

MODULUS OF CONTINUITY OF NEMYTSKIĬ OPERATORS WITH APPLICATION TO THE PROBLEM OF OPTION PRICING

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ABSTRACT. We introduce and analyze moduli of continuity for specific classes of Nemytskiĭ operators on spaces of continuous functions, which are given by kernels, strictly monotone in their second argument. Such operators occur as non-linear (outer) mappings for certain problems of option pricing within the Black-Scholes model for time-dependent volatility. This nonlinear mapping can be seen to be continuous, however its convergence properties are poor. Our general results allow to bound the related moduli of continuity, both for the forward and backward non-linear mappings. In particular we explain the observed ill-conditioning of the nonlinear backward problem. The analysis uses some abstract local analysis of index functions, which may be of independent interest.

1. INTRODUCTION, PROBLEM FORMULATION

The present study is motivated by the following variant of the classical Black-Scholes model. Denoting the asset price at time τ by P_τ the price process P is assumed to satisfy the stochastic differential equation

$$(1) \quad dP_\tau = \mu P_\tau d\tau + \sigma(\tau) dW_\tau,$$

with a time-dependent volatility function $\sigma(\tau) > 0$, $\tau \geq 0$, a drift coefficient $\mu \in \mathbb{R}$ and a standard Wiener process W . Furthermore, the existence of a bond with constant interest rate $r \geq 0$ is assumed.

At the initial time $\tau = 0$ we denote the current asset price by P and consider a family of European vanilla call options with fixed strike $K > 0$ and maturities t varying in a finite time interval $I := [0, T]$. Provided trading is frictionless and continuous and the asset price follows (1) the formula for the price $C(t)$ of such a European call option with maturity $t \in I$ is well-known (cf. [12, p. 71]).

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To reformulate this formula we denote by Φ and ϕ the cumulative distribution function and the density function of the standard normal distribution, respectively, i. e.,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{x^2}{2}} dx, \quad x, z \in \mathbb{R}.$$

Then we introduce the function u_{BS}^{call} as

$$(2) \quad u_{BS}^{call}(P, K, r, t, s) = \begin{cases} P\Phi(d + \sqrt{s}) - Ke^{-rt}\Phi(d) & , \text{if } s > 0, \\ (P - Ke^{-rt})_+ & , \text{for } s = 0, \end{cases}$$

where we use the parameters $c = c(t)$ and $d = d(t, s)$ given by

$$(3) \quad c(t) := \log \frac{P}{Ke^{-rt}}, \quad \text{and}$$

$$(4) \quad d(t, s) := \frac{c(t) - s/2}{\sqrt{s}}.$$

Now, the price $C(t)$ of a European call option with strike $K > 0$ and maturity $t \geq 0$ is given by

$$(5) \quad C(t) = u_{BS}^{call}(P, K, r, t, S(t)), \quad t \in I,$$

where $S(t)$ is given as $S(t) := \int_0^t \sigma^2(\tau) d\tau$, $0 \leq t \leq T$, hence $C(t)$ depends on the volatility σ only through the function $S(t)$, $t \in I$. In our subsequent analysis the quantities P, K, r will be fixed.

1.1. The forward Black-Scholes operator. We find it convenient to introduce the (forward) *Black-Scholes operator* as Nemytskiĭ-operator $N : \mathcal{D}_+(N) \subset \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ by

$$(6) \quad [N(S)](t) = u_{BS}^{call}(P, K, r, t, S(t)), \quad t \in I,$$

on the domain $\mathcal{D}_+(N) := \{f \in \mathcal{C}(I), f(0) = 0, f(t) > 0, t \in (0, T]\}$, and with range $\mathcal{R}(N)$.

Introducing the kernel function

$$(7) \quad k(t, s) := u_{BS}^{call}(P, K, r, t, s), \quad t > 0, s > 0,$$

we can rewrite this as

$$[N(S)](t) = k(t, S(t)), \quad t \in I.$$

The option price $C(t)$, $t \in I$, is obtained from the time dependent (squared) volatility function $\sigma^2(\tau)$, $0 \leq \tau \leq t$, as the *composition* of the linear mapping $\sigma^2 \in \mathcal{C}(I) \mapsto S \in \mathcal{C}(I)$ and the non-linear Nemytskiĭ-operator N , see [6, 7] or [11] for more details.

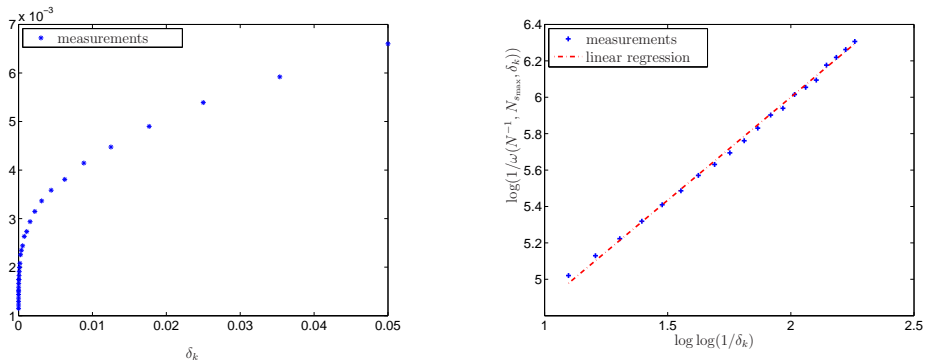


FIGURE 1. (left) Behavior of modulus of continuity $\omega(N^{-1}, u^\dagger, \delta)$ as $\delta \rightarrow 0$. (right) Linear regression on the model $\omega(N^{-1}, u^\dagger, \delta) \approx \log^{-\mu}(1/\delta)$ yields $\hat{\mu} = 1.1124$. (parameters $P = 1, K = 0.85, r = 0.05, T = 0.2$)

It is well-known that both mappings N and S are continuous on their domains in $\mathcal{C}(I)$, and hence so is their composition, say $F := N \circ S: \mathcal{C}(I) \rightarrow \mathcal{C}(I)$.

1.2. The inverse Black-Scholes operator. However, for the identification of the volatility function σ or the anti-derivative S we are concerned with the corresponding inverse operators. The properties of the inverse of the anti-derivative operator S are known, and lead to ill-posedness in appropriate spaces. It was not clear whether for N from (6) the inverse mapping $N^{-1}: \mathcal{R}(N) \subset \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ was continuous. In [10, 11] it has been shown, that this inverse operator is again of Nemytskiĭ-type, and that this mapping acts continuously from $\mathcal{R}(N) \rightarrow \mathcal{C}(I)$. We shall measure continuity in terms of global and local moduli of continuity, see (8) and (9) for the precise introduction.

For every $u^\dagger \in \mathcal{R}(N)$ the modulus of continuity of N^{-1} turned out to be poor, see Figure 1(left). The poor convergence behavior to zero of this modulus of continuity in the numerical studies suggested a *logarithmic model* of the form $\omega(N^{-1}, u^\dagger, \delta) \approx \log^{-\mu}(1/\delta)$ (Details are given below) for some $\mu > 0$. Indeed, regression based on the simulated data, see Figure 1(right), led to conjecture that this model is true for $\mu \approx 1$, independently of $u^\dagger \in \mathcal{R}(N)$.

1.3. Outline of the material. The present study extends the previous discussion on ill-posedness vs. ill-conditioning in [11]. These authors first exhibited the continuity of the inverse Black-Scholes operator and observed a *delayed convergence* due to ill-conditioning. Here

we go one step further, as we shall introduce and analyze the local and global moduli of continuity for the mappings $N: \mathcal{D}(N) \subset \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ and $N^{-1}: \mathcal{R}(N) \subset \mathcal{C}(I) \rightarrow \mathcal{C}(I)$. This allows to quantify the ill-conditioning.

