Exercise Sheet 7

Exercise 25 (in written form). Fourier series.
Let $\Omega =]0, 2\pi[$ and $k \in \mathbb{N}_0$ and consider the Hilbert spaces

$$\mathrm{H}^k(\Omega) = \{\, f \in \mathrm{L}^2(\Omega) \mid f, f', ..., f^{(k)} \in \mathrm{L}^2(\Omega)\,\},$$
$$\mathrm{H}^k_{\mathrm{per}}(\Omega) = \{\, f \in \mathrm{H}^k(\Omega) \mid f^{(j)}(0) = f^{(j)}(2\pi) \text{ for } j = 0, ..., k{-}1\,\},$$
$$\mathrm{H}^k_0(\Omega) = \text{closure of } \mathrm{C}^\infty_c(\Omega) \text{ in } \mathrm{H}^k(\Omega).$$

Further let $S_n(t) = s_n \sin(nt)$ and $C_n(t) = c_n \cos(nt)$. Then, $\mathrm{L}^2(\Omega)$ has the following three complete orthonormal systems (ONS)

$$O_1 = \{\, C_n, S_m \mid m \in \mathbb{N},\ n \in \mathbb{N}_0\,\},\quad O_2 = \{\, C_{m/2} \mid m \in \mathbb{N}_0\,\},\quad O_3 = \{\, S_{m/2} \mid m \in \mathbb{N}\,\}.$$

Recall that an ONS $O = \{\, \phi_j \mid j \in \mathbb{N}\,\}$ in H is called *complete*, if for all $u \in H$ we have $\|u\|^2 = \sum_{j=1}^\infty \langle u, \phi_j\rangle^2$ and hence $u = \sum_{j=1}^\infty \langle u, \phi_j\rangle \phi_j$.

One may use without proof: $\mathrm{H}^{k+1}(]0, 2\pi[) \subset \mathrm{C}^k([0, 2\pi])$ for $k = 0, 1, \ldots$ and
$\mathrm{H}^1_0(]0, 2\pi[) = \{\, f \in \mathrm{H}^1(]0, 2\pi[) \mid f(0) = 0 = f(2\pi)\,\}$.

(a) For O_1 show that $f = \sum_1^\infty a_m S_m + \sum_0^\infty b_n C_n \in \mathrm{L}^2(\Omega)$ lies in $\mathrm{H}^1_{\mathrm{per}}(\Omega)$ if and only if $\sum_1^\infty l^2(|a_l|^2{+}|b_l|^2)$ is finite and that in this case we may differentiate the series representation of f term by term.

(b) For O_2 show that $f = \sum_0^\infty b_m C_{m/2}$ lies in $\mathrm{H}^1(\Omega)$ iff $\sum_0^\infty m^2 b_m^2 < \infty$.

(c) For O_3 show that $f = \sum_1^\infty a_m S_{m/2}$ lies in $\mathrm{H}^1_0(\Omega)$ iff $\sum_1^\infty m^2 a_m^2 < \infty$.

(General hint: Compare the series differentiated term by terms with a suitable new expansion of the derivative. Beware of boundary terms in integrations by parts.)

Exercise 26. Poisson's formula for a disc. Let $\Omega = B_R(0) \subset \mathbb{R}^2$, $g \in \mathrm{C}^0(\partial\Omega)$, and

$$u(x) = \int_{|y|=R} P(x, y)\, g(y)\, \mathrm{d}a \quad \text{with } P(x, y) = \frac{R^2 - |x|^2}{2\pi R\, |x{-}y|^2}. \tag{PI}$$

(a) Show that (PI) defines a function $u \in \mathrm{C}^2(\Omega)$ satisfying $\Delta u = 0$ in Ω.

(b) Establish $u \in \mathrm{C}(\overline{\Omega})$ and $u(y) = g(y)$ for $y \in \partial\Omega$.
(Hint: Show $P(x, y) \geq 0$, $\int_{\partial\Omega} P(x, y)\, \mathrm{d}a = 1$ for all $x \in \Omega$, and $P(x, y) \to 0$ for $x \to y_* \in \partial\Omega \backslash \{y\}$. Polar coordinates $x = r(\cos\phi, \sin\phi)$ and $y = R(\cos\psi, \sin\psi)$ may come in handy.)

