

Exercise Sheet 3

Exercise 9 (Complete Orthonormal Systems) - written

Let $\mathcal{O} = \{e_j \mid j \in \mathbb{N}\}$ be an orthonormal system in $(X, \langle \cdot, \cdot \rangle)$. Show that the following are equivalent.

- (i) \mathcal{O} is complete, i.e. $\text{span } \mathcal{O}$ is dense in X .
- (ii) $\langle x, e_j \rangle = 0$ for all $j \in \mathbb{N}$ implies $x = 0$.
- (iii) For all $x \in X$, $\sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2 = \|x\|^2$.
- (iv) For all $x \in X$, $\sum_{j=1}^n \langle x, e_j \rangle e_j \xrightarrow{n \rightarrow \infty} x$.

Exercise 10 (HAAR Wavelets) - oral

In the Hilbert space $L^2((0, 1))$ with the usual scalar product, we define the functions

$$h_{1,1} \equiv 1, \quad h_{1,2}(t) = \begin{cases} 1 & \text{for } t \in (0, \frac{1}{2}] \\ -1 & \text{for } t \in (\frac{1}{2}, 1), \end{cases} \quad h_{k,m}(t) = \begin{cases} 1 & \text{for } t \in (\frac{2m-2}{2^k}, \frac{2m-1}{2^k}) \\ -1 & \text{for } t \in (\frac{2m-1}{2^k}, \frac{2m}{2^k}) \\ 0 & \text{otherwise.} \end{cases}$$

for $k \geq 2$ and $m = 1, \dots, 2^{k-1}$.

- (a) Sketch the first six functions and show that they form an orthogonal system.
- (b) Prove by induction that $V_n = \text{span}\{h_{k,m} \mid k = 1, \dots, n, m = 1, \dots, \max\{2, 2^{k-1}\}\}$ is the 2^n -dimensional space of functions which are constant on the intervals $((m-1)2^{-n}, m2^{-n})$.
- (c) Define $P_n : L^2((0, 1)) \rightarrow V_n$ by $P_n f := \sum_{k=1}^n \sum_{m=1}^{\max\{2, 2^{k-1}\}} \frac{\langle f, h_{k,m} \rangle_{L^2}}{\|h_{k,m}\|_{L^2}^2} h_{k,m}$. Show that $P_n f$ assumes on $((m-1)2^{-n}, m2^{-n})$ the mean value of f restricted to this interval.
- (d) Use (c) to conclude that the $\{h_{k,m}\}$ are a complete orthogonal system. (Hint: For polynomials q , one can show $\|P_n q - q\|_{L^2} \rightarrow 0$.)

Exercise 11 (Tensor Product Systems) - oral

Let $\Omega_1 \subseteq \mathbb{R}^{n_1}$ and $\Omega_2 \subseteq \mathbb{R}^{n_2}$ be open domains. For $f \in L^2(\Omega_1)$ and $g \in L^2(\Omega_2)$, the prescription

$$(f \otimes g)(x, y) := f(x)g(y), \quad x \in \Omega_1, y \in \Omega_2 \quad (\text{tensor product})$$

defines an element $f \otimes g \in L^2(\Omega^1 \times \Omega^2)$. Let $\mathcal{S}_1 = \{e_i \mid i \in I\}$ and $\mathcal{S}_2 = \{f_j \mid j \in J\}$ be orthonormal systems in $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively.

- (a) Show that $\mathcal{S} = \{e_i \otimes f_j \mid i \in I, j \in J\}$ is an orthonormal system in $L^2(\Omega^1 \times \Omega^2)$.
- (b) Show that \mathcal{S} is complete if \mathcal{S}_1 and \mathcal{S}_2 are.
- (c) Let $\Omega = (0, a) \times (0, b) \times (0, c) \subseteq \mathbb{R}^3$. Construct a complete orthonormal system of $L^2(\Omega)$.

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Exercise 12 (Separability) - oral

- (a) Show that the Banach spaces $(c_0, \|\cdot\|_\infty)$ and $(\ell_1, \|\cdot\|_1)$ are separable.
- (b) Show that the Banach space $BC^0((0,1)) = \{f \in C^0((0,1)) \mid f \text{ bounded}\}$ equipped with the norm $\|\cdot\|_\infty$ is not separable. (Hint: Find functions which correspond to sequences in $\{0,1\}^{\mathbb{N}}$.)
- (c) Show that every closed subspace V of a separable Banach space $(X, \|\cdot\|)$ is in turn a separable Banach space. Moreover, find a non-separable Banach space Y which has a closed, infinite dimensional and separable subspace U .