To this end we provide some general framework for Nemytskiĭ operators acting on continuous functions in Section 2. For certain classes of kernels which fall into one of the categories introduced below (Case 1 and Case 2), upper bounds for the global and local moduli of continuity are given in propositions 2.2–2.5. In Section 3 we then provide a detailed analysis for the specific kernels related to the forward and inverse Black-Scholes operators.

These results are then used in Section 4 to determine the moduli of continuity, for several constellations of the parameters P, K and r , in theorems 4.1–4.6. The major result is Theorem 4.5 where we confirm the conjecture mentioned above with exponent $\mu = 1$. These results are accompanied with lower bounds in § 4.3. In Section 5 we extend the analysis to European put options. Furthermore we summarize our finding with a short discussion.

An important ingredient will be some local analysis for index functions. This may be of independent interest, it is therefore presented in Appendix A.

2. MODULI OF CONTINUITY OF NEMYTSKIĬ OPERATORS

As outlined in Section 1 the operators under consideration are of Nemytskiĭ-type, and we shall provide some preliminary analysis for classes of such. For a general introduction to Nemytskiĭ-operators we refer to [2] and [1, Section 1.2]. Here we fix $T > 0$, the corresponding interval $I := [0, T]$, and we suppose that there is a function $t \in I \mapsto s_{\max}(t) \in (0, \infty)$ such that the real-valued kernel k is defined on $I \times [0, s_{\max}(t))$.

Remark 2.1. Below we shall frequently view the bivariate kernel k as a family of functions, say $(k_t)_{t \in I}$ of its second argument, acting on $[0, s_{\max}(t))$, respectively.

We assign the Nemytskiĭ operator $N: \mathcal{D}(N) \subset \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ by

$$[N(S)](t) := k(t, S(t)), \quad t \in I, S \in \mathcal{D}(N).$$

Thus its domain $\mathcal{D}(N)$ consists of continuous functions $f \in \mathcal{C}(I)$ with $0 \leq f(t) < s_{\max}(t)$.

It is well-known, see e.g. [2, p. 205ff] that such operator N is continuous (in $\mathcal{C}(I)$) if the kernel is continuous on $I \times [0, s_{\max}(t))$. Of particular

interest are continuous kernels which are strictly monotone in their second argument. It is clear that in this case we can define the *inverse* kernel g from $g(t, k(t, s)) = s$, such that the appropriately defined inverse operator is of Nemytskiĭ-type. A recent result states, see [10, § 5.3], that Nemytskiĭ operators with such kernels have a continuous inverse on the space of continuous functions. For the convenience of the reader we recall the core of the proof as follows.

Lemma 2.1. *Suppose that the kernel k is jointly continuous and that each k_t is strictly increasing. If $s_0, s_n \in \mathcal{C}(I)$ are such that uniformly for $t \in I$ it holds $k(t, s_n(t)) \rightarrow k(t, s_0(t))$, then necessarily $\|s_n - s_0\|_{\mathcal{C}(I)} \rightarrow 0$.*

Sketch of the proof. Suppose to the contrary that there are $\varepsilon > 0$ and a sequence $t_n \in I$ such that $|s_n(t_n) - s_0(t_n)| \geq \varepsilon$. Without loss of generality we may and do assume that $s_n(t_n) \geq s_0(t_n) + \varepsilon$. Consider the function $F(t) := k(t, s_0(t) + \varepsilon) - k(t, s_0(t))$, $t \in I$. This is a continuous function on I and positive, there. By compactness of I we deduce that $m := \min_{t \in I} F(t) > 0$. This implies that

$$k(t_n, s_n(t_n)) - k(t_n, s_0(t_n)) \geq k(t_n, s_0(t_n) + \varepsilon) - k(t_n, s_0(t_n)) = F(t_n) \geq m,$$

which contradicts the assumption. \square

Remark 2.2. It is clear from the proof that such reasoning does not extend to $L^p(I)$ -spaces, $1 \leq p < \infty$. Indeed, it can be seen that the inverse need not be continuous, there. We refer to the recent paper [11] for more details.

A typical way to measure smoothness is provided by certain *moduli of continuity*. For the local modulus of continuity we fix some $S^\dagger \in \mathcal{D}(N)$ and study

$$(8) \quad \omega(N, S^\dagger, \delta) = \sup \{ \|N(S) - N(S^\dagger)\|_{\mathcal{C}(I)}, S \in \mathcal{D}(N), \|S - S^\dagger\|_{\mathcal{C}(I)} \leq \delta \}.$$

Accordingly, the global modulus of continuity is defined for any non-empty subset $D \subset \mathcal{D}(N)$ as

$$(9) \quad \omega(N, D, \delta) := \sup \{ \|N(S_1) - N(S_2)\|_{\mathcal{C}(I)}, S_1, S_2 \in D, \|S_1 - S_2\|_{\mathcal{C}(I)} \leq \delta \}.$$

Plainly, in case that $s_{\max}(t) = s_{\max} > 0$, given $S^\dagger \in \mathcal{D}(N)$ there is $\delta_0 > 0$ small enough such that $\|S^\dagger\|_{\mathcal{C}(I)} + \delta_0 < s_{\max}$, the local modulus of continuity can be bounded by the global one as

$$\omega(N, S^\dagger, \delta) \leq \omega(N, \mathcal{D}(N), \delta), \quad 0 < \delta \leq \delta_0.$$

The modulus of continuity cannot decay to zero faster than linearly, except in trivial cases.

Lemma 2.2. *Suppose that the domain D is convex. It holds $\omega(N, D, \delta)/\delta \rightarrow 0$ if and only if $k(t, s) = \kappa(t)$, $t \in I$, $0 \leq s < s_{\max}(t)$ (in which case $\omega(N, D, \delta) = 0$).*

Proof. If D contains only one point, then there is nothing to prove. Otherwise we fix $t \in I$. If now $S_0 \in \mathcal{D}(N)$ is such that $S_0(t) = s$ then for every $\delta > 0$ and other $S_\delta \in \mathcal{D}(N)$ we deduce from $\|S_0 - S_\delta\|_{C(I)} \leq \delta$ that

$$\frac{|k(t, s) - k(t, S_\delta(t))|}{\delta} \leq \frac{\omega(N, S_0, \delta)}{\delta} \rightarrow 0, \quad \text{as } \delta \rightarrow 0.$$

Thus, at every $t \in I$ the kernel k_t is constant with respect to s , which proves the lemma. \square

Therefore it is reasonable to call the operator N *well-conditioned* if $\omega(N, D, \delta)/\delta \leq L$ is uniformly bounded as $\delta \rightarrow 0$, or it is *ill-conditioned* if the quotient is unbounded (along a sub-sequence) as $\delta \rightarrow 0$. The first result is immediate.

Proposition 2.1. *If the family $(k_t)_{t \in I}$ is uniformly Lipschitz then the corresponding operator is well-conditioned.*

Proof. Plainly, for any pair $S_1, S_2 \in \mathcal{D}(N)$ we deduce, using the assumption (with Lipschitz constant L), that

$$|k(t, S_1(t)) - k(t, S_2(t))| \leq L |S_1(t) - S_2(t)|,$$

from which the proof can be completed. \square

For our analysis we shall concentrate on kernels which are well-behaved, possibly except the specific point $s = 0$. First we shall assume that $\underline{s} := \inf_{t \in I} s_{\max}(t) > 0$.

Definition 1 (see [8]). A strictly increasing continuous function $\varphi: [0, \underline{s}] \rightarrow \mathbb{R}^+$ is called *index function* if $\varphi(0) = 0$.

Throughout this section we shall impose the following assumption on the kernel $k(t, s)$.

