Exercise Sheet 10

Exercise 35 (Adjoint Maps) - written

Let X, Y be BANACH spaces and $M \subseteq \mathcal{L}(X,Y)$ be such that for all $x \in X$ and $f \in Y'$

$$\sup_{T\in M} |f(T(x))| < \infty$$

Prove that M is uniformly bounded, i.e.

$$\sup_{T\in M} \|T\| < \infty$$

Hint: Use a corollary of HAHN-BANACH'S theorem.

Exercise 36 (General Boundary Conditions) - oral

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. We define a bilinear form

$$\mathcal{A}(u,v) = \int_\Omega (A\nabla u \cdot \nabla v + B \cdot \nabla u\, v + c\, u\, v)\, \mathrm{d}x + \int_{\partial\Omega} k\, u\, v\ \mathrm{d}a,$$

with $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^d$ and $c, k \in \mathbb{R}$ (constant in Ω).

(a) Prove the estimate $|\mathcal{A}(u,v)| \le C\|u\|_{\mathrm{H}^1}\|v\|_{\mathrm{H}^1}$ for all $u, v \in \mathrm{H}^1(\Omega)$.
Hint: Use (but do not prove) continuity of the map $\Gamma : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\partial\Omega);\ u \mapsto u|_{\partial\Omega}$.

(b) Let $u \in \mathrm{C}^2(\overline{\Omega})$ be a solution of the weak boundary value problem

$$\forall\, v \in \mathrm{H}^1(\Omega):\ \ \mathcal{A}(u,v) = \int_\Omega f_{\mathrm{vol}}\, v\, \mathrm{d}x + \int_{\partial\Omega} f_{\mathrm{surf}}\, v\, \mathrm{d}a. \qquad \text{(WBVP)}$$

where A, B, c, k, f_{vol}, and f_{surf} are now continuous on $\overline{\Omega}$. Show that u is a solution of the following classical boundary value problem:

$$-\operatorname{div}\big(A\nabla u\big) + B\cdot\nabla u + c\, u = f_{\mathrm{vol}} \ \text{ in } \Omega, \qquad \big(A\nabla u\big)\cdot\nu + k\, u = f_{\mathrm{surf}} \ \text{ on } \partial\Omega. \qquad \text{(CBVP)}$$

where ν is the outward-pointing surface normal.
Hint: Start with $v \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$. Then use $v \in \mathrm{C}^\infty(\overline{\Omega})$ which is dense in $\mathrm{H}^1(\Omega)$.

(c) Let $B \in \mathbb{R}^d$ and $k \in \mathbb{R}$. Show that there exists a constant $\lambda_0 = \lambda_0(k) \in \mathbb{R}$ such that for all positive definite symmetric matrices A (i.e. $A = A^\top > 0$) with all eigenvalues $\lambda > \lambda_0$ the following holds: There exists a constant $c_0 = c_0(A, B, k) \in \mathbb{R}$ such that (WBVP) has, for all $c > c_0$ and all $f_{\mathrm{vol}} \in \mathrm{L}^2(\Omega)$ und $f_{\mathrm{surf}} \in \mathrm{L}^2(\partial\Omega)$, a unique solution.

(please turn over)

Exercise 37 (Dual Spaces of c_0 and ℓ_1) - oral

Consider the well-known spaces $(c_0, \|\cdot\|_\infty)$, $(\ell_1, \|\cdot\|_1)$ and $(\ell_\infty, \|\cdot\|_\infty)$ with the usual norms $\|(a_\gamma)_{\gamma\in\mathbb{N}}\|_1 = \sum_{\mathbb{N}} |a_\gamma|$ and $\|(a_\gamma)_{\gamma\in\mathbb{N}}\|_\infty = \sup_{\mathbb{N}} |a_\gamma|$. Show the following dual space relations:

$$\text{(a)} \quad c_0' = \ell_1 \qquad\qquad \text{(b)} \quad \ell_1' = \ell_\infty.$$

To show $X' = Y$, proceed in both cases as follows: (i) define a linear embedding $E : Y \to X'$, (ii) show that E preserves norms and (iii) that E is surjective.
Hint:: The unit vectors $e^j = (0, \ldots, 0, 1, 0, \ldots)$ and the density of $A = \operatorname{span}\{\, e^j \mid j \in \mathbb{N}\,\}$ are helpful.

(c) Find $F \in \ell_\infty'$ which cannot be written in the form $Ea : x \mapsto \sum_{\mathbb{N}} a_j x_j$ with $a \in \ell_1$.