Höhere Analysis II: Partial Differential Equations
Sommersemester 2009
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Exercise Sheet 3

Exercise 9. Traffic model I: We study the one-dimensional traffic model

$$\partial_t \rho + \partial_x\big(\rho V(\rho)\big) = 0 \text{ for } (t,x) \in [0,\infty[\, \times \mathbb{R}; \quad \rho(0,x) = \rho_0(x),$$

where $V(\rho) = \frac{v_{\max}}{\rho_{\max}}(\rho_{\max} - \rho)$. For a given solution $\rho : [0,T] \times \mathbb{R} \to [0,\rho_{\max}]$ we want to investigate the paths $x = X(t,\xi)$ of individual vehicles, where $t \in [0,T]$ and $\xi = X(0,\xi)$ is the starting position of the vehicle labeled by ξ. Thus, we have to solve the ODE

$$\dot{x}(t) = V(\rho(t,x(t))) \text{ and } x(0) = \xi.$$

We study two cases.

(a) *Entering of a traffic jam from behind:* For $0 < \rho_- < \rho_+ \le \rho_{\max}$ we have the solution

$$\rho(t,x) = \begin{cases} \rho_+ & \text{for } x \ge c_* t, \\ \rho_- & \text{for } x < c_* t, \end{cases} \quad \text{with } c_* = \frac{v_{\max}}{\rho_{\max}}\big(\rho_{\max} - \rho_+ - \rho_-\big).$$

Calculate $X(t,\xi)$ explicitly.

(b) *Starting from rest after traffic light switched to green*: With $s(t) = -\ell$ for $t \in [0, \ell/v_{\max}]$ and $s(t) = v_{\max}t - 2\sqrt{\ell v_{\max} t}$ for $t \ge \ell/v_{\max}$ the traffic density is given as

$$\rho(t,x) = \begin{cases} \rho_{\max} & \text{for } t \in [0,\ell/v_{\max}] \text{ and } x \in [-\ell, -v_{\max}t], \\ \frac{\rho_{\max}}{2}\Big(1 - \frac{x}{v_{\max}\, t}\Big) & \text{for } x \in [s(t), v_{\max}t], \\ 0 & \text{otherwise.} \end{cases}$$

Calculate all $X(t,\xi)$ and discuss the relative distance of the first and the last vehicle in the queue.

Exercise 10. Traffic model II: For the PDE $\partial_t \rho + \partial_x(\rho - \rho^2) = 0$ we consider the initial condition $\rho(0,x) = f(x)$ with

$$f(x) = \begin{cases} 1/2 & \text{for } x < -\ell, \\ 1 & \text{for } x \in [-\ell, 0], \\ 0 & \text{for } x > 0. \end{cases}$$

(a) Construct the entropy solution for $t \in [0, 2\ell]$ by piecing together known solutions.

(b) For $t > 2\ell$ derive an ODE for the curve $s(t)$ such that $\rho(t,x) = 1/2$ for $x < s(t)$. Use the Rankine-Hugoniot relation and check that entropy condition.

(c) Give the full entropy solution and determine $\max \rho(t,\cdot)$ explicitly.

please turn

Exercise 11 [in written form] Burgers equation (named after Johannes Martinus Burgers, 1895-1981):

$$\partial_t u + u\partial_x u = 0 \qquad \text{(inviscid case)}.$$

We consider the piecewise constant functions

$$\widetilde{u}(t,x) = \begin{cases} u_+ & \text{for } x > ct, \\ u_- & \text{for } x < ct, \end{cases} \qquad \widehat{u}(t,x) = \begin{cases} u_1 & \text{for } x < c_1 t, \\ u_2 & \text{for } c_1 t < x < c_2 t, \\ u_3 & \text{for } x > c_2 t \end{cases}$$

(a) Show that $\widetilde{u}$ is a weak solution if and only if $2c = u_+ + u_-$. Under what conditions does the shock satisfy the entropy condition?

(b) Given u_1 and u_3 (and hence $\widehat{u}(0,x)$) find all u_2, c_1, and c_2 such that $c_1 < c_2$ and that $\widehat{u}$ is a weak solution for $t > 0$. When do we have shocks satisfying the entropy condition?

(c) Construct a piecewise constant solution $u^{(1)}$ satisfying $u^{(1)}(0,x) = f_1(x)$ with $f_1(x) = 6$ for $x < -1$, $f_1(x) = 4$ for $|x| < 1$, and $f_1(x) = 2$ for $x > 1$.

(d) Give a second initial condition f_2 such that the solution $u^{(2)}$ satisfies $u^{(2)}(t,x) = u^{(1)}(t,x)$ for $t \geq 4$ and $x \in \mathbb{R}$.

Exercise 12. General jump conditions: Let $A : \Omega\times\mathbb{R} \to \mathbb{R}^d$ and $b : \Omega\times\mathbb{R} \to \mathbb{R}$ define the quasilinear PDE in conservation form

$$\begin{aligned} &\big(\operatorname{div} A(\cdot, u(\cdot))\big)(x) = \widetilde{b}(x, u(x)) \text{ for } x \in \Omega \subset \mathbb{R}^d, \\ &\text{with } \big(\operatorname{div} A(\cdot, u(\cdot))\big(x) := \textstyle\sum_{j=1}^d \partial_{x_j}\alpha_j(x) \text{ where } \alpha_j(x) = A_j(x, u(x)) \big) \end{aligned}$$

(a) Write the equation in the quasilinear form $a(x,u)\cdot\nabla u = b(x,u)$ by assuming that everything is sufficiently smooth.

(b) A function $u \in \mathrm{L}^\infty(\Omega)$ is called weak solution, if

$$\int_\Omega A(x,u(x))\cdot\nabla\phi(x) + \widetilde{b}(x,u(x))\phi(x)\,\mathrm{d}x = 0 \qquad \text{for all } \phi \in \mathrm{C}_\mathrm{c}^\infty(\Omega).$$

Show that classical solutions are weak solutions.

(c) Let $\mathcal{C}$ be a smooth hypersurface separating Ω into the two pieces Ω_+ and Ω_-. Let $u : \Omega \to \mathbb{R}$ be such that the restrictions $u_\pm = u|_{\Omega_\pm}$ lie in $\mathrm{C}^1(\overline{\Omega}_\pm)$. Derive the *jump relations*

$$\Big(A(y, u_+(y)) - A(y, u_-(y))\Big)\cdot\nu(y) \ \text{ for all } y \in \mathcal{C},$$

where ν is the normal vector to $\mathcal{C}$. (Hint: First show that in Ω_+ and Ω_- we have classical solutions. Then do integration by parts in Ω_+ and Ω_- separately and compare the boundary terms.)

(c) Discuss the jump relations in the special two-dimensional case

$$\partial_t\big(\alpha(x,u)\big) + \partial_x\big(\beta(x,u)\big) = b(t,x,u),$$

if $\mathcal{C}$ is given in the form $x = s(t)$.