined by $H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. They satisfy the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{1}$$

and are orthogonal with respect to the weighted scalar product

$$\langle f, g \rangle_w = \int_{\mathbb{R}} f(x)g(x)e^{-x^2} dx$$

i.e. they satisfy

$$\langle H_n, H_m \rangle_w = C_n \delta_{n,m} \tag{2}$$

with a constant C_n depending on n.

(a) Show (1). Moreover, explicitly calculate H_n and C_n for $n = 0, 1, 2, 3$.

(b) Prove (2) and calculate C_n for general $n \in \mathbb{N}$. Conclude that the rescaled polynomials $C_n \cdot H_n$ are (weighted) orthonormal.

(c) Use the Hermite polynomials to construct functions ψ_n which are orthonormal with respect to the usual L^2-scalar product. Establish a differential equation for the ψ_n.

<u>Remark:</u> The ψ_n are called *Hermite functions*. They play an important role as solutions of the Schrödinger equation of the quantised harmonic oscillator.