A Posteriori Error Estimation for Elliptic PDEs in Energy Norms

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This project concentrates on upper bounds for the error between the exact solution u and some finite element approximation u_h in energy norms. The unified approach of [Carstensen, 2005] leads to the estimation of the dual norm of some residual of the form

$$\mathrm{Res} \in V^\star, \quad \mathrm{Res}(v) = \int_\Omega f \cdot v \, dx - \int_\Omega \sigma_h : Dv \, dx$$

Different Classes of A Posteriori Error Estimators η

- Explicit residual-based error estimator η_{R}
- Averaging error estimators, e.g. $\eta_{\mathrm{A}}, \eta_{\mathrm{MP1}}$
- Equilibration error estimators, e.g. $\eta_{\mathrm{B}}, \eta_{\mathrm{MFEM}}, \eta_{\mathrm{LW}}, \eta_{\mathrm{EQL}}$
- Localisation error estimator η_{CF}

$$\|\mathrm{Res}\|_\star := \sup_{v \in V} \mathrm{Res}(v)/\|\nabla v\|_{L^2(\Omega)}$$

Equilibration Error Estimators

For any $q \in H(\mathrm{div}, \Omega)$ and $\gamma \in H_1(\Omega)/\mathbb{R}$ it holds

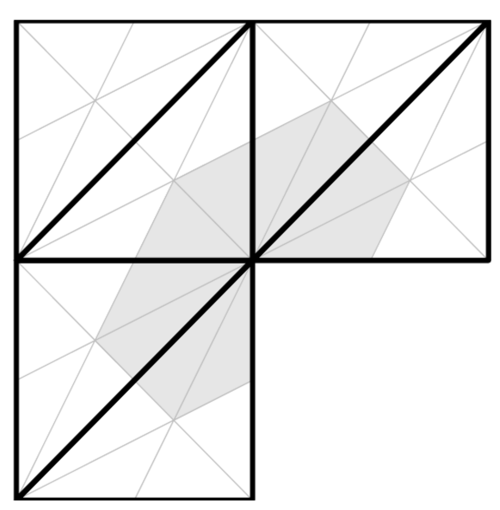
$$\|\mathrm{Res}\|_\star \leq \|f + \mathrm{div}\, q\|_\star + \|\sigma_g - q - \mathrm{Curl}\,\gamma\|_{L^2(\Omega)}$$

Novel equilibration techniques offer designs of Raviart-Thomas elements with higher-order upper bound of

$$\|f + \mathrm{div}\, q\|_\star \lesssim \mathrm{osc}(f, \mathcal{T}) := \|h_\mathcal{T}(f - f_\mathcal{T})\|_{L^2(\Omega)}$$

How to design such q?

- Mixed FEM
- Least-Square FEM, [Repin]
- [Braess, 2005]
- [Luce-Wohlmuth, 2004] on a finer mesh →



Postprocessings $\gamma \in H_1(\Omega)/\mathbb{R}$ allow for more efficient Equilibration estimators [Carstensen-Merdon, in prep.].

Error Estimators for Nonconforming FEM

With the nonconforming residual

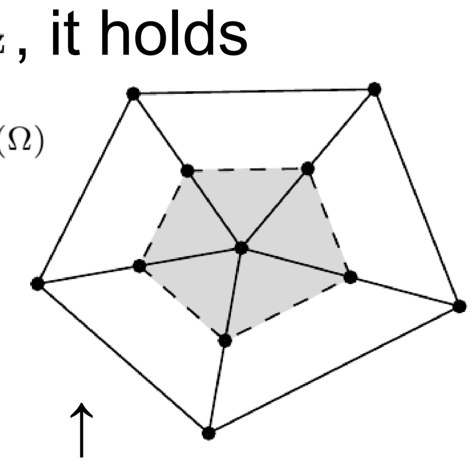
$$\mathrm{Res}_{\mathrm{NC}}(v) := -\int_\Omega \nabla_{\mathrm{NC}} u_{\mathrm{CR}} \cdot \mathrm{Curl}\, v \, dx \quad \text{for} \quad v \in V = H^1(\Omega)$$

the energy error for the Crouzeix-Raviart finite element solution u_{CR} can be estimated by

$$\|\nabla u - \nabla_{\mathrm{NC}} u_{\mathrm{CR}}\|^2_{L^2(\Omega)} \leq \underbrace{\|\mathrm{Res}_{\mathrm{NC}}\|^2_\star}_{\leq \eta} + \underbrace{\left(\|f_\mathcal{T}/2\,(\bullet - \mathrm{mid}(\mathcal{T}))\|_{L^2(\Omega)} + \mathrm{osc}(f, \mathcal{T})/\pi\right)^2}_{\text{overhead}}$$