Assumption 1. The kernel k is

- (1) continuously differentiable with respect to s and t for $0 < t \leq T$ and $0 < s < s_{\max}(t)$,
- (2) for each $t \in I$ the function k_t is an index function.

Remark 2.3. We restrict to kernels with $k(t, 0) = 0$. If this were not the case, then we may let $\tilde{k}(t, s) := k(t, s) - k(t, 0)$, $t \in I$, $0 \leq s < s_{\max}(t)$. It is easy to see that the original Nemytskiĭ operator has the same modulus of continuity as the one corresponding to the shifted kernel.

For its modulus of continuity it is important whether the kernels increase from $s = 0$ in a concave or convex manner for all t , see Figure 2, and we distinguish these cases.

Case 1: For each $t \in I$ there is $0 < s(t) \leq s_{\max}(t)$ such that the function $k(t, s)$ is convex on $[0, s(t))$.

Case 2: For each $t \in I$ there is $0 < s(t) \leq s_{\max}(t)$ such that the function $k(t, s)$ is concave on $[0, s(t))$.

In addition, for each $0 < t_0 \leq T$ there is $m := m(t_0) \leq T$, such that for each $0 \leq t \leq t_0$ and $0 \leq s < s_{\max}(t)$ it holds that $k(t, s) \leq k(m(t_0), s)$.

The crucial additional assumption in Case 2 is that the function m does not depend on s , which holds true for the applications we have in mind.

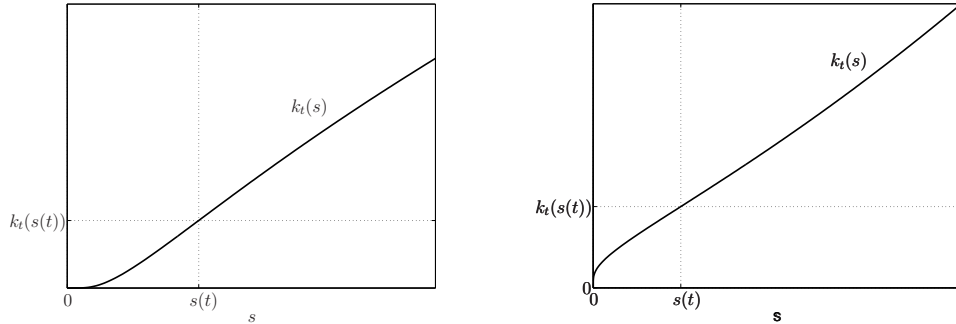


FIGURE 2. Kernels corresponding to Case 1 (left) and Case 2 (right), respectively. The point $s(t)$ is indicated.

As it will turn out Case 1 corresponds to well-conditioning of the operator, whereas in Case 2 we shall provide a concave bound in δ , which reflects ill-conditioning. To this end the following property proves important. For each $0 < t_0 \leq T$ there is $L = L(t_0) < \infty$ such that

$$(10) \quad \sup_{0 \leq t \leq t_0} \left\{ \frac{\partial k}{\partial s}(t, s), \quad s(t) \leq s < s_{\max}(t) \right\} \leq L(t_0).$$

We agree to abbreviate $L := L(T)$. Such uniform bound has various implications and we start with the following.

Lemma 2.3. *Granted Case 1, and if (10) is satisfied for $t_0 = T$ then the family $(k_t)_{t \in I}$ is uniformly Lipschitz continuous with constant L .*

Proof. In Case 1 the continuous functions $\partial k / \partial s$ are increasing on $(0, s(t))$. Therefore

$$\frac{\partial k}{\partial s}(t, s) \leq \sup_{s(t) \leq s < s_{\max}(t)} \frac{\partial k}{\partial s}(t, s) \leq L.$$

Using the mean value theorem we obtain for $0 < s' < s'' < s_{\max}(t)$ that

$$k(t, s'') - k(t, s') = \int_{s'}^{s''} \frac{\partial k}{\partial s}(t, \zeta) d\zeta \leq L(s'' - s'),$$

and hence the Lipschitz continuity uniformly for $t \in I$. \square

Remark 2.4. An important special sub-case for Case 1 occurs, if the kernel is concave on $[s(t), s_{\max}(t)]$. Then

$$\sup_{s(t) \leq s < s_{\max}(t)} \frac{\partial k}{\partial s}(t, s) = \frac{\partial k}{\partial s}(t, s(t)),$$

a fact which will prove useful, later.

In Case 2 we shall decompose the kernel into a globally concave and a globally Lipschitz one, as this is indicated in Figure 3.

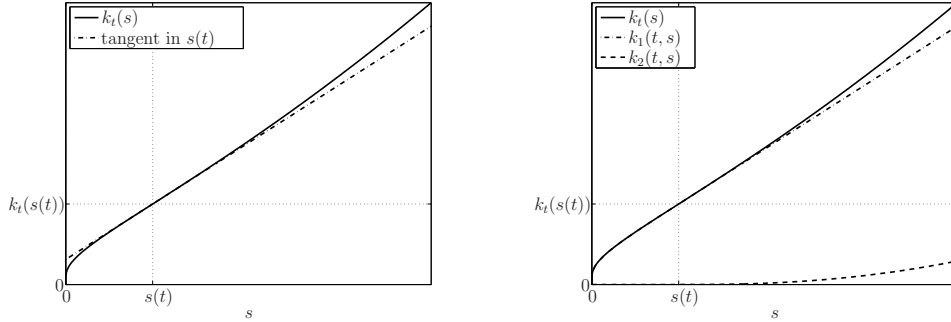


FIGURE 3. Construction in Case 2. (left) Indicating the tangent. (right) Showing the decomposition.

Lemma 2.4. *Fix $t \in I$. Granted Case 2, let $k(t, s) = k_1(t, s) + k_2(t, s)$ be the decomposition into*

$$k_1(t, s) := \begin{cases} k(t, s) & , 0 \leq s < s(t) \\ k(t, s(t)) + \frac{\partial k}{\partial s}(t, s(t))(s - s(t)) & , s(t) \leq s < s_{\max}(t), \end{cases}$$

and

$$k_2(t, s) := k(t, s) - k_1(t, s), \quad 0 \leq s < s_{\max}(t).$$

The following assertions hold true.

- (1) The function $k_1(t, s)$ is a concave index function.
- (2) If (10) is fulfilled for $t_0 = T$ then the functions $k_2(t, s)$ are Lipschitz continuous with respect to s with common constant L .

Proof. The first item follows by construction, since $k_1(t, s)$ is concave on $[0, s(t))$ and has an increasing linear continuation. To prove the second item we notice that $\frac{\partial k_2}{\partial s}(t, s) = 0$ for $0 < s \leq s(t)$, whereas for $0 < s(t) \leq s' < s'' < s_{\max}(t)$ we have for some $\tilde{s} \geq s(t)$ that

$$k_2(t, s'') - k_2(t, s') = \frac{\partial k}{\partial s}(t, \tilde{s})(s'' - s').$$

Thus, under (10) the Lipschitz continuity can be proved. □

2.1. Global modulus. In both cases we can bound the global modulus of continuity of the operator N .

Proposition 2.2 (Case 1). *Under (10) for $t_0 = T$ it holds that*

$$\omega(N, \mathcal{D}(N), \delta) \leq L\delta.$$

Proof. This is an immediate consequence of Lemma 2.3 and Proposition 2.1. □

To treat Case 2 we require some specific result for concave index functions, which appears to be crucial in our analysis. For the convenience of the reader we provide the proof, and we refer to [9, § 6.1] for additional information.

