Exercise Sheet 12

Exercise 38 (Adjoint Maps) - written

Let X, Y and Z be real BANACH spaces and $T, T_1, T_2 \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$.

(a) For $\lambda_1, \lambda_2 \in \mathbb{R}$, show the identities

(i) $(\lambda_1 T_1 + \lambda_2 T_2)' = \lambda_1 T_1' + \lambda_2 T_2'$ (ii) $(ST)' = T'S'$ (iii) $(\mathrm{id}_X)' = \mathrm{id}_{X'}$ (iv) $J_Y T = T'' J_X$

where $J_V : V \to V''$ denotes the canonical embedding into the bidual space.

(b) Conclude that bijectivity of T implies bijectivity of T' and, in this case, $(T')^{-1} = (T^{-1})'$ holds.

(c) For $j = 1, \dots, N$, let $x_j' \in X'$ and $y_j \in Y$ be given. Show that $Ax = \sum_{j=1}^N x_j'(x) y_j$ defines an operator in $\mathcal{L}(X,Y)$ and explicitly calculate the adjoint operator A'.

Exercise 39 (HAHN-BANACH'S THEOREM) **- oral**

This exercise proves HAHN-BANACH'S THEOREM in the separable case by induction, without using the axiom of choice:
Let X be a *separable* real BANACH space and

- $p : X \to \mathbb{R}$ be sublinear, i.e.

  $$p(x+y) \leq p(x) + p(y)\ , \qquad p(\alpha x) = \alpha p(x) \qquad \text{for } x, y \in X \text{ and } \alpha \geq 0$$

  For simplicity, we further assume that p is continuous, i.e. $|p(x)| \leq C\|x\|$ for all $x \in X$.

- Y be a subspace of X and $\varphi : Y \to \mathbb{R}$ be a linear function such that $\varphi(x) \leq p(x)$ for $x \in Y$.

Then there exists a linear function $l : X \to \mathbb{R}$ such that $l|_Y = \varphi$ and $l(x) \leq p(x)$ for all $x \in X$.
<u>Hint:</u> Let $\{z_j\} \subseteq X$ be countable and dense. Define subspaces $Y_{k+1} = Y_k \oplus \mathrm{span}\{z_{k+1}\}$ (if $z_{k+1} \notin Y_k$) and extend $\varphi_k : Y_k \to \mathbb{R}$ by the ansatz $\varphi_{k+1}(y + \alpha z_{k+1}) = \varphi_k(y) + c_{k+1}\alpha$ with a constant c_{k+1} such that $\varphi_{k+1} \leq p$. Proceed by induction to obtain a linear function φ_∞ on a dense subspace of X and construct l by extension.

(please turn over)

Exercise 40 (Reflexivity) - oral

This exercise shows that a BANACH space X is reflexive if and only if its dual space X' is. Let $J_X : X \to X''$ and $J_{X'} : X' \to X'''$ denote the canonical embeddings of X and X' into the respective bidual space.

(a) In general, let $T : X \to Y$ be an isomorphism of BANACH spaces (i.e. a norm-preserving and bijective linear map). Prove that the adjoint map $T' : Y' \to X'$ is an isomorphism.

(b) Let $E = (J_X)' : X''' \to X'$ denote the adjoint $x''' \mapsto x''' \circ J_X$ of J_X. Show that $E \circ J_{X'} = \mathrm{id}_{X'}$. Moreover, for reflexive X, show that also $(J_X^{-1})' \circ E = \mathrm{id}_{X'''}$ holds and conclude $J_{X'} = (J_X^{-1})'$ and reflexivity of X'.

(c) Deduce from reflexivity of X' that of X. (Consider $J_X(X)$ as a subspace of X''.)