Alternatively, for any conforming approximation v_{xyz}, it holds

$$\|\mathrm{Res}_{\mathrm{NC}}\|_\star = \min_{v \in H^1_0(\Omega)} \|\nabla_{\mathrm{NC}} u_{\mathrm{CR}} - \nabla v\|_{L^2(\Omega)} \leq \|\nabla_{\mathrm{NC}} u_{\mathrm{CR}} - \nabla v_{\mathrm{xyz}}\|_{L^2(\Omega)}$$

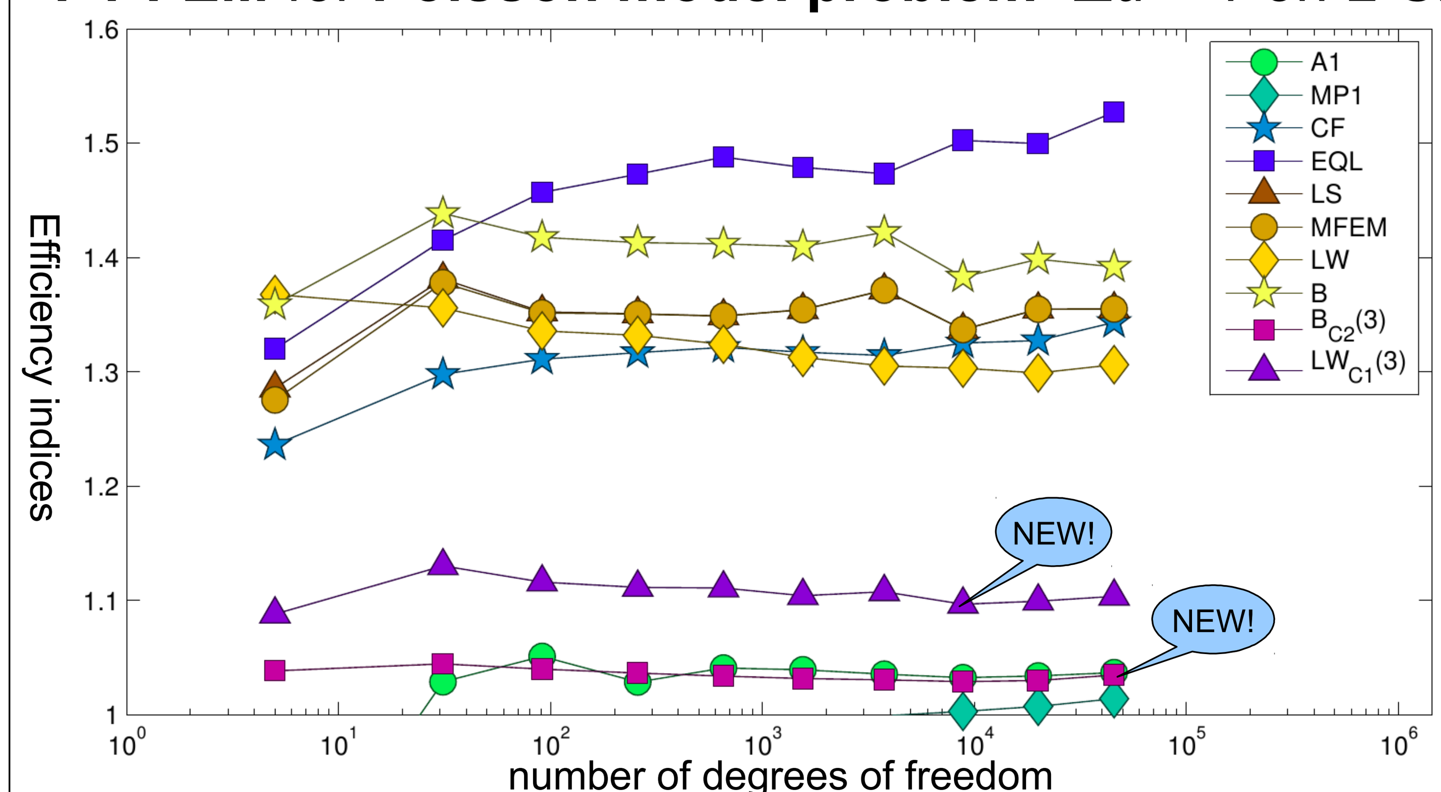


How to design v_{xyz} ?

