

Exercise Sheet 2

Exercise 5. Scalar equation of first order:

(a) Find the solution of the semilinear PDE $u_t + u_y = u^2$, which satisfies the initial condition $u(0, y) = e^y \sin(y^2+3) + y^5$.

(b) For arbitrary, differentiable $f : \mathbb{R} \rightarrow \mathbb{R}$ give the solution of the Cauchy problem

$$u_t + yu_y = y^2, \quad u(0, y) = f(y).$$

Exercise 6 [in written form]. Transformation of quasilinear equations:

We consider the quasilinear problem

$$a(x, u(x)) \cdot \nabla u(x) = g(x, u(x)) \text{ for } x \in \Omega \subset \mathbb{R}^d, \quad (\text{Q})$$

where the function $u : \Omega \rightarrow \mathbb{R}$ is to be determined.

(a) Consider a coordinate change $x = \Phi(y)$ with a bijective mapping $\Phi : \tilde{\Omega} \rightarrow \Omega$, where Φ and Φ^{-1} are in C^1 . We let $v(y) = u(\Phi(y)) = (u \circ \Phi)(y)$. Which equation holds for v , if u is a solution of (Q)?

(b) Moreover, consider a bijection $\psi : \mathbb{R} \rightarrow \mathbb{R}$, such that ψ and ψ^{-1} are in C^1 . Which equation holds for $w : \Omega \rightarrow \mathbb{R}; x \mapsto \psi(u(x))$, if u is a solution of (Q)?

(c) For the special case $\partial_t u + \tilde{a}(t, x, u) \cdot \nabla_x u = b(t, x, u)$ give the equations for v and w from (a) und (b) in the form $\partial_t v + \dots$ and $\partial_t w + \dots$, respectively. Here, the coordinate change Φ in (a) should not depend on t .

(d) Which bijections Φ and ψ transform the equation $\partial_t u + \partial_{x_1} u + x_1 \partial_{x_2} u = u$ into the equation $\partial_t w + \partial_{x_1} w = 1$? Construct the general solution w and provide a formula for the general solution u .

Exercise 7. Linear transport problem:

For constants $a, b \in \mathbb{R}$ we consider the scalar, linear equation

$$u_t + au_x = bu \text{ for } t > 0 \text{ and } x \in \Omega =]0, \infty[$$

together with the Cauchy data $u(0, x) = f(x)$.

(a) Show that for $a \leq 0$ there is a unique solution for this Cauchy problem (recall $t > 0$).

(b) Provide all solutions of the above Cauchy problem for the case $a > 0$. Confirm *local uniqueness* near $(t, x) = (0, x_*)$ and show *global nonuniqueness*.

(c) Under what conditions does the above Cauchy problem (with $t > 0$) has a solution that satisfies the additional *boundary condition* $u(t, 0) = h(t)$?

please turn

Exercise 8. Fundamental lemma of the calculus of variations:

Consider an open domain $\Omega \subset \mathbb{R}^d$, a scalar function $b \in C(\Omega)$ and a vector field $v \in C^1(\Omega; \mathbb{R}^d)$.

(a) Assume that for all closed balls $B_r(x) \subset \Omega$ (as test volumes) we have $\int_{B_r(x)} b(y) dy = 0$. Conclude that $b \equiv 0$.

(b) Assume that for all closed balls $B_r(x) \subset \Omega$ we have $\int_{B_r(x)} b(y) dy = \int_{\partial B_r(x)} v(\eta) \cdot \nu(\eta) da(\eta)$. Conclude $b = \operatorname{div} v$ in Ω .

(c) Assume that $\int_{\Omega} b(x)\psi(x) dy = 0$ for all $\psi \in C_c^\infty(\Omega)$. Show that $b \equiv 0$. (Here $C_c^\infty(\Omega)$ denotes the space of all infinitely often differentiable functions $\psi : \Omega \rightarrow \mathbb{R}$ such that the support $\operatorname{spt}(\psi) = \operatorname{closure}(\{x \in \Omega \mid \psi(x) \neq 0\})$ is compact and contained in Ω .)