Exercise Sheet 10

Exercise 37 (in written form). A corner singularity. In $\mathbb{R}^2$ use the polar coordinates $x = r(\cos\phi, \sin\phi)$ and define the domain Ω as the three-quarter disc

$$\Omega = \{\, r(\cos\phi, \sin\phi) \in \mathbb{R}^2 \mid r \in \,]0,1[\,,\ \phi \in \,]0, 3\pi/2[\,\}.$$

(a) Check that the function $w(x) = r^{2/3}\sin(2\phi/3)$ is harmonic in Ω and lies in $\mathrm{H}^1(\Omega)$. Does it also lie in $\mathrm{H}^2(\Omega)$?

(b) Find a function $f \in \mathrm{C}^0(\overline{\Omega}) \subset \mathrm{L}^2(\Omega)$ such that the unique solution u of the Dirichlet problem $\Delta u = f$ in Ω and $u|_{\partial\Omega} = 0$ lies in $\mathrm{H}^1(\Omega)$ but not in $\mathrm{H}^2(\Omega)$. (Hint: Determine the values of w on the boundary $\partial\Omega$ and write $u = w - v$ for a smooth function v.)

Exercise 38. The Rellich-Kondrachov Theorem of compact embedding: For a bounded domain $\Omega \subset \mathbb{R}^d$ each bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{H}^1_0(\Omega)$ has a subsequence $(u_{n_l})_{l\in\mathbb{N}}$ and a limit $u \in \mathrm{H}^1_0(\Omega)$ such that $u_{n_l} \rightharpoonup u$ weakly in $\mathrm{H}^1_0(\Omega)$ and $u_{n_l} \to u$ in$\mathrm{L}^2(\Omega)$.

(a) Prove this result for the case $\Omega = \,]0,\pi[^d$ using a suitable orthonormal Fourier basis $\{\, \phi_k \mid k \in \mathbb{N}^d \,\}$ and by looking at convergence of the coefficients.

(b) Generalize the result to all bounded domains. (Hint: Embed $\mathrm{H}^1_0(\Omega)$ into $\mathrm{H}^1_0(]-R,R[^d)$.)

Exercise 39. Difference quotients. For $u \in \mathrm{L}^2(\mathbb{R}^d)$ and $h \in \mathbb{R}^d \setminus \{0\}$ we write

$$u^h(x) = \tfrac{1}{|h|}\big(u(x{+}h) - u(x)\big). \tag{1}$$

(a) Show that for $u \in \mathrm{H}^1(\Omega)$ we always have $\|u^h\|_{\mathrm{L}^2} \le \|\nabla u\|_{\mathrm{L}^2}$.

(b) Assume that $u \in \mathrm{L}^2(\mathbb{R}^d)$ satisfies $C_u := \sup\{\, \|u^h\|_{\mathrm{L}^2} \mid h \in \mathbb{R}^d \setminus \{0\}\,\} < \infty$. Show that $u \in \mathrm{H}^1(\mathbb{R}^d)$ with $\|\nabla u\|_{\mathrm{L}^2} \le dC_u$.

Exercise 40. Regularity. Assume $a, c \in \mathrm{BC}^1(\mathbb{R}^d)$ (= functions and first derivatives are bounded and continuous) with $a(x), c(x) \ge \alpha > 0$ for all $x \in \mathbb{R}$. Let u be the unique weak solution $u \in \mathrm{H}^1(\mathbb{R}^d) = \mathrm{H}^1_0(\mathbb{R}^d)$ of

$$-\operatorname{div}\Big(a(x)\nabla u(x)\Big) + c(x)u(x) = f(x) \tag{2}$$

for some $f \in \mathrm{L}^2(\mathbb{R}^d)$.

(a) Show that u^h defined in (1) is a weak solution of (2) with a suitable right-hand side F^h satisfying the estimate $\sup\{\, \|F^h\|_{\mathrm{H}^{-1}} \mid h \in \mathbb{R}^d \setminus \{0\}\,\} \le C(a,c)\|f\|_{\mathrm{L}^2}$.

(b) Conclude $u \in \mathrm{H}^2(\mathbb{R}^d)$ and derive a PDE for the partial derivatives $\partial_{x_j} u$.