- Conforming FEM
- Interpolation [Ainsworth, 2005]
- Interpolation on red-refinement [Carstensen-Merdon, in prep.] ↑

Error Estimator Competitions

P1-FEM for **Poisson model problem** *-Δu = 1* on L-Shaped domain:

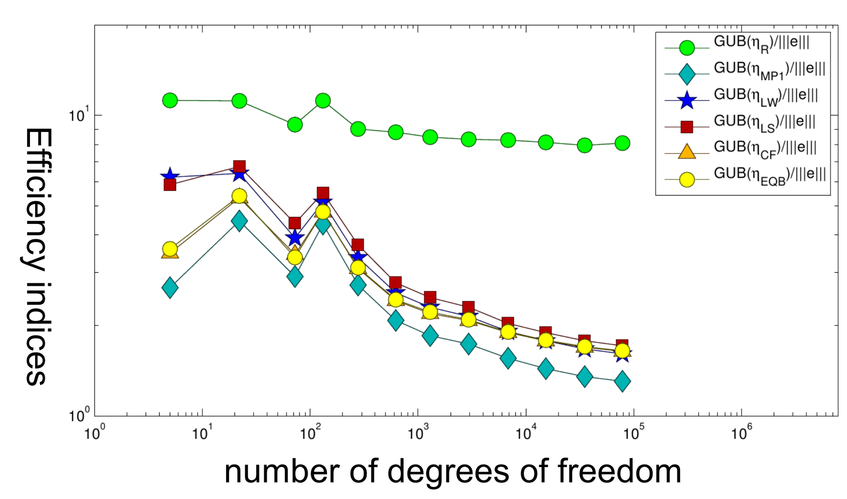


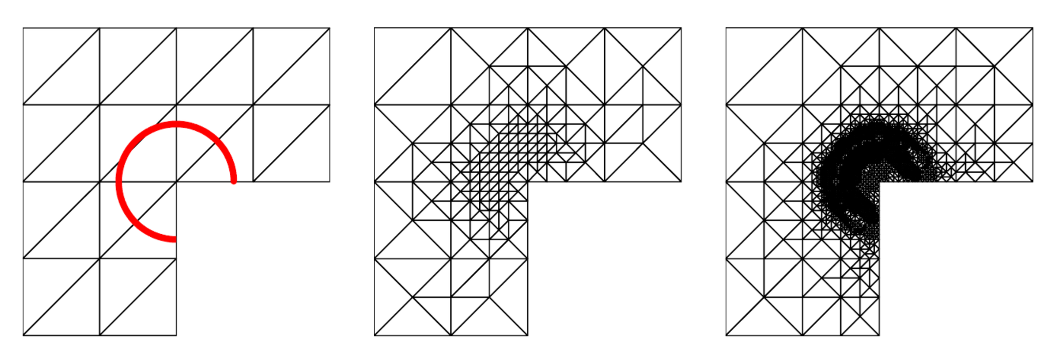
Efficiency indices $\eta_{\mathrm{xyz}}/\|\nabla(u - u_h)\|_{L^2(\Omega)}$ between 1 and 2 are possible. Averaging estimators A1 and MP1 are sharp but no guaranteed upper bound. New post-processed equilibration error estimators yield very efficient upper bounds!

P1-FEM for **obstacle problem** *-Δu = f* and *u ≤ χ* :

The Galerkin orthogonality of this nonlinear problem is restored by the introduction of some auxiliary Poisson problem [Braess, 2005] that can be estimated by all known error estimators.

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \mathrm{GUB} := \frac{a}{2} + \Big(\frac{a^2}{4} + \underbrace{\int_\Omega (\chi - u_h)(J\Lambda_h)\, dx}_{\text{overhead}}\Big)^{1/2} \qquad a := \underbrace{\|\mathrm{Res}_{\mathrm{AUX}}\|_\star}_{\leq \eta_{\mathrm{xyz}}} + \underbrace{\|\Lambda_h - J\Lambda_h\|_\star}_{\text{overhead}}$$





Adaptively generated mesh-refinements for a L-shape example, the contact zone $\{u = \chi\}$ outside the red circle is not refined.

CR-FEM for **Poisson Model Problem** *-Δu = 1* on L-Shaped domain:

The novel interpolation estimators yield very good efficiency. A preconditioned conjugate gradient postprocessing of PMRED into direction of the optimal conforming approximation MP1RED allows further improvement.