Lemma 2.5. *If on some interval $[0, a]$ the function φ is a concave index function then*

$$(11) \quad |\varphi(s') - \varphi(s)| \leq \varphi(|s' - s|), \quad 0 \leq s, s' \leq a.$$

Proof. If $s < s'$, then $s = \frac{s}{s'}s' + (1 - \frac{s}{s'})0$, and concavity yields that $\varphi(s)/s$ is a non-increasing function. This implies, again for $s \leq s'$ that

$$\begin{aligned} \varphi(s') &= \frac{s}{s'}\varphi(s') + \frac{s' - s}{s'}\varphi(s') \\ &\leq s\frac{\varphi(s)}{s} + (s' - s)\frac{\varphi(s' - s)}{s' - s} = \varphi(s) + \varphi(s' - s), \end{aligned}$$

which is equivalent to (11). □

Remark 2.5. Index functions which obey (11) are called *modulus of continuity*, and the lemma states that concave index functions are moduli of continuity, reproving [9, Prop. 6.1.1], and we refer to [9] for more details on this connection.

Proposition 2.3 (Case 2). *Under (10) for $t_0 = T$ and with $m = m(t_0)$ as required in Case 2 it holds that*

$$(12) \quad \omega(N, \mathcal{D}(N), \delta) \leq k(m(t_0), \delta) + 2L\delta, \quad 0 < \delta \leq \underline{s}.$$

Proof. For each $t \in I$ and $|S_1(t) - S_2(t)| \leq \delta$ we have from Lemma 2.4 and Lemma 2.5 that

$$\begin{aligned} |[N(S_1)](t) - [N(S_2)](t)| &\leq k_1(t, |S_1(t) - S_2(t)|) + L|S_1(t) - S_2(t)| \\ &\leq k_1(t, \delta) + L\delta \leq \max_{t \in I} k_1(t, \delta) + L\delta. \end{aligned}$$

Now we consider two cases. If $t \in I$ is such that $s(t) \geq \delta$ then $k_1(t, \delta) = k(t, \delta) \leq k(m(t_0), \delta)$. Otherwise, if $s(t) \leq \delta$, we have from the definition of k_1 that $k_1(t, \delta) \leq k(m(t_0), \delta) + L\delta$. In any case this results in

$$|[N(S_1)](t) - [N(S_2)](t)| \leq k(m(t_0), \delta) + 2L\delta.$$

Taking suprema with respect to $t \in I$ the proof is complete. \square

2.2. Local modulus. We complete these preliminary considerations with companions of Propositions 2.2 and 2.3 for the local modulus of continuity under an additional restriction on the domain of definition. We restrict $\mathcal{D}(N)$ to the subset

$$(13) \quad \mathcal{D}_+(N) = \{S \in \mathcal{D}(N), \quad S(0) = 0, \quad S(t) > 0, \quad 0 < t \leq T\}.$$

One consequence of this restriction is as follows.

Lemma 2.6. *For each $S^\dagger \in \mathcal{D}_+(N)$ and each $0 < t_0 \leq T$ there is $\delta_0 > 0$ such that for every $S \in \mathcal{D}(N)$ with $\|S - S^\dagger\|_{\mathcal{C}(I)} \leq \delta_0$ it holds*

$$\delta_0 \leq S(t) \leq \|S^\dagger\|_{\mathcal{C}(I)} + \delta_0, \quad t_0 \leq t \leq T.$$

Proof. Since the function $S^\dagger \in \mathcal{D}_+(N)$ is continuous, we can find for every $t_0 > 0$ a positive number $0 < s_{min}^+ < s_{max}(t)$ such that

$$s_{min}^+ \leq S^\dagger(t) \leq \|S^\dagger\|_{\mathcal{C}(I)}, \quad t_0 \leq t \leq T.$$

Consequently, we let $\delta_0 := s_{min}^+/2$ and use the triangle inequality twice, to deduce that $\delta_0 \leq S(t) \leq \|S^\dagger\|_{\mathcal{C}(I)} + \delta_0$ if $\|S - S^\dagger\|_{\mathcal{C}(I)} \leq \delta_0$. \square

This may be used in conjunction with another property: For each $0 < t_0 \leq T$ and $0 < s_0 < S_0 < s_{\max}(t)$ there is $C = C(t_0) < \infty$ such that

$$(14) \quad \sup_{t \in [t_0, T]} \left\{ \frac{\partial k}{\partial s}(t, s), \quad s \in [s_0, S_0] \right\} \leq C(t_0),$$

which expresses that the kernel is well-behaved away from the axes $s =$ and $t = 0$. The following lemma asserts that under (14) the problem is well conditioned at any time point beyond $t_0 \leq T$.

Lemma 2.7. *Suppose that condition (14) holds for some $0 < t_0 \leq T$. Then for each $S^\dagger \in \mathcal{D}_+(N)$ there is $\delta_0 > 0$ such that*

$$(15) \quad \sup_{t \in [t_0, T]} |[N(S)](t) - [N(S^\dagger)](t)| \leq C_0 \delta, \quad 0 < \delta \leq \delta_0.$$

Proof. We apply Lemma 2.6 and find $\delta_0 > 0$ with properties as asserted. Let $s_0 := \delta_0$ and $S_0 := \|S^\dagger\|_{C(I)} + \delta_0$. For any $0 < \delta \leq \delta_0$ and $S \in \mathcal{D}(N)$ with $\|S^\dagger - S\|_{C(I)} \leq \delta$ we then have that $s_0 \leq S(t), S^\dagger(t) \leq S_0$, hence

$$[N(S)](t) - [N(S^\dagger)](t) = k(t, S(t)) - k(t, S^\dagger(t)) = \int_{S(t)}^{S^\dagger(t)} \frac{\partial k}{\partial s}(t, s) ds,$$

which yields that

$$|[N(S)](t) - [N(S^\dagger)](t)| \leq C_0 |S(t) - S^\dagger(t)| \leq C_0 \delta,$$

which completes the proof. \square

This allows to establish the following bounds for the local moduli, both in Cases 1 and 2, based on the following trivial split

$$(16) \quad \sup_{t \in [0, T]} |[N(S)](t) - [N(S^\dagger)](t)| \\ \leq \sup_{t \in [0, t_0]} |[N(S)](t) - [N(S^\dagger)](t)| + \sup_{t \in [t_0, T]} |[N(S)](t) - [N(S^\dagger)](t)|,$$

and we state

Proposition 2.4 (Case 1). *Suppose that there is $0 < t_0 \leq T$ such that (10) and (14) hold true. Then, for each $S^\dagger \in \mathcal{D}_+(N)$ there is $\delta_0 > 0$ such that*

$$\omega(N, S^\dagger, \delta) \leq (L(t_0) + C(t_0))\delta, \quad 0 < \delta \leq \delta_0.$$

Proof. We use (16). Under (10) Proposition 2.2 allows to bound the first term by $L(t_0)\delta$, while Lemma 2.7 provides us with a bound for the second term by $C(t_0)\delta$, which completes the proof. \square

Proposition 2.5 (Case 2). *Suppose that the bounds (10) and (14) are valid for some $t_0 \leq T$. For $m(t_0)$ as required in Case 2 there is $\delta_0 > 0$ such that*

$$(17) \quad \omega(N, S^\dagger, \delta) \leq k(m(t_0), \delta) + (2L(t_0) + C(t_0))\delta, \quad 0 < \delta \leq \delta_0.$$

Proof. Again we use (16). As in the previous proof, for $t \in [0, t_0]$ we can apply Proposition 2.3, while for $t \in [t_0, T]$ Lemma 2.7 applies, and yields another $C(t_0)\delta$, which allows to complete the proof. \square

Thus in the specific applications we need to check that Assumption 1 on the kernel is fulfilled, that some bound (10) can be proved, and that one of the two cases applies. For the local modulus the condition (14) must be checked.

