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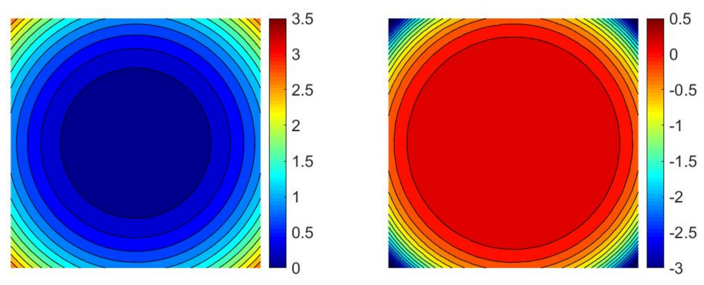
Pressure-Robustness and Acceleration of Navier–Stokes Solvers

Navier-Stokes Equations

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} + \nabla p = \mathbf{f}$$
$$\nabla\cdot\mathbf{u} = 0$$

$|\mathbf{u}|$ p



Stabilisation at Work

$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0}, \nabla\cdot\mathbf{u} = 0$

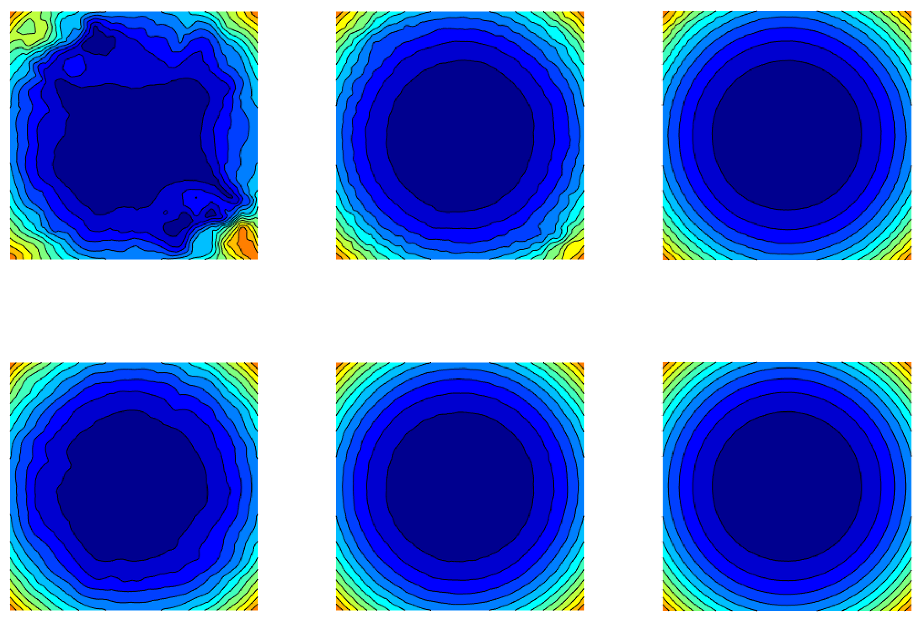
$\nu = 10^{-2}, \ \mathbf{u} = \nabla h, \ h = x^3y - y^3x$

Classical solver:

- Bernardi–Raugel FEM
- spurious velocity oscillations

Stabilised solver:

- stab. Bernardi–Raugel FEM
- pressure-robust velocity



Stabilised Solver

Modifies test functions in:

- exterior force f
- convection term $(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h$
- Coriolis force $2\boldsymbol{\Omega}\times\mathbf{u}_h$
- discrete time derivative $\dot{\mathbf{u}}_h$

Main properties:

- universal approach (FEM, FV, DG, ...)
- works on unstructured grids
- no artifical diffusion

A Continuous L^2 Orthogonality

$X := H_0^1(D;\mathbb{R}^d)$: velocity space

$Q := L^2(D)$: pressure space

$V_0 := \{v \in X : \nabla\cdot\mathbf{v} = 0\}$: divergence-free vectorfields

$L^2_\sigma(D) := \left\{\mathbf{v} \in L^2(D;\mathbb{R}^d) : \nabla\cdot\mathbf{v} = 0, \mathbf{v}\cdot\mathbf{n} = 0 \text{ along } \partial D\right\}$

Divergence:

$\operatorname{div} : X \to Q, \quad \mathbf{v} \mapsto \nabla\cdot\mathbf{v}$ lin., bd. & surj. (inf-sup stable)

$(\nabla\varphi, \mathbf{v}) = -(\varphi, \nabla\cdot\mathbf{v})$ for all $(\mathbf{v}, q) \in X \times H^1(D)$

L^2 orthogonality:

$$\forall(\mathbf{v}, q) \in V_0 \times H^1 : \qquad (\nabla\varphi, \mathbf{v}) = 0$$

Helmholtz decomposition:

