Exercise Sheet 12

Exercise 45. Parabolic problem on a half-line

(a) Use the reflection principle to construct a solution formula in the form $u(t,x) = \int_0^\infty K_{\rm D}(t,x,y)u_0(y)\,{\rm d}y$ for the heat equation in the half-line $\Omega =]0,\infty[$:

$$\begin{gathered} \partial_t u = \partial_x^2 u \text{ for } (t,x) \in]0,\infty[\times\Omega, \\ \text{(IC) } u(0,x) = u_0(x) \text{ for } x\in\Omega, \quad \text{(BC) } u(t,0) = 0 \text{ for } t>0. \end{gathered} \tag{Dir}$$

Find similarly the a kernel $K_{\rm N}$ is the Dirichlet Boundary Conditions in (BC) are replaced by the Neumann Boundary conditions $\partial_x u(t,0) = 0$.

(b) Assume $u_0 \in {\rm C}_{\rm c}^0(\Omega)$. In which case do we have energy conservation $\frac{\rm d}{{\rm d}t}\int_0^\infty u(t,x)\,{\rm d}x = 0$ and in which case entropy production $\frac{\rm d}{{\rm d}t}\int_0^\infty \eta(u(t,x))\,{\rm d}x \geq 0$? Here $\eta : \mathbb{R} \to \mathbb{R}$ is a concave entropy density, i.e. $\eta'' \leq 0$.

Exercise 46. Temperature problems when laking a shower.

How much colder is the water after shampooing your hair?

We model the water pipe by the one-dimensional interval $\Omega =]0,l[$, where the water reservoir with fixed temperature 60°C is at $x = l$, whereas at $x = 0$ is the outlet. The water is flowing with velocity $v = 0$ during shampooing or $v = V$ during running water. The equations are:

$$\begin{aligned} &cu_t = ku_{xx} + vu_x - \alpha(u{-}20) \quad \text{for } t>0,\ x\in\Omega, \\ &u_x(t,0) = 0 \quad u(t,l) = 60 \quad \text{for } t>0. \end{aligned}$$

Here $\alpha > 0$ models the heat losses through into the wall, which has 20°C.

(a) Let $k = l = 1$ and $\alpha = 4$ and calculate the steady states (time-independent solutions) for the two different flow velocities $v = 0$ and $v = 3$. Draw a picture!

(b) Show that the solution for $v = 0$ is always colder than the solution for $v > 0$.
Hint: Use maximum principles and so on.

please turn

Exercise 47. Gronwall's lemma [1919]
(after Thomas Hakon Grönwall (1877-1932)).

(A) Differential form: Let $x \in \mathrm{W}^{1,1}([0,T])$ and $x(t) \geq 0$. Moreover, let $\alpha \in \mathrm{L}^1([0,T])$ be such that $\dot{x}(t) \leq \alpha(t)x(t)$. Show the Gronwall inequality $x(t) \leq x(0)\mathrm{e}^{\int_0^t \alpha(s)\,\mathrm{d}s}$.

(B) Assume that $y \in \mathrm{C}^1([0,\infty[)$ and $\phi \in \mathrm{L}^1([0,\infty[)$ satisfy $\phi(t) \geq 0$ and $\frac{\mathrm{d}}{2\,\mathrm{d}t}y^2 \leq y\phi$ for all $t \geq 0$. Show that y remains bounded by $|y(0)| + \int_0^\infty \phi(t)\,\mathrm{d}t$.

Exercise 48 (in written form). A priori estimates).

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. We consider a general parabolic equations

$$
\begin{aligned}
&c(x)u_t = \operatorname{div}\big(A(x)\nabla u(x)\big) + a(x)\cdot\nabla u(x) + \alpha(x)u(x) + f(t,x), \quad (t,x) \in \,]0,\infty[\, \times \Omega,\\
&u(0,x) = u_0(x) \text{ in } \Omega, \qquad (A(x)\nabla u(t,x))\cdot\nu(x) + \beta(x)u(x) = 0 \text{ for } t>0,\ x \in \partial\Omega,
\end{aligned}
$$

where the divergence is understood in the weak sense. For the coefficients assume $A \in \mathrm{L}^\infty(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$, $a \in \mathrm{L}^\infty(\Omega,\mathbb{R}^d)$, $c,\alpha \in \mathrm{L}^\infty(\Omega)$, $\beta \in \mathrm{L}^\infty(\partial\Omega)$ with $\beta \geq 0$. Moreover, $A(x)\xi\cdot\xi \geq A_{\min}|\xi|^2$ and $c(x) \geq c_{\min}$ with $A_{\min}$, $c_{\min} > 0$ and $f \in \mathrm{BC}([0,\infty[\,, \mathrm{L}^2(\Omega))$.

(a) Let $E_1(t) = \int_\Omega c(x)u(t,x)^2\,\mathrm{d}x$ and assume that u is a sufficiently smooth solution. Derive the estimate

$$
\dot{E}_1(t) + c_1\|\nabla u\|^2_{\mathrm{L}^2(\Omega)} \leq C_2E_1(t) + C_3\|f(t)\|^2_{\mathrm{L}^2(\Omega)},
$$

where the constants c_1, C_2, C_3 may only depend on the coefficients.

(b) Use the Gronwall lemma to show the a priori estimates

$$
\|u(t)\|^2_{\mathrm{L}^2(\Omega)} + \int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \;\leq\; C_4\big(\|u(0)\|^2_{\mathrm{L}^2(\Omega)} + F^2\big)\mathrm{e}^{C_5 t},
$$

where $F = \sup\{\, \|f(t)\|_{\mathrm{L}^2(\Omega)} \mid t \geq 0 \,\}$.