3. PRELIMINARY ANALYSIS FOR THE KERNELS

We turn to the kernels corresponding to the forward Black-Scholes operators N from (6) and its inverse N^{-1} . For each $t \in I$ the kernel

$$k(t, s): (0, \infty) \rightarrow ((P - Ke^{-rt})_+, P),$$

from (7) is continuous and strictly increasing, with $k(t, 0) = (P - Ke^{-rt})_+$ and $\lim_{s \rightarrow \infty} k(t, s) = P$, see e.g. [5, 6].

The kernel $g(t, s)$ for the inverse mapping N^{-1} is *implicitly* defined by

$$(18) \quad g(t, u) = s, \quad t \in I, \quad (P - Ke^{-rt})_+ \leq u = k(t, s) < P.$$

By Lemma 2.1 the operator corresponding to the kernel g is continuous from $\mathcal{R}(N) \subset \mathcal{C}(I)$ to $\mathcal{C}(I)$.

In view of Assumption 1 we find it convenient to introduce the auxiliary shifted kernels for $t \in I$ as

$$(19) \quad \tilde{k}(t, s) := k(t, s) - (P - Ke^{-rt})_+, \quad s > 0,$$

and with $u_{\max}(t) := P - (P - Ke^{-rt})_+$ we obtain that the inverse kernel \tilde{g} is obtained, using that $g(t, k(t, s)) = s$, as

$$(20) \quad \begin{aligned} \tilde{g}(t, u) &= \tilde{g}(t, \tilde{k}(t, s)) = s = g(t, k(t, s)) = g(t, \tilde{k}(t, s) + k(t, 0)) \\ &= g\left(t, u + (P - Ke^{-rt})_+\right), \quad 0 \leq u < u_{\max}(t). \end{aligned}$$

Figures of the forward kernels can be found in Figure 4, below. We introduce the auxiliary family of quadratic functions

$$(21) \quad p_c(s) := s^2 + 4s - 4c^2, \quad c \in \mathbb{R}, \quad s > 0,$$

and we recall the parameters c and d from (3) and (4). We shall use the following representations for derivatives of the kernels \tilde{k} and \tilde{g} .

Lemma 3.1 ([5, 6]). *For $t \geq 0$, $s > 0$ it holds that*

$$(22) \quad \frac{\partial \tilde{k}}{\partial s}(t, s) = P\phi(d + \sqrt{s})\frac{1}{2\sqrt{s}},$$

$$(23) \quad \frac{\partial^2 \tilde{k}}{\partial s^2}(t, s) = -P\frac{\phi(d + \sqrt{s})}{16s^2\sqrt{s}}p_{c(t)}(s),$$

$$(24) \quad \frac{\partial \tilde{k}}{\partial t}(t, s) = rKe^{-rt} \begin{cases} \Phi(d) & , \text{if } P - Ke^{-rt} \leq 0 \\ \Phi(d) - 1 & , \text{if } P - Ke^{-rt} > 0, \end{cases}$$

The corresponding derivatives of \tilde{g} are obtained from $\tilde{g}(t, \tilde{k}(t, s)) = s$ using the implicit function theorem.

3.1. Monotonicity, concavity and convexity. As can be seen from the definition of the kernels $k(t, s)$ and the implicitly defined inverse kernel $g(t, s)$, they are differentiable for $s, t > 0$ and part of the analysis is devoted to study domains of monotonicity, concavity and convexity.

We first examine convexity properties of the auxiliary kernels, and recall the auxiliary family of quadratic functions from (21). For each $t \in I$ we assign as $s_0(t)$ the non-negative root of (21) at value $c(t)$, thus

$$(25) \quad s_0(t) := 2 \left(\sqrt{1 + c^2(t)} - 1 \right), \quad u_0(t) = k(t, s_0(t)), \quad t \geq 0.$$

Lemma 3.2. *For each $t \geq 0$ the functions \tilde{k}_t are convex on $[0, s_0(t))$ and concave on $[s_0(t), \infty)$. Consequently the kernels $\tilde{g}(t, u)$ are concave on $[0, u_0(t))$ and convex on $[u_0(t), u_{max}(t))$.*

Proof. We use the representation from (23). Given $t \in I$ the polynomial $p_{c(t)}$ is negative on $[0, s_0(t))$ and positive on $[s_0(t), \infty)$ which provides us with the convexity properties for \tilde{k}_t . \square

We continue with the following bound for the derivatives of the kernels \tilde{k}_t . For any constant $\bar{s} < \infty$ we introduce the concavity set (with respect to s) for $\tilde{k}(t, s)$ as

$$(26) \quad B(\bar{s}) := \{(t, s) : t \in I, s_0(t) \leq s \leq \bar{s}\} \subset I \times [0, \infty).$$

Lemma 3.3. *There exists a constant $C > 0$ such that on the set $B(\bar{s})$ we have*

$$(27) \quad \frac{\partial \tilde{k}}{\partial s}(t, s) \geq C.$$

Consequently, by the implicit function theorem it holds

$$0 \leq \frac{\partial \tilde{g}}{\partial u}(t, u) \leq \frac{1}{C} < \infty \quad \text{for } t \in I, u \geq u_0(t).$$

Proof. By Lemma 3.2 the function \tilde{k}_t is concave on $[s_0(t), \bar{s}]$, thus

$$\frac{\partial \tilde{k}}{\partial s}(t, s) \geq \frac{\partial \tilde{k}}{\partial s}(t, \bar{s}) = P\phi(d + \sqrt{\bar{s}}) \frac{1}{2\sqrt{\bar{s}}}.$$

As P , K , r and \bar{s} are fixed we find a constant D such that

$$|d + \sqrt{\bar{s}}| \leq D, \quad t \in I.$$

Using the monotonicity of ϕ this yields $\frac{\partial \tilde{k}}{\partial s}(t, s) \geq \phi(D) \frac{P}{2\sqrt{\bar{s}}} =: C$, hence (27), and finally the second assertion. \square

We finally examine monotonicity properties of the forward, and hence inverse kernels. It turns out that it is important whether there is $\mu \in I$ for which $c(\mu) = 0$, where $c(t)$ is defined in (3).

Lemma 3.4. *Given $t_0 \in I$ let*

$$(28) \quad \mu_0 := \begin{cases} 0 & , K \leq P, r > 0, \\ \min \left\{ \frac{1}{r} \log(K/P), t_0 \right\} & , K > P, r > 0, \\ t_0 & , K > P, r = 0. \end{cases}$$

For each $s > 0$ the function $\tilde{k}(t, s)$ is increasing on $[0, \mu_0]$ and decreasing on $[\mu_0, T]$. In case that $r = 0$ the kernel is constant with respect to t .

Proof. We only provide the proof in case that $0 < \mu_0 < t_0$. Then $K > P$ but $Ke^{-rT} < P$. If now $0 \leq t \leq \mu_0$ then $P - Ke^{-rt} \leq 0$. Representation (24) yields that kernel is increasing, there. If $\mu_0 \leq t \leq T$ then $P - Ke^{-rt} \geq 0$ and (24) shows that the kernel is decreasing there, since $\Phi(d) - 1 < 0$. \square

Remark 3.1. The identities in the beginning of (20) and Lemma 3.4 imply that the inverse kernel \tilde{g} is decreasing on $[0, \mu_0]$ and increasing on $[\mu_0, T]$, such that for any $t_0 \in I$ the corresponding m_0 in Case 2 must be one of the endpoints 0 or t_0 of the interval $[0, t_0]$. More specifically, if $K \leq P$ and $r > 0$ we have that $m(t_0) = t_0$, while for $K > P$ and $0 < t_0 \leq \frac{1}{r} \log(K/P)$ we always have that $m(t_0) = 0$.

