Höhere Analysis II: Partial Differential Equations
Sommersemester 2009
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Exercise Sheet 4

Exercise 13. Wave equation: We consider the Cauchy problem

$$u_{tt} = u_{xx} + bu \text{ for } (t,x) \in \mathbb{R}^2, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),$$

where b is a real parameter.

(a) Use a power-series expansion to find the solution of the CAUCHY problem for $u_0(x) = \mathrm{e}^{\alpha x}$ and $u_1 \equiv 0$, where $\alpha \in \mathbb{C}$.

(b) Use the linearity to find the solution for $u_0(x) = \frac{1}{h}\big(\mathrm{e}^{(\alpha+h)x} - \mathrm{e}^{\alpha x}\big)$ and $u_1 \equiv 0$. Consider and justify the limit $h \to 0$ to find the solution v for the CAUCHY data $v_0(x) = x\sin(kx)$ and $v_1 \equiv 0$ mit $k \in \mathbb{R}$.

(c) Return to the solution u from Part (a). Show that $w = u_t$ is again a solution. What are the CAUCHY data for w at $t = 0$? Give the solution for the CAUCHY data $u_0 \equiv 0$ and $u_1(x) = \mathrm{e}^{\alpha x}$.

Exercise 14. Divergence form versus quasilinear form.

Quasilinear form: (QF) $\mathbb{A}(x,u,\mathrm{D}u) : \mathrm{D}^2 u = b(x,u,\mathrm{D}u)$
Divergence from: (DF) $\operatorname{div} A(x,u,\mathrm{D}u) = \widetilde{b}(x,u,\mathrm{D}u)$

(a) Assume that $\mathbb{A}$ does not depend on $\mathrm{D}u$. Show that (QF) can be rewritten as (DF). [Hint: Search for $A(x,u,\xi)$ which is linear in ξ.]

(b) For $d = 2$ and $A(x,u,\xi) = (A_1(\xi_1,\xi_2), A_2(\xi_1,\xi_2))^\mathsf{T}$ in (DF) derive the associated $\mathbb{A}$ in (QF).

(c) Provide an elliptic example of (QL) that cannot be written in as (DF). [Hint: It suffices to consider $\mathbb{A}(\xi)$ in diagonal form.]

(d) For $f \in \mathrm{C}^2(\mathbb{R}^d,\mathbb{R})$ let $A(\xi) = \mathrm{D}f(\xi)$ in (DF) and calculate the associated $\mathbb{A}$ for (QL). Show that (QL) is elliptic if f is uniformly convex, i.e. $\mathrm{D}^2 f(\xi) \geq c\mathrm{I}$ with $c > 0$ in the sense of positive definite matrices.

please turn

Exercise 15 [turn in in written form]. Weak solutions for the wave equation: We consider the linear wave equation $u_{tt} = u_{xx}$, which has the classical solutions $u(t,x) = g(x+t)+h(x-t)$ for $g, f \in \mathrm{C}^2(\mathbb{R})$. We define weak solutions $u \in \mathrm{C}^{\mathrm{Lip}}(\mathbb{R}^2)$ via the condition

$$\forall\, \phi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^2): \quad \int_{\mathbb{R}^2} \big(u_t\phi_t - u_x\phi_x\big)\,\mathrm{d}t\,\mathrm{d}x = 0,$$

and distributional solutions $u \in \mathrm{L}^\infty(\mathbb{R}^2)$ via the condition

$$\forall\, \phi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^2): \quad \int_{\mathbb{R}^2} \big(u\phi_{tt} - u\phi_{xx}\big)\,\mathrm{d}t\,\mathrm{d}x = 0.$$

We use here that for every function $u \in \mathrm{C}^{\mathrm{Lip}}(\mathbb{R}^2)$ the gradient ∇u is a well-defined function in $\mathrm{L}^\infty(\mathbb{R}^2)$ (Rademacher's theorem).

(a) Show that weak and distributional solutions satisfying additionally $u \in \mathrm{C}^2(\mathbb{R}^2)$ are classical solutions.

(b) For which g and h is $u(t,x) = g(x+t) + h(x-t)$ a weak or distributional solution? [Hint: Use linearity to discuss g and h independently.]

(c) Find the solution with the initial data $u(0,x) = f_0(x)$ and $u_t(0,x) \equiv 0$, where f_0 is the piecewise affine and periodic function with $f_0(x) = 1 - |x-1|$ for $x \in [-1,3]$ and $f_0(x+4k) = f_0(x)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.

(d) Show that the solution u in (c) satisfies $u(t,0) = 0 = u(t,2)$ for $t \in \mathbb{R}$ and $u(t+p,x) = u(t,x)$ for a suitable period p. Hence it describes the vibration of a guitar string, which is clamped at the points $x_0 = 0$ and $x_1 = 2$ and released from an initial triangle position. Draw a picture or plot u for $(t,x) \in [0,10]\times[0,2]$ and discuss the relation to experiments.

Exercise 16. The Tricomi equation (after Francesco Giacomo Tricomi 1897–1978) was important in aeronautics in flight conditions near the speed of sound. It reads

$$\partial_{x_1}^2 u - x_2\partial_{x_1}^2 u = 0.$$

(a) Write the equation in the form $\mathbb{A}(x) : \mathrm{D}^2 u = 0$ and discuss the type of the equation.

(b) Calculate all characterisic curves in the hyperbolic region, i.e. curves $s \mapsto x(s)$ such that $x'(s)\cdot\mathbb{A}(x(s))x'(s) \equiv 0$.

(c) Transform the equation via $y = Y(x)$ and $u(x) = v(Y(x))$ with $Y_1(x) = x_1 + \frac{2}{3}x_2^{3/2}$ and $Y_2(x) = x_1 - \frac{2}{3}x_2^{3/2}$ into the form

$$\partial_{y_1}\partial_{y_2}v(y) = \beta_1(y)\partial_{y_1}v(y) + \beta_2(y)\partial_{y_2}v(y).$$

(d) Find all solutions of the Tricomi equation in the form $u(x) = x_2^\alpha(x_1^2+ax_2^3)^\beta$, where $\alpha, \beta, a \in \mathbb{R}$.