Exercise Sheet 8

Exercise 29. Friedrichs' inequality. A domain $\Omega \subset \mathbb{R}^d$ is said to satisfy a FRIEDRICHS' INEQUALITY, if there exists a constant $C > 0$ such that

$$(*) \qquad C \int_\Omega u(x)^2 \,\mathrm{d}x \le \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x \text{ for all } u \in \mathrm{C}_\mathrm{c}^\infty(\Omega).$$

The largest such C is called the Friedrichs constant $C_{\text{Fried}}(\Omega)$ of the domain.

(a) Show that (*) holds if and only if the same inequality holds for all $u \in \mathrm{H}_0^1(\Omega)$ (which is the completion of $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ in the norm of $\mathrm{H}^1(\Omega)$, i.e., $\|u\|_{\mathrm{H}^1(\Omega)}^2 = \int_\Omega u^2 + |\nabla u|^2 \,\mathrm{d}x$).

(b) Show that every bounded domain satisfies a Friedrichs inequality. For unbounded domains show that both cases can occur.

(c) Using Fourier series find the Friedrichs' constant for the domain $\Omega =]0, \ell_1[\times \cdots \times]0, \ell_d[$.

Exercise 30. Maximum principle. Assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain (i.e. open and connected). Moreover, let $u \in \mathrm{C}^0(\overline{\Omega})$ be subharmonic, i.e. $\forall\, x \in \Omega\ \forall\, r > 0$ mit $B_r(0) \subset \Omega$: $u(x) \le \frac{1}{\omega_d r^{d-1}} \int_{\partial B_r(x)} u(y) \,\mathrm{d}a(y)$. Show the following statements:

(a) *Weak form of the maximum principle:* u attains its maximum on $\partial\Omega$.

(b) *Strong form of the maximum principle:* If u attains the maximum (also) in the interior of Ω, then u is constant.

Exercise 31 (in written form). Green's function for the strip $\Omega = \mathbb{R} \times]0, \pi[$:
(a) For $\alpha > 0$ consider the ODE $-u'' + \alpha^2 u = f$ on $\mathbb{R}$ (one-dimensional elliptische problem). Show that for $f \in \mathrm{BC}^0(\mathbb{R})$ (bounded continuous functions) the unique bounded solution is given via

$$u(x) = \int_{y \in \mathbb{R}} G_\alpha(x-y) f(y) \,\mathrm{d}y \quad \text{with } G_\alpha(z) = \frac{1}{2\alpha} \mathrm{e}^{-\alpha|z|}.$$

(b) To solve the DIRICHLET-Problem $\Delta u = f$ in Ω with $u = 0$ on $\partial\Omega$ decompose u and f in FOURIER series with respect to the x_2 direction ($u(x_1, x_2) = \sum_1^\infty u_k(x_1) \sin(k x_2)$). Derive solution formulas for the coefficients u_k and construct a series representation of the Green's function G.

(c) Find an explicit formula for G and check that $(x, y) \mapsto G(x, y) - K_2(x-y)$ is analytic on $\Omega \times \Omega$.

Exercise 32. General properties of Green's functions. Assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^2 boundary. Denote by $G : \Omega \times \Omega \to \mathbb{R} \cup \{-\infty\}$ the Green's function for the Laplace operator on Ω with pure Dirichlet boundary conditions. Derive the following general properties:

(i) For all $x, y \in \overline{\Omega}$ we have $G(x, y) = G(y, x)$ and further $\Delta_x G(x, y) = \Delta_y G(x, y) = 0$ for $x \ne y$. (Hint: $\int_\Omega u_1 \Delta u_2 \,\mathrm{d}x = \int_\Omega u_2 \Delta u_1 \,\mathrm{d}x$ and choose ssuitable $f_j = \Delta u_j$.)

(ii) $\int_{\partial\Omega} \nabla_y G(x, y) \cdot \nu(y) \,\mathrm{d}a(y) = 1$ for all $x \in \Omega$.

(iii) $G(x, y) < 0$ for $x, y \in \Omega$. (Hint: Show that $G(x, \cdot) = 0$ is subharmonic and use a maximum principle.)

(iv) For $y \in \partial\Omega$ we have $\nabla_y G(x, y) = \alpha(x, y)\nu(y)$ for some $\alpha(x, y) \ge 0$, i.e. $\nabla_y G(x, y) \cdot \nu(y) \ge 0$.