3.2. Local analysis of the kernels. In the asymptotic analysis of moduli of continuity emphasis is on properties in a (right) neighborhood of zero, it is thus local. In [13] the notion of a *germ* of an index function proved to be useful. Here the major tool is provided by a different *equivalence relation*, which will be studied below. We shall study functions which are inverse to index functions and we find it convenient to denote $\text{id}(s) := s$, the identical mapping on \mathbb{R}^+ . We recall

the following definition, specializing the corresponding one from [3] for the filter of right neighborhoods of zero.

Definition 2. Two index functions φ, ψ are called *equivalent at zero*, and we write $\varphi \sim \psi$, if $\lim_{s \rightarrow 0+} \varphi(s)/\psi(s) = 1$.

Remark 3.2. We shall also use equivalence at infinity, abbr. as \sim_∞ , which is defined for functions, say $f, g > 0$ in a neighborhood of ∞ . Thus we write $f \sim_\infty g$ if $\lim_{s \rightarrow \infty} f(s)/g(s) = 1$. Again, we refer to [3, Chapt. V] for more details.

3.2.1. *The forward kernel.* To describe the asymptotics of the forward kernel we introduce the auxiliary kernel

$$h_t(s) = h(t, s) := \sqrt{\frac{PKe^{-rt}}{2\pi}} \frac{1}{c(t)^2} s^{3/2} e^{-\frac{c(t)^2}{2s}} \quad s > 0, t > 0.$$

The following asymptotics of the forward kernels \tilde{k}_t is basic for the following analysis. We indicate the kernels and their asymptotics in Figure 4.

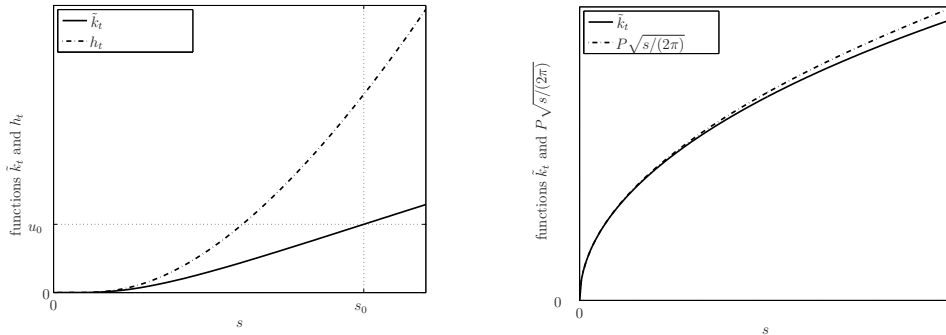


FIGURE 4. The forward kernels and their asymptotics for $c(t) \neq 0$ (left) and $c(t) = 0$ (right).

Lemma 3.5. For any $t \in I$ the following asymptotics holds true.

$$(29) \quad \tilde{k}_t(s) \sim \begin{cases} h_t(s) & , \text{ if } c(t) \neq 0, \\ P\sqrt{\frac{s}{2\pi}} & , \text{ if } c(t) = 0, \end{cases} \quad \text{as } s \rightarrow 0.$$

Proof. In the first case we shall provide the proof for $c(t) > 0$, only. The case $c(t) < 0$ is treated similarly. A simple manipulation of the kernel \tilde{k}_t , based on u_{BS}^{call} from (2), yields that

$$(30) \quad \tilde{k}_t(s) = Ke^{-rt}(1 - \Phi(d)) - P(1 - \Phi(d + \sqrt{s})), \quad s > 0.$$

The definition of d from (4) implies that $d + \sqrt{s} = \sqrt{d^2 + 2c}$, and we arrive at

$$\tilde{k}_t(s) = \frac{1}{\sqrt{2\pi}} \left(K e^{-rt} \int_d^\infty e^{-\tau^2/2} d\tau - P \int_{\sqrt{d^2+2c}}^\infty e^{-\tau^2/2} d\tau \right),$$

such that we let

$$(31) \quad F(d) := K e^{-rt} \int_d^\infty e^{-\tau^2/2} d\tau - P \int_{\sqrt{d^2+2c}}^\infty e^{-\tau^2/2} d\tau.$$

Substituting $\tau := \sqrt{u^2 + 2c}$ in the integral on the right yields

$$F(d) = K e^{-rt} \int_d^\infty e^{-u^2/2} \left(1 - \sqrt{\frac{u^2}{u^2 + 2c}} \right) du.$$

Elementary calculus proves that

$$e^{-u^2/2} \left(1 - \sqrt{\frac{u^2}{u^2 + 2c}} \right) \sim_\infty \frac{c}{u^2} e^{-u^2/2}.$$

By [3, Chapt. V, §3.3, Prop. 6] this implies that

$$(32) \quad F(d) \sim_\infty K e^{-rt} c \int_d^\infty \frac{1}{u^2} e^{-u^2/2} du.$$

Plainly we have that

$$\left(1 - \frac{15}{u^4} \right) \frac{1}{u^2} e^{-u^2/2} \leq \frac{1}{u^2} e^{-u^2/2} \leq \left(1 + \frac{1}{u^2} \right) \frac{1}{u^2} e^{-u^2/2}, \quad u > 2,$$

but the integrals with integrands on the left and right, respectively, can explicitly be evaluated, which yields

$$\left(1 - \frac{3}{d^2} \right) \frac{1}{d^3} e^{-d^2/2} \leq \int_d^\infty \frac{1}{u^2} e^{-u^2/2} du \leq \frac{1}{d^3} e^{-d^2/2}.$$

Both, the upper and lower bounds are equivalent at infinity and hence

$$\int_d^\infty \frac{1}{u^2} e^{-u^2/2} du \sim_\infty \frac{1}{d^3} e^{-d^2/2}.$$

This yields $F(d) \sim_\infty K e^{-rt} c \frac{1}{d^3} e^{-d^2/2}$. Inserting this and performing the coordinate transformation (4), which preserves equivalence, see [3,

Chapt. V, § 1.3], we derive that

$$\begin{aligned}
 \tilde{k}_t(s) &\sim \frac{Ke^{-rt}c}{\sqrt{2\pi}} \frac{s^{3/2}}{(c-s/2)^3} e^{-\frac{(c-s/2)^2}{2s}} \\
 &= \frac{Ke^{-rt}c}{\sqrt{2\pi}} \frac{1}{(c-s/2)^3} s^{3/2} e^{-\frac{c^2}{2s} + \frac{c}{2} - \frac{s}{8}} \\
 &= \frac{Ke^{-rt}c}{\sqrt{2\pi}} \frac{\sqrt{P/(Ke^{-rt})}}{(c-s/2)^3} s^{3/2} e^{-\frac{c^2}{2s} - \frac{s}{8}} \\
 &\sim \frac{c}{\sqrt{2\pi}} \frac{\sqrt{PK}e^{-rt}}{(c-s/2)^3} s^{3/2} e^{-\frac{c^2}{2s}},
 \end{aligned}$$

from which the proof of (29) can be completed in the first case.

The proof of (29) is simpler in the second case, since in this case $P = Ke^{-rt}$, thus

$$\tilde{k}_t(s) = P (\Phi(\sqrt{s}/2) - \Phi(-\sqrt{s}/2)) = \frac{P}{\sqrt{2\pi}} \int_{-\sqrt{s}/2}^{\sqrt{s}/2} e^{-x^2/2} dx,$$

from which the proof of (29) can easily be accomplished. \square

Remark 3.3. The asymptotics of the integral in (32) was performed similar to the one for $1 - \Phi(x)$ as $x \rightarrow \infty$ in [4, Chapt. VII, §1, Lemma 2].

