Exercise Sheet 5

Exercise 17 [in written form]. Eikonal equation:

The eikonal equation in a medium with varying refraction index n reads

$$|\nabla u|^2 - n(x) = 0.$$

The associated equation for the characteristic strips reads

$$|p|^2 - n(x) = 0, \quad x' = 2p, \quad z' = 2n(x), \quad p' = -\nabla_x n(x),$$

which we want to solve with the initial data $u(x_1, 0) = 0$ and $\partial_{x_2} u(x_1, 0) > 0$.

We consider a two-dimensional problem with $\Omega = \mathbb{R}^2$, where the refraction index only depends on x_1, i.e. $n(x) = \mu(x_1)$. This corresponds to a layered material.

(a) Show that along characteristic strips the equation equation for $x_1(s)$ takes the form $x_1'' + 2\mu'(x_1) = 0$. Find a function of x_1 and x_1' that is constant along characteristic strips.

(b) *Focusing case:* We consider the case $\mu(x_1) = 2 - \frac{1}{1+x_1^2}$. Show that all characteristics starting at $x(0, \xi) = (\xi, 0)$ oscillate periodically around the x_2 axis and thus form a caustic. Try to estimate the smallest value $x_2 > 0$, for which two different characteristics meat in $(0, x_2^*)$. (The material acts as a wave guide, since light is guided along the x_2-axis.)

(c) [Voluntary] Write a computer program for solving the characteristic strips numerically. Plot a few characteristics $s \mapsto x(s; \xi)$ for the the case in (b) and for $\mu(x_1) = 1 + \frac{1}{1+x_1^2}$.

Exercise 18. Viscous conservation laws: For $\varepsilon > 0$ the semilinear parabolic equation

$$u_t + f'(u)u_x = \varepsilon u_{xx} \quad \text{or} \quad u_t + \big(f(u) - \varepsilon u_x\big)_x = 0$$

is called a viscous conservation law, as the right-hand side is in conservation form.

The positive sign of the viscosity ε is remanent to introducing a positive direction of time.

(a) We are looking for traveling waves $u(t, x) = U\big((x - ct)/\varepsilon\big)$. Derive an ODE of the form $U' = g(U)$, which has to be satisfies by such waves. Give its explicit form in the case of $f(u) = \frac{1}{2}u^2$.

(b) A wave U from (a) is called front if it satisfies $U(\xi) \to u_\pm$ for $\xi \to \pm\infty$. Discuss the existence of such fronts and relate it to the entropy condition and the Rankine-Hugoniot condition.

please turn

Exercise 19. The Cole-Hopf transformation. We consider the linear heat equation

$$u_t = k\Delta u \tag{1}$$

and the nonlinear transformation $u = \phi(v)$ or $v = \psi(u)$.

(a) Find a function g such that v solves

$$v_t = k\Delta v + g(v)|\nabla v|^2 \tag{2}$$

if and only if u solves (1). Which ϕ give $g \equiv$ const.?

(b) Use (a) to show that any solution of the initial value problem $v_t = \Delta v + b|\nabla v|^2$, $v(0,x) = v_0(x)$ with $v_- \leq v_0(x) \leq v_+$ satisfies $v_- \leq v(t,x) \leq v_+$ and $|\nabla v(t,x)| \leq C/t^{1/2}$ for all $t > 0$ and $x \in \mathbb{R}^d$, where C depends only on v_- and v_+. (Hint: Use Exercise 4.)

Exercise 20. Theorem of Cauchy (1789-1857 & Kovalevskaya (1850-1891)

Construct the solution of the following Cauchy problems via power-series expansion.

$$u_{xx} + \sigma u_{yy} = 0, \quad u(x,0) = u_0(x) = \frac{x^2-1}{(x^2+1)^2},\ u_y(x,0) = u_1(x) = \frac{2-6x^2}{(x^2+1)^3},$$

where $\sigma \in \{-1, 1\}$.

(a) Check that the Theorem of Cauchy–Kovalevskaya is applicable in both cases.

(b) Find the power-series expansion explicitly. (Hint: Write u_0 as real part and u_1 as imaginary part of simple complex-valued functions.) Discuss the radius of convergence and the singularities of the explicit solutions.