Prize exercise (not solved in tutorial), Prize = 20 € book coupon.
For some $d \geq 2$ find a function $f \in \mathrm{C}^0_c(\mathbb{R}^d)$, such that the solution u of the Poisson problem $\Delta u = f$ given via the convolution $u = K_d * f$ does not lie in $\mathrm{C}^2(\mathbb{R}^d)$.
[[Please turn in solutions in written form to A. Mielke by July 9, 2009, 13:15 h.]]

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Exercise 27. Green's function for a quadrant:
Consider the quarter plane $\Omega =]0,\infty[^2$ Dirichlet boundary $\Gamma_{\mathrm{Dir}} =]0,\infty[\times \{0\}$ and Neumann boundary $\Gamma_{\mathrm{Neu}} = \{0\} \times]0,\infty[$.

Construct the Green's functions G, G_{Dir}, and G_{Neu} such that

$$u(x) = \int_\Omega G(x,y)f(y)\,\mathrm{d}y + \int_0^\infty G_{\mathrm{Dir}}(x,y_1,0)u_{\mathrm{Dir}}(y_1,0)\,\mathrm{d}y_1 + \int_0^\infty G_{\mathrm{Neu}}(x,0,y_2)h_{\mathrm{Neu}}(0,y_2)\,\mathrm{d}y_2$$

provides the solution of the Poisson problem

$$\Delta u = f \text{ in } \Omega, \quad u = u_{\mathrm{Dir}} \text{ on } \Gamma_{\mathrm{Dir}}, \quad \nabla u \cdot \nu = h_{\mathrm{Neu}} \text{ on } \Gamma_{\mathrm{Neu}}$$

(Hint: Use appropriate reflections.)

Exercise 28. Linearized elasticity.
We treat an elliptic system on a bounded $\Omega \subset \mathbb{R}^3$, which describes an elastic body. By $u : \Omega \to \mathbb{R}^3$ we denote the displacement. Given the two material parameters $\lambda, \mu > 0$ (also called Lamé constants) the Lamé operator A is defined via

$$Au = -\mu\Delta u - (\lambda{+}\mu)\nabla(\operatorname{div} u) := \begin{pmatrix} -\mu\Delta u_1 - (\lambda{+}\mu)\partial_{x_1}(\operatorname{div} u) \\ -\mu\Delta u_2 - (\lambda{+}\mu)\partial_{x_2}(\operatorname{div} u) \\ -\mu\Delta u_3 - (\lambda{+}\mu)\partial_{x_3}(\operatorname{div} u) \end{pmatrix}$$

(a) Establish the counter parts to Green's formula, namely

$$\int_\Omega (Au)\cdot v\,\mathrm{d}x = \int_\Omega \Sigma(u){:}\frac{1}{2}(\nabla v{+}\nabla v^{\mathsf{T}})\,\mathrm{d}x - \int_{\partial\Omega} v\cdot\Sigma(u)\nu\,\mathrm{d}a,$$
$$\int_\Omega (Au)\cdot v - u\cdot(Av)\,\mathrm{d}x = \int_{\partial\Omega} u\cdot\Sigma(v)\nu - v\cdot\Sigma(u)\nu\,\mathrm{d}a,$$

where $\Sigma(u) = \lambda \operatorname{div} u\, I + \mu(\nabla u{+}\nabla u^{\mathsf{T}}) \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$.
(b) Find $\alpha, \beta \in \mathbb{R}$ such that for each vector $b \in \mathbb{R}^3$ the function $u(x) = K(x)b$ satisfies $Au(x) = 0$ for $x \neq 0$, where the matrix-valued function K is given via

$$K(y) = \frac{\alpha}{|y|}I + \frac{\beta}{|y|^3}\, y\times y \in \mathbb{R}^{3\times3}_{\mathrm{sym}}.$$

[Remark: For the correct choice of α and β we obtain a fundamental matrix such that $u = K * f$ with $f \in \mathrm{C}^2_{\mathrm{c}}(\mathbb{R}^3;\mathbb{R}^3)$ provides a solution of $Au = f$.]