3.2.2. *The inverse kernel.* We turn to the local analysis of the kernels \tilde{g}_t , and we introduce the index functions $\eta_t(u) := \frac{c(t)^2}{2 \log(1/u)}$.

Lemma 3.6. *For each $t \in I$ the following assertions hold true.*

- (1) $\eta_t(u) \sim h_t^{-1}(u)$ as $u \rightarrow 0$.
- (2) *Consequently, the asymptotics*

$$(33) \quad \tilde{g}_t(u) \sim \begin{cases} \eta_t(u) & , \text{ if } c(t) \neq 0, \\ 2\pi (u/P)^2 & , \text{ if } c(t) = 0, \end{cases}$$

is valid as $u \rightarrow 0$.

Proof. We first establish (1). In view of Corollary A.1 it is enough to show $\eta_t \circ h_t \sim id$, or equivalently $\lim_{s \rightarrow 0} \frac{\eta_t(h_t(s))}{s} = 1$. Indeed, we have

$$\begin{aligned}
 \frac{\eta_t(h_t(s))}{s} &= \frac{c^2}{2s \log \left(\frac{\sqrt{2\pi}c^2 e^{\frac{c^2}{2s}}}{\sqrt{PK}e^{-rt} s^{3/2}} \right)} \\
 &= \frac{c^2}{2} \frac{1}{s \log \left(\frac{\sqrt{2\pi}c^2}{\sqrt{PK}e^{-rt}} \right) + \frac{c^2}{2} - \frac{3}{2}s \log(s)}.
 \end{aligned}$$

Observing that

$$s \log \left(\frac{\sqrt{2\pi}c^2}{\sqrt{PK}e^{-rt}} \right) \rightarrow 0 \quad \text{and} \quad \frac{3}{2}s \log(s) \rightarrow 0$$

as $s \rightarrow 0$, yields the desired result.

Assertion (2) in case that $c(t) \neq 0$ will follow from assertion (1), once we have proved that $\tilde{k}_t^{-1} \sim h_t^{-1}$. To this end we first observe that the auxiliary function η_t is the $c(t)^2/2$ multiple of the function ϱ_2 , defined in the appendix. Example A.3 shows that η_t has property (R), as defined in the appendix. Consequently, by assertion (1) the kernel h_t^{-1} also has property (R), as this property is compatible with the equivalence relation \sim . Now, Lemma 3.5 together with Corollary A.2 gives $\tilde{k}_t^{-1} \sim h_t^{-1}$. The remaining case is trivial. \square

4. ASYMPTOTICS OF THE FORWARD AND INVERSE PROBLEMS

Here we shall establish the asymptotics of the moduli of continuity for the forward and inverse operators N and N^{-1} , respectively. Since the shift does not depend on s the corresponding moduli of continuity are not affected, see Remark 2.3.

4.1. Modulus of continuity of the forward problem. We first emphasize the validity of Assumption 1. The function $t \mapsto s_{\max}(t)$ equals infinity, here.

Lemma 4.1. *Let $s_0(t)$ be the non-negative root of $p_c(s) = 0$, with function p_c from (21), hence the root is given by (25). Given $t \in I$ the kernel $\tilde{k}(t, s)$ is convex on $[0, s_0(t))$ and concave on $[s_0(t), \infty)$.*

This shows that the situation of Case 1 is met when $s_0(t) > 0$, $t \in I$. To this end we require that the option is not at the money.

Assumption 2. There is a constant $c_{\min} > 0$ such that

$$|c(t)| = \left| \log \frac{P}{Ke^{-rt}} \right| \geq c_{\min} > 0 \quad t \in I.$$

Lemma 4.2. *Under Assumption 2 there is $s_{\min} > 0$ such that $s_0(t) \geq s_{\min} > 0$, $t \in I$.*

Proof. Under Assumption 2 we find from (25) that

$$s_0(t) \geq 2 \left(\sqrt{1 + c_{\min}^2} - 1 \right) =: s_{\min} > 0, \quad t \in I,$$

which completes the proof. \square

In order to apply Proposition 2.2 we need to bound the Lipschitz constants $\partial k/\partial s$. We start with some technical result.

Lemma 4.3. *Let $p_c(s)$ be from (21) and let $s > 0$, $c \in \mathbb{R}$ be related by $p_c(s) = 0$. Then*

$$(34) \quad \lim_{|c| \rightarrow 0} \frac{\sqrt{s}|c|}{(c + s/2)^2} = 1.$$

Proof. From $p_c(s) = 0$ we immediately see that $0 < s \leq |c|^2$. But, for each $0 < \varepsilon \leq 1$, and if $|c|^2 \leq \varepsilon$ then

$$|c|^2 = s + s^2/4 = s(1 + s/4) \leq (1 + \varepsilon)s.$$

In our proof we distinguish two cases. For $c > 0$ we conclude that

$$1 = \frac{c^2}{c^2} \geq \frac{\sqrt{sc}}{(c + s/2)^2} \geq \frac{s}{\left(\sqrt{(1 + \varepsilon)s} + s/2\right)^2} \rightarrow 1,$$

as ε , and hence $s \rightarrow 0$. If $c < 0$ then we rewrite the relation (21) as $(c + s/2)(c - s/2) = s$ which yields that $(s/2 - |c|)(s/2 + |c|) = -s$. Thus we obtain the identity

$$\frac{\sqrt{s}|c|}{(c + s/2)^2} = \frac{\sqrt{s}|c|}{(s/2 - |c|)^2} = \frac{\sqrt{s}|c|(s/2 + |c|)^2}{s^2}.$$

Form this we derive that

$$\frac{\sqrt{s}|c|}{(c + s/2)^2} = \frac{\sqrt{s}|c|(s/2 + |c|)^2}{s^2} \geq \left(\frac{|c|^2}{s}\right)^{3/2} \rightarrow 1, \quad \text{as } c \rightarrow 0.$$

Finally we obtain

$$\frac{\sqrt{s}|c|}{(c + s/2)^2} \leq \frac{\sqrt{s}|c|(|c|^2/2 + |c|)^2}{s^2} \leq (1 + \varepsilon)^{3/2} \frac{|c|(|c|^2/2 + |c|)^2}{|c|^3} \rightarrow 1,$$

as ε , hence c tend to zero. \square

Lemma 4.4. *For each $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\frac{\partial \tilde{k}}{\partial s}(t, s_0(t)) \leq (1 + \varepsilon) \frac{P}{e\sqrt{2\pi}|c(t)|}, \quad \text{if } |c(t)| \leq \delta.$$

Proof. The partial derivative of $\tilde{k}(t, s)$ with respect to s is given in (22) We rewrite the right hand side as

$$(35) \quad \frac{P}{2} \phi \left(\frac{c(t) + s/2}{\sqrt{s}} \right) \frac{1}{\sqrt{s}} \\ = \frac{P}{2} \phi \left(\frac{c(t) + s/2}{\sqrt{s}} \right) \left(\frac{c(t) + s/2}{\sqrt{s}} \right)^2 \times \frac{\sqrt{s}}{(c(t) + s/2)^2}.$$

The function $P\phi(x)x^2/2$, $x > 0$, is uniformly bounded by $P/(e\sqrt{2\pi})$. The functions $s_0(t)$ and $c(t)$ are related by (25), thus (21) holds, and we use Lemma 4.3 to bound the last factor and to complete the proof. \square

Lemma 4.5. *Under Assumption 2 there is $L < \infty$ such that (10) holds.*

Proof. By definition of $s_0(t)$ the function \tilde{k}_t is concave on $[s_0(t), \infty)$, and Remark 2.4 applies together with the bound from Lemma 4.4. \square

This immediately yields the following result for the forward operator.

