Höhere Analysis II: Partial Differential Equations
Sommersemester 2009
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Exercise Sheet 1

Exercise 1 (Navier–Stokes equation) We consider the Navier–Stokes equation

$$\partial_t u + (u\cdot\nabla)u + \nabla p = \Delta u, \qquad \operatorname{div} u = 0 \qquad \text{for } (t,x) \in (0,T)\times\Omega$$

in the two-dimensional strip $\Omega = \mathbb{R}\times(0,h)$. Here $u(t,x) \in \mathbb{R}^2$ describes the velocity of the fluid particles, and $p(t,x) \in \mathbb{R}$ is the pressure field. By Δu we denote the componentwise Laplace-Operator, and $(u\cdot\nabla)v = \sum_1^d u_j\partial_j v$. Moreover, we impose the boundary conditions $u = 0$ on the boundary $\partial\Omega$ of the flow domain Ω.

(a) Show that for all $k \in \mathbb{N}$ there are solutions of the form $u(t,x) = \mathrm{e}^{\alpha t}\sin(k\pi x_2/h)\Phi$ for suitable $\alpha \in \mathbb{R}$ and $\Phi \in \mathbb{R}^2$.

(b) Determine all solutions (u,p), for which u does not depend on (t,x_1).

(c) For general solutions show that the mass flux in x_1 direction is independent of x_1, i.e., $\int_0^h u_1(t,x_1,\eta)\,\mathrm{d}\eta = c(t)$. (Hint: Apply Gauß's theorem to a suitable subdomain of Ω.)

(d) Find a simple partial differential equation characterizing all solutions (u,p), for which u is independent of $x_1 \in \mathbb{R}$.

Exercise 2 (Schrödinger equation) A free quantummechanical particle is governed by the complex-valued Schrödinger equation without potential. The wave function $\Psi : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ has to satisfy

$$\mathrm{i}\frac{\partial}{\partial t}\Psi = \Delta\Psi. \tag{SE}$$

(a) We look for solutions Ψ of (SE) in the form

$$\Psi(t,x) = \gamma\mathrm{e}^{-a(t)|x|^2 + b(t)\cdot x + c(t)} \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^d.$$

Derive ordinary differential equations

$$\dot a = f(a,b,c), \quad \dot b = g(a,b,c), \quad \text{and } \dot c = h(a,b,c),$$

for $a : \mathbb{R} \to \mathbb{C}$, $b : \mathbb{R} \to \mathbb{C}^d$, and $c : \mathbb{R} \to \mathbb{C}$ such that this Ψ solves (SE).

(b) Write $a(t) = \alpha(t) + \mathrm{i}\beta(t)$ and show that $\dot a = f(a,b,c)$ gives the ODE

$$\dot\alpha = r(\alpha,\beta) \quad \text{and} \quad \dot\beta = s(\alpha,\beta).$$

Conclude that $\alpha(0) > 0$ implies that $\alpha(t) > 0$ for all t. Show that all solutions can be given explicity. (Hint: For a one obtains a complex Bernoulli equation.)

(c) Provide the general solution for the ODE system in (a).

(d) The function $\rho(t,\cdot) = |\Psi(t,\cdot)|^2 \geq 0$ has the physical interpretation of a density distribution of the particle's position in $\mathbb{R}^d$ at time $t \in \mathbb{R}$. Discuss the motion of the average position $X(t) \in \mathbb{R}^d$ and the variance $\sigma(t) > 0$ of the distribution.

please turn

Exercise 3 (Separation of variables) Consider the three PDEs

$$\text{(i) } u_{tt} = u_{xx}, \qquad \text{(ii) } u_{tt} + u_{xx} = 0, \qquad \text{(iii) } u_t = u_{xx}.$$

(a) For all three PDEs construct all possible solutions with *separated variables*, i.e. they have the form $u(t,x) = V(t)W(x)$ for some real-valued, twice differentiable functions V and W. (Hint: After inserting the ansatz into the PDE, separate the variables to the different sides of the equality. Argue then that both sides must be constant.)

(b) For the heat equation (iii) find all solutions of (a) satisfying additionally $W(0) = 0 = W(1)$. From this, construct a solution of (iii) that additionally satisfies the boundary conditions $u(t,0) = u(t,1) = 0$ and the initial condition $u(0,x) = \frac{\pi^2}{17}\sin(\pi x) - 1.2\sin(4\pi x)$.

(c) For the wave equation (i) find all solution of (a) satisfying additionally $W(0) = 0 = W(\ell)$, where $\ell > 0$ is given. Show that all these solutions satisfy $u(t+n\ell, x) = u(t,x)$ for $n \in \mathbb{Z}$.

Exercise 4 [written] (Heat equation) We want to investigate the heat equation on $\Omega = (t,x) \in (0,\infty) \times \mathbb{R}^d$:

$$u_t = \Delta u = \textstyle\sum_{j=1}^{d} \frac{\partial^2}{\partial {x_j}^2} u. \tag{HE}$$

(a) The function $H(t,x) = \frac{1}{(4\pi t)^{d/2}} \exp\big(-|x|^2/(4t)\big)$ is called *heat kernel*. Show that $u(t,x) = H(t,x)$ is a solution of (HE).

(b) For $f \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R}^d) = \{\, f \in \mathrm{C}^0(\mathbb{R}^d) \mid \mathrm{sppt}(f) \text{ compact in } \mathbb{R}^d \,\}$ check that

$$u(t,x) = \int_{y \in \mathbb{R}^d} H(t, x-y) f(y) \,\mathrm{d}y \tag{S}$$

is a solution of (HE), too. (Please, justify the interchange of limit processes $\int$ and ∂!)

(c) For the solutions u in (S) and $t > 0$ derive the estimates

$$\inf\{\, f(x) \mid x \in \mathbb{R}^d \,\} \le \inf\{\, u(t,x) \mid x \in \mathbb{R}^d \,\} \le \sup\{\, u(t,x) \mid x \in \mathbb{R}^d \,\} \le \sup\{\, f(x) \mid x \in \mathbb{R}^d \,\}.$$

Moreover, establish the conservation of the total heat energy, i.e. $\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x$ is independent of the time t.

(d) Establish for u in (S) and all $x \in \mathbb{R}^d$ the limit $\lim_{t \nearrow 0} u(t,x) = f(x)$. (Hint: Substitute $y = x + t^\alpha z$ for a suitable α.)