Examination Dates

Possible dates for an oral examination are as follows:

Fri, 24.02.2012
Tue, 03.04.2012
Wed, 04.04.2012

Please note: An oral examination is *not* a prerequisite for obtaining a course certificate ("Schein"). Please consult your curriculum ("Studienordnung" bzw. "Prüfungsordnung") whether you actually need (and want) one.

Exercise Sheet 4

Exercise 13 (Fourier Series) - written

In the Hilbert space $L^2((0,2\pi))$ with the usual scalar product, we define the functions

$$e_0(t) := \frac{1}{\sqrt{2\pi}}\,, \qquad e_j(t) := \frac{1}{\sqrt{\pi}}\cos(jt)\,, \qquad e_{-j}(t) := \frac{1}{\sqrt{\pi}}\sin(jt)$$

for $j \in \mathbb{N} \setminus \{0\}$. The Fourier series of $f \in L^2((0,2\pi))$ is defined by $f_n := \sum_{k=-n}^{n} \langle f, e_k \rangle e_k$.

(a) Show that $\{e_j |\, j \in \mathbb{Z}\}$ is an orthonormal system.

(b) Let $f \in C^1([0,2\pi])$. Prove pointwise convergence $f_n(x) \to f(x)$.

(c) Let $f \in C^1([0,2\pi])$ be a periodic function (i.e. $f(0) = f(2\pi)$). Prove the estimate $\|f - f_n\|_{L^2}^2 \le \frac{1}{n+1}\|f'\|_{L^2}^2$ and conclude that f_n converges in L^2 to f.

Hint: Use integration by parts to find suitable expressions for $\langle f, e_j \rangle$.

(d) Use (c) to prove that $\{e_j |\, j \in \mathbb{Z}\}$ is a complete orthonormal system.

Exercise 14 (A Differential Operator) - oral

On the interval $\Omega = (0,1)$, we define the differential operator

$$L : Y \to X; u \mapsto u'' \quad \text{with } Y \subset X = \mathrm{C}^0(\overline{\Omega})$$

where $Y := \{\, u \in \mathrm{C}^2(\overline{\Omega}) \mid u(0) = 0,\ u'(1) = 0 \,\}$.

(a) Determine all eigen-pairs $(\lambda_k, u_k) \in \mathbb{R} \times Y$ (i.e. pairs such that $Lu_k = \lambda_k u_k$).

(b) Show that for all $u, v \in Y$, $\langle Lu, v \rangle = \langle u, Lv \rangle$ holds, where $\langle \cdot, \cdot \rangle$ denotes the standard L^2-scalar product.

(c) Conclude that there are no eigen-pairs (λ, u) with $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and that eigenfunctions with respect to different eigenvalues are orthogonal on each other.

(d) Show that the normed eigenfunctions form a complete orthonormal system.

(bitte wenden)

Exercise 15 (Heat Conduction) - oral

In 1807, J.J. FOURIER established the **heat equation** $\frac{\partial}{\partial t}u = \operatorname{div}\big(\kappa\nabla u\big)$ as the fundamental law of heat conduction. Here, $\kappa > 0$ is the thermal diffusivity and u is the difference of the (absolute) temperature and a given reference temperature.

We want to study the one-dimensional heat equation (HE). Let an initial temperature distribution $u_0(x)$ be given in $\Omega = (0, b)$, and the boundary points are to be at the reference temperature at any time, i.e. $u = 0$. We are thus looking for a function $u : [0, \infty) \times [0, b] \to \mathbb{R}$ such that

$$\text{(HE)}\quad u_t = \kappa u_{xx} \text{ in } (0,\infty)\times\Omega,$$
$$\text{(IC)}\quad u(0,x) = u_0(x) \text{ for } x \in \Omega, \qquad \text{(BC)}\quad u(t,0) = u(t,b) = 0 \text{ for } t > 0.$$

(IC=initial condition, BC=boundary condition).

(a) Show that there are solutions of (HE) and (BC) in the form $u(t,x) = a(t)\sin(\mu x)$ with a and μ yet to be determined.

(b) Choose a suitable orthonormal system $\{\, e_j \mid j \in \mathbb{N} \,\}$ in $\mathrm{L}^2((0,b))$ such that there are solutions of (HE) and (BC) in the form $u(t,x) = \sum_{j=1}^{N} a_j(t)e_j(x)$.

(c) How can we find, for arbitrary $u_0 \in \mathrm{L}^2((0,b))$, a solution of (HE) which also satisfies (BC) and (IC)? (Additional question: Why is u thus constructed for $(t,x) \in (0,\infty)\times(0,b)$ suitably often differentiable?)

Exercise 16 (Convergence Theorems in LEBESGUE Theory) - oral

Consider the real-valued functions f_n, g_n, h_n, k_n on $\Omega := (0,\infty)$, defined as follows.

(a) $f_n(x) := F(x-n)$ with $F(x) := x\mathrm{e}^{-|x|}$,

(b) $g_n(x) := \begin{cases} 1/n & \text{for } x \in (n, n{+}1) \\ 0 & \text{otherwise,} \end{cases}$

(c) $h_n(x) := \begin{cases} n^\alpha & \text{for } x \in (0, 1/n) \\ 0 & \text{otherwise,} \end{cases}$

(d) $k_n(x) := \begin{cases} (1+x)^{-\beta} & \text{for } x \in (0, n^2) \\ -x^{-\gamma} & \text{for } x > n^2. \end{cases}$

Depending on $\alpha, \beta, \gamma \in (0,\infty)$, examine in which cases the theorems of Lebesgue (dominated convergence), Fatou (one-sided boundedness) and Beppo Levi (monotone convergence) are applicable. Moreover, calculate $\lim_{n\to\infty}\int_\Omega f_n(x)\,\mathrm{d}x$ and $\int_\Omega \lim_{n\to\infty} f_n(x)\,\mathrm{d}x$ as well as the corresponding expressions with f_n replaced by g_n, h_n, k_n.