$\mathbf{f} \in L^2(D;\mathbb{R}^d)$ can be decomposed into

$$\mathbf{f} = \nabla\varphi + \mathbf{w} \quad \text{with } \varphi \in H^1(D) \text{ and } \mathbf{w} \in L^2_\sigma(D)$$

Helmholtz projector:

$$\mathbb{P}(\mathbf{f}) = \mathbb{P}(\nabla\varphi + \mathbf{w}) := \mathbf{w} = \operatorname*{arginf}_{\beta\in V_0} \|f - \beta\|_{L^2(D)}$$
$$\left\|\mathbb{P}(\nabla\varphi)\right\|_{V_0^\star} = 0$$

Repairing the Discrete L^2 Orthogonality and the Divergence-Free Momentum Balance

$X_h \subset X$: discrete velocity space

$Q_h \subset Q$: discrete pressure space

$V_{0,h} := \{v_h \in X_h : \nabla_h\cdot v_h = 0\}$: discretely divergence-free vectorfields

Discrete divergence:

$\operatorname{div}_h : X_h \to Q_h, \quad \mathbf{v}_h \mapsto \nabla_h\cdot\mathbf{v}_h := \pi_{Q_h}(\nabla\cdot\mathbf{v}_h)$ inf-sup stable

$$\forall(\mathbf{v}_h, q) \in X_h \times H^1(D) : \qquad (\nabla\varphi, \mathbf{v}_h) = -(\varphi, \nabla\cdot\mathbf{v}_h) \neq -(\varphi, \nabla_h\cdot\mathbf{v}_h)$$

Incomplete L^2 orthogonality:

$$\forall(\mathbf{v}_h, q_h) \in V_{0,h} \times Q_h : \qquad (q_h, \nabla_h\cdot\mathbf{v}_h) = 0$$

Reconstruction operator:

$\pi : X_h \to H(\operatorname{div}, D)$ such that $\nabla\cdot(\pi V_{0,h}) \equiv 0$ and

$$\forall(\mathbf{v}_h, q) \in X_h \times H^1(D) : \qquad (\nabla\varphi, \pi\mathbf{v}_h) = -(\varphi, \nabla\cdot(\pi\mathbf{v}_h)) = -(\varphi, \nabla_h\cdot(\pi\mathbf{v}_h))$$

Discrete Helmholtz projectors:

classical

$$\mathbb{P}_h(\mathbf{f}) := \operatorname*{argmin}_{\beta_h\in V_{0,h}} \|f - \beta_h\|_{L^2(D)}$$
$$\left\|\mathbb{P}_h(\nabla\varphi)\right\|_{V_{0,h}^\star} \leq \min_{q_h\in Q_h} \|\varphi - q_h\|_{L^2(D)}$$
$$\mathbb{P}_h(\mathbf{f}) = \mathbb{P}_h(\mathbb{P}\mathbf{f}) + \mathbb{P}_h(\nabla\varphi)$$

stabilised

$$\mathbb{P}_h^\star(\mathbf{f}) := \operatorname*{argmin}_{\pi\beta_h\in\pi(V_{0,h})} \|f - \pi\beta_h\|_{L^2(D)}$$
$$\left\|\mathbb{P}_h^\star(\nabla\varphi)\right\|_{V_{0,h}^\star} = 0$$
$$\Rightarrow \quad \forall\mathbf{v}_h \in V_{0,h} : (\mathbf{f}, \pi\mathbf{v}_h) = (\mathbb{P}\mathbf{f}, \pi\mathbf{v}_h)$$

Stabilisation by Variational Crime

Seek $\mathbf{u}_h \in V_{0,h}$ (plus apropriate boundary conditions) such that for all $\mathbf{v}_h \in V_{0,h}$

Classical solver:

$$(\dot{\mathbf{u}}_h, \mathbf{v}_h) + (\nu\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) + ((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h, \mathbf{v}_h) + ((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h, \mathbf{v}_h) + (2\boldsymbol{\Omega}\times\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)$$

$\Updownarrow$ (theoretically)

$$(\mathbb{P}_h(\dot{\mathbf{u}}_h), \mathbf{v}_h) + (\nu\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) + (\mathbb{P}_h((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h), \mathbf{v}_h) + (2\mathbb{P}_h(\boldsymbol{\Omega}\times\mathbf{u}_h), \mathbf{v}_h) = (\mathbb{P}_h(\mathbf{f}), \mathbf{v}_h)$$

Stabilised solver:

$$(\dot{\mathbf{u}}_h, \pi\mathbf{v}_h) + (\nu\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) + ((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h, \pi\mathbf{v}_h) + (2\boldsymbol{\Omega}\times\mathbf{u}_h, \pi\mathbf{v}_h) = (\mathbf{f}, \pi\mathbf{v}_h)$$

