Exercise Sheet 13

Exercise 49. Smoothing properties of the heat equation.

Consider the heat equation $u_t = u_{xx}$ on the interval $\Omega =]0, 2\pi[$ with periodic boundary conditions $u(t, 2\pi) = u(t, 0)$ and $u_x(t, 2\pi) = u_x(t, 0)$. The initial-value problem has the solution

$$u(t, \cdot) = \mathrm{e}^{tA} u_0 \quad \text{with} \quad \mathrm{e}^{tA} v := \sum\nolimits_{n\in\mathbb{Z}} \mathrm{e}^{-n^2 t} \langle v, E_n \rangle_{\mathrm{L}^2} E_n,$$

where $E_n(x) = (2\pi)^{-1/2} \mathrm{e}^{\mathrm{i}nx}$ forms a complete ONS in $\mathrm{L}^2(\Omega)$.

(a) Show that the operator e^{tA} a bounded linear operator from $\mathrm{L}^2(\Omega)$ to $\mathrm{L}^2(\Omega)$ which has operator norm 1.

(b) For $k \in \mathbb{N}$ construct C_k and α_k such that $\|\mathrm{e}^{tA}\|_{\mathrm{L}^2(\Omega)\to\mathrm{H}^k_{\mathrm{per}}(\Omega)} \leq C_k(1+t^{-\alpha_k})$ holds for all $t > 0$.

Exercise 50. Quadratic membrane. On the square $\Omega = (0, \pi) \times (0, \pi)$ we consider the wave equation $u_{tt} = \Delta u = u_{x_1x_1} + u_{x_2x_2}$ with Dirichlet boundary conditions $u(t, \cdot)|_{\partial\Omega} = 0$.

(a) Determine a complete orthonormal system $\{ \Phi_j \mid j \in \mathbb{N} \}$ in $\mathrm{L}^2(\Omega)$, such that $u(t, x) = \cos(\omega_j t)\Phi_j(x)$ provides a solution to the wave equation for suitable ω_j.

(b) Show that for $\omega^2 = 10$ there exist two different eigenfunctions Φ_j, such that $\Phi_i(x_1, x_2) = \Phi_j(x_2, x_1) \neq \Phi_j(x_1, x_2)$ holds for $(x_1, x_2) \in \Omega$. Discuss the nodal lines (= zero set) of the solution $u(t, \cdot) = \cos(\sqrt{10}\, t)\big(\alpha\Phi_i+\beta\Phi_j\big)$. In particular study the cases $\alpha/\beta \in \{\infty, 1, -1, 0\}$.

Exercise 51 (in written form). An explicit solution for the wave equation.

On $\Omega = \mathbb{R}^d$ consider the wave equation

$$u_{tt} = \Delta u, \qquad u(0, x) = u^0(x), u_t(0, x) = u^1(x).$$

(a) Calculate the explicit solution for $u^0 \equiv 0$ and $u^1(x) = 1$ for $|x| \leq R$ and $u^1(x) = 0$ for $|x| > R$. What is the support of $u(t, \cdot)$? Where is u continuous? (Hint: The initial condition is radially symmetric.)

(b) Show that $\|u(t)\|_{\mathrm{L}^\infty} \to 0$ and that the difference between the potential energy $E_{\mathrm{pot}}(t)$ and the kinetic energy $E_{\mathrm{kin}}(t)$ decays to 0, where

$$E_{\mathrm{pot}}(t) = \int_{\mathbb{R}^d} \Big|\frac{1}{2}\nabla u(t, x)\Big|^2 \mathrm{d}x \quad \text{and} \quad E_{\mathrm{kin}}(t) = \int_{\mathbb{R}^d} \Big|\frac{1}{2} u_t(t, x)\Big|^2 \mathrm{d}x.$$

Try to determine the decay rates.

please turn

Exercise 52. Finite speed of propagation in general wave equations. On $\Omega = \mathbb{R}^d$ consider sufficiently smooth solutions of the general homogeneous wave equation

$$\rho(x)u_{tt} = \operatorname{div}\big(A(x)\nabla u(t,x)\big) - c(x)u(t,x),$$

where $\rho, c \in \mathrm{BC}(\mathbb{R}^d)$, $A \in \mathrm{BC}^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ such that $\rho(x) \geq \rho_{\min} > 0$, $c(x) \geq 0$, and $A(x)\xi\cdot\xi \geq A_{\min}|\xi|^2$ with $A_{\min} > 0$.

(a) Define the regions $R(t) = \{\, x \in \mathbb{R} \mid |x - x_*| < r_* - vt \,\}$ for $x_* \in \mathbb{R}^d$, $r_*, v > 0$, and $t \in \,]0, r_*/v[$. Show the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R(t)} \frac{\rho}{2}u_t^2 + \frac{1}{2}A\nabla u\cdot\nabla u + \frac{c}{2}u^2 \,\mathrm{d}x = \int_{\partial R(t)} A\nabla u\cdot\nu\, u_t - v\Big(\frac{\rho}{2}u_t^2 + \frac{1}{2}A\nabla u\cdot\nabla u + \frac{c}{2}u^2\Big)\,\mathrm{d}a.$$

(b) Construct the smallest v, such that $u(t,x) = 0$ for all $x \in R(t)$ whenever $u(0,x) = u_t(0,x)$ for $x \in R(0)$. Conclude further that $\operatorname{sppt}((u(0,\cdot), u_t(0,\cdot)) \subset B_R(x_*)$ implies $\operatorname{sppt}((u(t,\cdot), u_t(t,\cdot)) \subset B_{R+v|t|}(x_*)$ for all $t \in \mathbb{R}$.