Höhere Analysis II: Partial Differential Equations
Sommersemester 2009
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21. Mai 2009

Exercise Sheet 6

Exercise 21. The Korteweg–de Vries equation describes the evolution of the water surface in cases where the surface remains sufficiently smooth and surface tension can be neglected:

$$u_t + 6uu_x + u_{xxx} = 0 \qquad \text{for } t, x \in \mathbb{R}. \tag{1}$$

(a) Construct all nontrivial solution of the form $u(t,x) = U(x{-}ct)$ satisfying $U(\xi) \to 0$ for $|\xi| \to \infty$. (These solitary waves are used to model tsunamis.)

(b) Transform the equation into a first order system $A_0\mathbf{w}_t + A_1\mathbf{w}_x = \mathbf{g}(\mathbf{w})$ and find all characteristic curves. Can the initial value problem with $u(0,x) = u_0(x)$ be solved with the theorem of CAUCHY-KOVALEVSKAYA?

Lemma 6.1 (LAGRANGE 1736–1813) *Consider* $a \in \mathbb{R}$ *and a function* $m \in \mathrm{C}(\mathbb{R})$*, for which there exists* $C > 0$ *such that* $m(y) \leq C{-}|y|/C$ *for* $y \in \mathbb{R}$*. Then, we have*

$$I(\varepsilon) = 2\varepsilon \log\Big(a \int_{\mathbb{R}} \exp\big(m(y)/(2\varepsilon)\big)\,\mathrm{d}y\Big) \longrightarrow m_* := \max\{\, m(y) \mid y \in \mathbb{R}\,\} \textit{ for } \varepsilon \to 0^+.$$

Exercise 22. The viscous Burgers equation is given via

$$u_t + uu_x = \varepsilon u_{xx}, \quad u(0,x) = f(x). \tag{2}$$

(a) Show that $(t,x) \mapsto v(t,x) = \int_{-\infty}^{x} u(t,\xi)\,\mathrm{d}\xi$ solves $v_t + \frac{1}{2}v_x^2 = \varepsilon v_{xx}$, use the Cole-Hopf transform, and the explicit solution of the heat equation $w_t = \varepsilon w_{xx}$ (see Exercise 4) to establish the following exact solution formula for the viscous Burgers equation (2):

$$u(t,x) = \frac{\partial}{\partial x} V_\varepsilon(t,x) \;\text{ with } F(x) = \int_{-\infty}^{x} f(\xi)\,\mathrm{d}\xi \text{ and}$$
$$V_\varepsilon(t,x) = -2\varepsilon \log\Big(\int_{\mathbb{R}} (2\varepsilon\pi t)^{-1/2} \exp\big(-(x{-}y)^2/(4\varepsilon t) - F(y)/(2\varepsilon)\big)\,\mathrm{d}y\Big),$$

(b) Use Lemma 6.1 for passing to the limit of vanishing viscosity, i.e. $\varepsilon \to 0$, and derive formula (3) below.

please turn

General fact (for proofs see the book of EVANS):
For the Burgers equation the entropy solutions and the solutions obtained via the vanishing-viscosity limit are identical.

Exercise 23. [in written form] The LAX-OLEINIK formula

$$u(t,x) = \partial_x V(t,x) \quad \text{with } V(t,x) = \min\{\, \tfrac{(x-y)^2}{2t} + F(y) \mid y \in \mathbb{R} \,\} \tag{3}$$

by PETER LAX (*1926, Abel Prize 2005) and OLGA OLEINIK (1925-2001) gives the unique entropy solution for the initial value problem for the BURGERS equation

$$u_t + u u_x = 0 \text{ for } t > 0,\ x \in \mathbb{R}; \quad u(0,\cdot) = f \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R}). \tag{4}$$

(a) Evaluate this formula for $f = f_\alpha$, where $f_\alpha(x) = 0$ for $x < 0$ and $f_\alpha(x) = \alpha$ for $x > 0$. Check that in all cases the entropy solution is obtained.

(b) Assume that locally near (t_*, x_*) the minimum in the definition of V (see (3)) is attained at a unique $y = Y(t,x)$. Show that we then have a classical solution. (Hint: Define $G(t,x,y) = (x-y)^2/(2t) + F(y)$, use $V(t,x) = G(t,x,Y(t,x)) = \min G(t,x,\cdot)$, and conclude $\partial_t V + \frac{1}{2}(\partial_x V)^2 = 0$.)

Exercise 24. Qualitative properties of the solutions to Burgers equation:

Again consider the initial value problem (4) with the solution formula (3).

(a) Show that the solution formula can also be written as $u(t,x) = \frac{x - Y(t,x)}{t}$, where $Y(t,x)$ is any of the minimizers. When do we have more than one minimizer?

(b) Using the solution formula (3) or (a) show that for all $t > 0$ we have the relations

$$\text{(i) } \int_{\mathbb{R}} u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{d}x, \qquad \text{(ii) } \|u(t,\cdot)\|_{\mathrm{L}^\infty(\mathbb{R})} \le C \|f\|_{\mathrm{L}^1(\mathbb{R})} / t^{1/2},$$

where an optimal value for $C > 0$ has to be found in terms of f.