$\Updownarrow$ (theoretically)

$$(\mathbb{P}(\dot{\mathbf{u}}_h), \mathbf{v}_h) + (\nu\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) + (\mathbb{P}((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h), \mathbf{v}_h) + (2\mathbb{P}(\boldsymbol{\Omega}\times\mathbf{u}_h), \pi\mathbf{v}_h) = (\mathbb{P}(\mathbf{f}), \pi\mathbf{v}_h)$$

A Priori Error Estimates for Stokes Equations

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0$$

Classical solver:

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leq C_1 \inf_{\mathbf{v}_h\in V_h} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(D)} + \frac{1}{\nu} \inf_{\mathbf{q}_h\in Q_h} \|p - p_h\|_{L^2(D)}$$

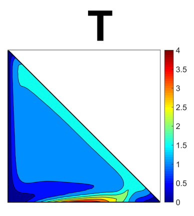
Stabilised/Pressure-robust solver:

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(D)} \leq C_1 \inf_{\mathbf{v}_h\in V_h} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(D)} + C_2 h^k |\mathbf{u}|_{H^{k+1}(D)}$$

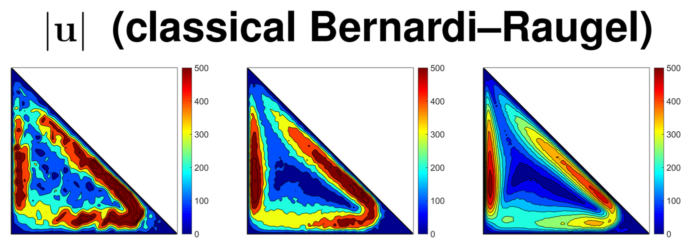
Example: Natural Convection

$$-\Delta\mathbf{u} + \nabla p = 10^6 \begin{pmatrix} 0 \\ T \end{pmatrix},$$
$$\nabla\cdot\mathbf{u} = 0,$$
$$-\Delta T + \mathbf{u}\cdot\nabla T = 0$$

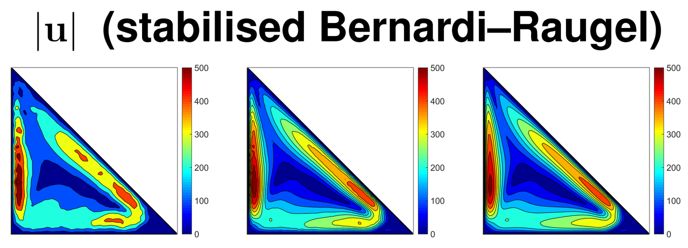
T



$|\mathbf{u}|$ **(classical Bernardi–Raugel)**



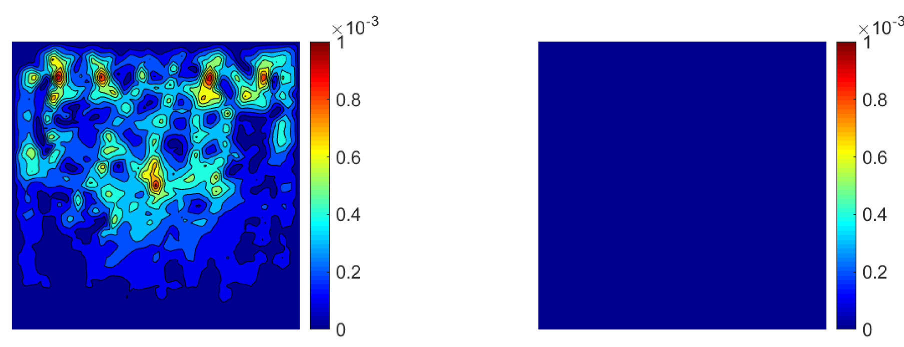
$|\mathbf{u}|$ **(stabilised Bernardi–Raugel)**



Outlook: Pressure-Robust Solvers & Coriolis Force

$$-\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2y\mathbf{u}^\perp + \nabla p = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u} = (1,0)^T, \ p = y^2$$

classical Bernardi–Raugel **stabilised Bernardi–Raugel**



References

- A. Linke, *On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime*, Comput. Methods Appl. Mech. Engrg. **268** (2014), 782–800.
- A. Linke, G. Matthies, L. Tobiska, *Robust arbitrary order mixed finite element methods for the incompressible Stokes equations*, ESAIM: Mathematical Modelling and Numerical Analysis, in press.
- A. Linke, C. Merdon, *On spurious oscillations due to irrotational forces in the Navier–Stokes momentum balance*, WIAS Preprint 2132, submitted
- V. John, A. Linke, C. Merdon, M. Neilan, L. Rebholz, *On the divergence constraint in mixed finite element methods for incompressible flows*, WIAS Preprint 2177, submitted
- A. Linke, C. Merdon, *Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier–Stokes equations*, in preperation

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