Exercise Sheet 9

Exercise 33 (in written form). Weak derivatives. For $\alpha \in \mathbb{N}_0^d$ a function $g_\alpha \in \mathrm{L}^1(\Omega)$ is called the weak α-derivative of $u \in \mathrm{L}^p(\Omega)$ if

$$\forall\, \phi \in \mathrm{C}_\mathrm{c}^\infty(\Omega): \quad \int_\Omega u(x)\, \mathrm{D}^\alpha \phi(x)\, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g_\alpha(x)\, \phi(x)\, \mathrm{d}x$$

We then write $\mathrm{D}^\alpha u := g_\alpha$ (in the sense of distributions).

(a) Show that the weak derivative is unique and that for $u \in \mathrm{C}^n(\overline{\Omega})$ all weak derivatives up to order n exist and coincide with the classical derivatives $\mathrm{D}_\mathrm{cl}^\alpha u$.

(b) Let $\Omega = B_1(0) \subset \mathbb{R}^d$ with $d \geq 2$ and $u(x) = 1/|x|$. For which p do we have $u \in \mathrm{L}^p(\Omega)$ and for which q does the classical gradient $\nabla_\mathrm{cl} u$ lie in $\mathrm{L}^q(\Omega)$? Show that for these values the weak gradient ∇u coincides with $\nabla_\mathrm{cl} u$.

Exercise 34. Sobolev spaces. Let $p \in [1, \infty[$. For general domains $\Omega \subset \mathbb{R}^d$ one defines the Sobolev spaces $W^{n,p}(\Omega)$ using the weak derivatives as follows:

$$W^{n,p}(\Omega) = \{\, u \in \mathrm{L}^p(\Omega) \mid \forall\, \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq n\ \exists\, g_\alpha \in \mathrm{L}^p(\Omega): \ \mathrm{D}^\alpha u = g_\alpha \,\},$$

$$\|u\|_{n,p} = \Big(\sum_{|\alpha| \leq n} \|\mathrm{D}^\alpha u\|_{\mathrm{L}^p}^p \Big)^{1/p}.$$

Moreover, $W_0^{n,p}(\Omega)$ is defined as the completion of $\mathrm{C}_\mathrm{c}^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{n,p}$. For bounded domains $\Omega \subset \mathbb{R}^d$ we define $\widetilde{W}^{n,p}(\Omega)$ as the completion of $\mathrm{C}^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{n,p}$.

(a) Show the inclusion $\widetilde{W}^{n,p}(\Omega) \subset W^{n,p}(\Omega)$. (Hint: For Cauchy sequences $(u_k)_{k\in\mathbb{N}}$ in $W^{n,p}(\Omega)$ consider the limits of $\mathrm{D}^\alpha u_k$.)

(b) Show that $W^{n,p}(\Omega)$ is a Banach space.

(Remark: For bounded domains Ω with Lipschitz boundary, one has $\widetilde{W}^{n,p}(\Omega) = W^{n,p}(\Omega)$.)

please turn

Exercise 35. DIRICHLET **problem on the unit disc.** Let $\Omega = \{ x \in \mathbb{R}^2 \mid |x| < 1 \}$ and consider the DIRICHLET problem

$$\Delta u(x) = 0 \text{ for } x \in \Omega, \quad u(y) = g(y) \text{ for } y = (\cos\phi, \sin\phi) \in \partial\Omega =: \mathbb{S}. \qquad \text{(DP)}$$

(a) Construct for $g_N : \phi \mapsto \mathrm{Re}\big(\sum_{n=0}^{N} c_n \mathrm{e}^{\mathrm{i}n\phi}\big)$ the associated solution u_N of (DP) and calcualte the norm of u_N in $\mathrm{L}^2(\Omega)$, $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^2(\Omega)$ explicitly. (Hint: $x \mapsto \mathrm{Re}\big(c_n(x_1{+}\mathrm{i}x_2)^n\big)$ solves (DP) for suitable boundary data and satisfies some orthogonality condition.)

(b) Investigate via the limit $N \to \infty$ for which functions $g : \partial\Omega \to \mathbb{R}$ the solution of (DP) exists and lies in $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^2(\Omega)$, respectively. For $s \in [0, \infty[$ use the function spaces

$$\mathrm{H}^s_{\mathrm{per}}(\mathbb{S}) = \Big\{ f \in \mathrm{L}^2(\mathbb{S}) \mid \exists\, (c_k)_{k\in\mathbb{Z}}\colon \sum_{k\in\mathbb{Z}} (1{+}|k|^2)^s |c_k|^2 < \infty \text{ and } f(\phi) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}k\phi} \Big\}.$$

Exercise 36. The LAX-MILGRAM **lemma in an unbounded domain.**
Let $\Omega \subset \mathbb{R}^d$ be any domain (i.e., open and connected). On $H = \mathrm{H}^1_0(\Omega)$ define the bilinear form $a : H \times H \to \mathbb{R}$ via

$$a(u, v) = \int_\Omega \nabla u(x) \cdot A(x) \nabla v(x) + c(x) u(x) v(x) \,\mathrm{d}x$$

which $A \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $c \in \mathrm{L}^\infty(\Omega)$. Further there is an $\alpha > 0$ with $\xi \cdot A(x)\xi \geq \alpha|\xi|^2$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^d$.

(a) Show that a is a symmetric and continuous bilinear form. Give sufficient conditions on c that guarantee that a is also coercive.

(b) Consider the Schrödinger operator $A_\lambda u = -\Delta u {+} V u {+} \lambda u$ in $\mathrm{H}^1_0(\Omega)$ with $\Omega = \mathbb{R}\times]0, \pi[$, where $V(x) = 1/(1{+}x_1^2)$. Show that the bilinear form a_λ associated with A_λ is coercive for $\lambda > -1$ but not for $\lambda \leq -1$.