Exercise Sheet 11

Exercise 41. Elliptic and parabolic equation via Fourier transform in $\mathbb{R}^d$.
(a) For $A \in \mathbb{R}^{d\times d}$ symmetric and positive definite, $b \in \mathbb{R}^d$, and $c \in]0,\infty[$ show that the elliptic equation

$$-\sum_{i,j=1}^{d} A_{ij}\tfrac{\partial^2}{\partial x_i \partial x_j}u + \sum_{k=1}^{d} b_k \tfrac{\partial}{\partial x_k}u + c\,u = f \text{ auf } \mathbb{R}^d$$

has a unique solution $u \in \mathrm{H}^2(\mathbb{R}^d)$ for each $f \in \mathrm{L}^2(\mathbb{R}^d)$.
(c) For $c > 0$ consider the parabolic equation

$$\tfrac{\partial}{\partial t}u = \Delta u - c\,u + f \quad \text{for } (t,x) \in \mathbb{R}^{1+d}.$$

Show that for each $f \in \mathrm{L}^2(\mathbb{R}^{1+d})$ there exists a unique solution $u \in \mathrm{L}^2(\mathbb{R}^{1+d})$, which satisfyies $\frac{\partial}{\partial t}u, \frac{\partial}{\partial x_j}u, \frac{\partial^2}{\partial x_i \partial x_j}u \in \mathrm{L}^2(\mathbb{R}^{1+d})$.

Exercise 42 (in written form). Cea's Lemma (Jean Céa 1962) Consider a symmetric, bounded and coercive bilinear form $B : H \times H \to \mathbb{R}$ in a real Hilbert space H:

$$B(u,v) = B(v,u), \quad |B(u,v)| \le C_1\|u\|\|v\|, \quad B(u,u) \ge c_0\|u\|^2.$$

(a) Let the approximation space H_N be a closed subspace of H. For $F \in H^* = \mathrm{Lin}(H,R)$ consider the solution $u \in H$ and the approximation $u_N \in H_N$ to be defined via

$$B(u,v) = F(v) \text{ for all } v \in H \quad \text{and} \quad B(u_N,v_N) = F(v_N) \text{ for all } v_N \in H_N,$$

respectively. Show that u_N is uniquely defined and satisfies the Céa estimate

$$\|u - u_N\| \le \sqrt{C_1/c_0}\,\inf\{\,\|u - w_N\| \mid w_N \in H_N\,\}, \tag{Céa}$$

i.e., u_N is, up to a constant, as good as the best approximant in H_N.
Hint: Show $B(u-u_N, u-u_N) + B(w_N-u_N, w_N-u_N) = B(u-w_N, u-w_N)$.
(b) Let $H = \mathrm{H}^1_0(]0,\pi[)$ and $H_N = \mathrm{span}\{\,\sin(k\cdot) \mid k = 1,...,N\,\}$. For $a \in \mathrm{C}^1([0,\pi])$ with $a(x) \in [1,2]$ and $f \in \mathrm{L}^2(]0,\pi[)$ we set $B(u,v) = \int_0^\pi a(x)u'(x)v'(x)\,\mathrm{d}x$ and $F(v) = \int_0^\pi f(x)v(x)\,\mathrm{d}x$. For u and u_N be as in (a) provide the estimates

$$\|u_N - u\|_{\mathrm{H}^1} \le \frac{1}{N}C_2\|u\|_{\mathrm{H}^2} \quad \text{and} \quad \|u\|_{\mathrm{H}^2} \le C_a\|f\|_{\mathrm{L}^2}.$$

Hint: Combine Exercise 25 and (Céa).

please turn

Exercise 43. Resolvents and eigenvalues of elliptic operators. Consider a bounded Lipschitz domain$\Omega \subset \mathbb{R}^d$. Further, $A \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, $a \in \mathrm{L}^\infty(\Omega; \mathbb{R}^d)$ and $\alpha \in \mathrm{L}^\infty(\Omega)$ are given such that for $\delta > 0$ we have $A(x)\xi \cdot \xi \geq \delta|\xi|^2$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^d$. On the complex Hilbert space $\mathrm{H}^1_0(\Omega) \oplus \mathrm{i}\mathrm{H}^1_0(\Omega)$ we define the sesquilinear form

$$B(u,v) = \int_\Omega A(x)\nabla u(x) \cdot \overline{\nabla v(x)} + a(x) \cdot \nabla u(x)\, \overline{v(x)} + \alpha(x)u(x)\overline{v(x)}\,\mathrm{d}x$$

and call $\lambda \in \mathbb{C}$ an eigenvalue of B, if there exists $u \in \mathrm{H}^1_0(\Omega)\backslash\{0\}$ such that

$$B_\lambda(u,v) := B(u,v) - \lambda \int_\Omega u\,\overline{v}\,\mathrm{d}x = 0 \quad \text{for all } v \in \mathrm{H}^1_0(\Omega).$$

If for some $\lambda \in \mathbb{C}$ there exists $c_0 > 0$ such that $\big|B_\lambda(u,u)\big| \geq c_0\|u\|^2_{\mathrm{H}^1}$, then λ lies in the resolvent set of B. Our aim is to show that the resolvent set is large.

(a) Find $C_1 > 0$, $c_0 > 0$, and λ_* such that $|B(u,v)| \leq C_1\|u\|_{\mathrm{H}^1}\|v\|_{\mathrm{H}^1}$ and $B_\lambda(u,u) \geq c_0\|u\|^2_{\mathrm{H}^1}$ holds for all $u, v \in \mathrm{H}^1_0(\Omega)$ and $\lambda \leq \lambda_*$.

(b) Construct postive numbers β, γ such that the set $\{\, \lambda \in \mathbb{R} \mid |\mathrm{Im}\,\lambda| \geq \gamma(\mathrm{Re}\,\lambda + \beta)\,\}$ is contained in the resolvent set. For this consider $B_{\theta,\lambda}(u,v) = \mathrm{e}^{\mathrm{i}\theta}B_\lambda(u,v)$ and bound $|B_{\theta,\lambda}(u,u)|$ from below via $\mathrm{Re}\,\big(B_{\theta,\lambda}(u,u)\big)$.

Exercise 44. A bilinear form with Robin boundary conditions Consider the two-dimensional domain $\Omega =]0,\pi[^2$, the Dirichlet boundary $\Gamma_{\mathrm{D}} = \{\, (x_1,x_2) \,|\, x_1 \in \{0,\pi\},\ x_2 \in]0,\pi[\,\}$ and the symmetric bilinear form

$$B(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) + \alpha u(x)v(x)\,\mathrm{d}x + \int_{\partial\Omega\backslash\Gamma_{\mathrm{D}}} \rho u(x)v(x)\,\mathrm{d}a \quad \text{with } \alpha, \rho \in \mathbb{R}$$

on the Hilbert space $\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) = \{\, u \in \mathrm{H}^1(\Omega) \mid u|_{\Gamma_{\mathrm{D}}} = 0 \text{ in the trace sense}\,\}$.

(a) Show that for $\alpha > 0$ and $\rho \geq 0$ the bilinear form is bounded and coercive. For the weak problem $B(u,v) = \int_\Omega fv\,\mathrm{d}x$ write the associated linear problem $L_{\alpha,\rho}u = f$ in strong form.

(b) Let $\alpha = \rho = 0$ and find all eigenfunctions u for $L_{0,0}$ (i.e. $L_{0,0}u = \lambda u$), which have the form $u(x) = \psi(x_1)\phi(x_2)$. Is the set of these eigenfunctions a complete ONS?