Exercise Sheet 13

Exercise 41 (Different Types of Convergence) - written

(a) Let H be a Hilbert space. Show the following implication:

$$\Big(y_k \rightharpoonup y \ \text{ und } \ \|y_k\| \to \|y\| \Big) \qquad \Longrightarrow \qquad y_k \to y.$$

(b) Let $X = \mathrm{L}^1((0,\pi))$, $f_* \equiv 1$ and $f_k(t) = 1+\sin(kt)$. Which of the following three convergence statements is true?

$$\text{(i) } \|f_k\|_1 \to \|f_*\|_1 \qquad \text{(ii) } f_k \to f_* \text{ in } X \qquad \text{(iii) } f_k \rightharpoonup f_* \text{ in } X.$$

For (iii) use that $X' = \mathrm{L}^\infty((0,\pi)) \subset \mathrm{L}^2((0,\pi))$.

Exercise 42 (Weak Convergence) - oral

Let X be a Banach space with dual space X'. Let $(y_k)_{k\in\mathbb{N}}$ be a sequence in X und $(x'_j)_{j\in\mathbb{N}}$ be a sequence in X'. Prove the following.

(a) $y_k \rightharpoonup y$ in X and $x'_j \to x'$ in X' (strong) implies $x'_k(y_k) \to x'(y)$ in $\mathbb{K}$.

(b) $y_k \to y$ in X (strong) and $x'_j \overset{*}{\rightharpoonup} x'$ in X' implies $x'_k(y_k) \to x'(y)$ in $\mathbb{K}$.

(c) Construct an example with $y_k \rightharpoonup y$ and $x'_j \overset{*}{\rightharpoonup} x'$ such that $x'_k(y_k)$ does not converge.

Exercise 43 (Counterexample for Separation) - oral

Consider the Hilbert space $X = (l_2, \langle\cdot,\cdot\rangle)$ and the subsets

$$A = \{(a_n)_n \,|\, \sum_{n\in\mathbb{N}} n^2 a_n^2 < 1\} \quad \text{and} \qquad B = \{b = (b_n)_n\} \quad \text{with} \quad b_n = \sqrt{\frac{6}{\pi^2}} \cdot \frac{1}{n^2}$$

(a) Show that A and B are both convex and $A \cap B = \emptyset$.
<u>Hint:</u> Use that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

(b) Show that A and B cannot be separated by a functional: There exists no $x' \in l_2' \cong l_2$ such that for all $a \in A$ we have $x'(a) < x'(b)$.
<u>Hint:</u> Assume, by contradiction, there exists such x'. Identify x' with $c = (c_n)_n \in l^2$. Considering $a^{(1)} := (0, b_2, b_3, \ldots) \in A$, show that $c_1 > 0$. Furthermore, consider $\alpha > 0$ such that $a^{(m)}(\alpha) := (\alpha, b_2, \ldots, b_{m-1}, 0, b_{m+1}, \ldots) \in A$ and $x'(b) > x'(a^{(m)}(\alpha))$ to obtain an estimate for c_m. Conclude that this contradicts $c \in l^2$.

(c) Use a general principle to conclude that A is not open.