Theorem 4.1. *Under Assumption 2 the forward operator N from (6) is Lipschitz continuous, hence there is $L < \infty$ for which*

$$\omega(N, \mathcal{D}(N), \delta) \leq L\delta.$$

Specifically, for each $\varepsilon > 0$ there is $\eta > 0$ such that

$$\omega(N, \mathcal{D}(N), \delta) \leq (1 + \varepsilon) \frac{P}{e\sqrt{2\pi}} \frac{\delta}{c_{min}}, \text{ provided that } 0 < c_{min} \leq \eta.$$

Remark 4.1. The above result reproves previous results from [5, § 4.4.2] with emphasis on the quantity c_{min} .

There is a local variant which allows to describe the modulus of continuity of the forward operator if the option is *not at the money*.

Theorem 4.2. *Suppose that $P \neq K$. For each $S^\dagger \in \mathcal{D}_+(N)$ there are $\delta_0 > 0$ and $L < \infty$ such that*

$$\omega(N, S^\dagger, \delta) \leq L\delta, \quad 0 < \delta \leq \delta_0.$$

Proof. We first check that (10) is fulfilled for some interval $t \in [0, t_0]$. If $P > K$ then Assumption 2 is fulfilled $t_0 := T$ and $c_{min} := \log(P/K)$, and Theorem 4.1 applies immediately. If $P < K$ then there is $0 < t_0 \leq T$ for which $c(t) \leq c(t_0) < 0$, $t \in [0, t_0]$. Then we may apply Lemma 4.5 to obtain a uniform bound for the derivatives $\frac{\partial \tilde{k}}{\partial s}(t, s_0(t))$ for $0 \leq t \leq t_0$. We need to check (14), and use representation (22) to bound the derivatives. The density ϕ is uniformly bounded by $1/\sqrt{2\pi}$, and for any $s_0 > 0$ and $s \geq s_0$ the quotient $1/\sqrt{s}$ is bounded from above, which

shows that (14) is valid. Thus, the assumptions of Proposition 2.4 are fulfilled and the proof follows from this. \square

In the specific (practically irrelevant) case that $P = K$ and $r = 0$ we have that $c(t) = 0$ for all $t \in I$ and thus $s_0(t) = 0$. In this case the kernel $\tilde{k}(t, s)$ does not depend on t , and is concave. We are thus in Case 2 and apply Proposition 2.3.

Theorem 4.3. *Suppose that $P = K$ and $r = 0$. For each $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that*

$$\omega(N, \mathcal{D}(N), \delta) \leq (1 + \varepsilon) \frac{P}{\sqrt{2\pi}} \sqrt{\delta}, \quad 0 < \delta \leq \delta(\varepsilon).$$

Proof. Here the kernel does not depend on t and Proposition 2.3 yields that

$$(36) \quad \omega(N, \mathcal{D}(N), \delta) \leq k(0, \delta) + 2L\delta, \quad \delta > 0.$$

Now we apply the local analysis and see from Lemma 3.5 that $k(0, s) \sim \frac{P}{\sqrt{2\pi}} \sqrt{s}$ as $s \rightarrow 0$. Plainly, we have that $s = O(k(0, s))$ as $s \rightarrow 0$, such that we apply the reasoning from § A.1 to complete the proof. \square

There is one case left for which the asymptotics could not be established. This is when $P = K$ but $r > 0$.

4.2. Modulus of continuity of the inverse problem. We finally establish bounds for the modulus of continuity for the inverse Black-Scholes operator $N^{-1}: \mathcal{R}(N) \subset \mathcal{C}(I) \rightarrow \mathcal{C}(I)$. By the reasoning in the beginning of § 2 this inverse mapping is again a continuous Nemytskiĭ operator. The kernel \tilde{g} is defined on $I \times [0, u_{\max}(t))$, where $u_{\max}(t)$ is given before (20). Also, as can be seen from the analysis in § 3.1 the domains of convexity and concavity interchange when switching from the forward to the inverse problems. We first need to check that $u_0(t)$ from (25) is strictly positive. This holds globally on I under Assumption 2.

Lemma 4.6. *Under Assumption 2 there is $u_{\min} > 0$ with $u_0(t) \geq u_{\min}$.*

Proof. With the constant s_{\min} from the proof of Lemma 4.2 we can bound

$$u_0(t) \geq \tilde{k}(t, s_{\min}) \geq \min \left\{ \tilde{k}(0, s_{\min}), \tilde{k}(T, s_{\min}) \right\} := u_{\min} > 0,$$

completing the proof. \square

We next examine the value $m(T)$ in Case 2. This is either zero or equals T , as asserts the following

Lemma 4.7. *Let $s_0(t)$ and $u_0(t)$ from (25). The kernel $\tilde{g}(t, u)$ obeys Assumption 1. Furthermore, the kernel $\tilde{g}(t, u)$ is concave on $[0, u_0(t))$ and convex on $[u_0(t), u_{max}(t))$. The function $m(t_0)$ is given as*

$$(37) \quad m(t_0) := \begin{cases} 0 & , \text{if } P < K \text{ and Assumption 2 holds,} \\ t_0 & , \text{if } P \geq K. \end{cases}$$

Proof. We only verify that the function m is given as in (37). The final assertion follows from Lemma 3.2. If $P < K$ and Assumption 2 holds then μ_0 from (28) equals t_0 , thus the forward kernel is increasing (in t) and the inverse kernel is decreasing, and hence we obtain that $m(t_0) = 0$. Otherwise, if $P \geq K$ then $\mu_0 = 0$, hence $m(t_0) = t_0$. \square

In order to apply Proposition 2.3 we need to bound the derivatives $\frac{\partial \tilde{g}}{\partial u}(t, u)$ for $u \geq u_0(t)$. However, as $u \rightarrow u_{max}(t)$, see (20), these explode. Hence for the analysis of the global modulus of continuity of the inverse problem we restrict the domain of definition of N^{-1} by imposing a norm bound on $\|S\|_{C(I)}$ by \bar{s} and consider as domain

$$\mathcal{D}_{\bar{s}}(N^{-1}) := N(\{S, \|S\|_{C(I)} \leq \bar{s}\}),$$

the image of the ball of radius \bar{s} under N . By doing so we can use Lemma 3.3 to obtain that

$$(38) \quad \frac{\partial \tilde{g}}{\partial u}(t, u) \leq L \quad \text{for } u_0(t) \leq u \leq \bar{u} := \tilde{k}(m(T), \bar{s}).$$

Theorem 4.4. *The following bounds for the modulus of continuity hold true.*

(1) *If $K < P$ then for all $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that*

$$\omega(N^{-1}, \mathcal{D}_{\bar{s}}(N^{-1}), \delta) \leq (1 + \varepsilon) \frac{\log^2(P/K + rT)}{2 \log(1/\delta)}, \quad 0 < \delta \leq \delta(\varepsilon).$$

(2) *If $P < K$ and, in addition, Assumption 2 holds then for all $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that*

$$\omega(N^{-1}, \mathcal{D}_{\bar{s}}(N^{-1}), \delta) \leq (1 + \varepsilon) \frac{\log^2(P/K)}{2 \log(1/\delta)}, \quad 0 < \delta \leq \delta(\varepsilon).$$

Proof. As discussed above we may apply Proposition 2.5 with the values of $m(T)$ given in Lemma 4.7. Hence, for $K < P$ this yields

$$\omega(N^{-1}, \mathcal{D}_{\bar{s}}(N^{-1}), \delta) \leq \tilde{g}(T, \delta) + 2L\delta.$$

Now we use the asymptotics of the inverse kernel from Lemma 3.6 to see that $\tilde{g}(T, u) \sim h(T, u)$ as $u \rightarrow 0$. The reasoning from § A.1 allows to complete the proof in this case. Otherwise, if $P < K$ then we can continue in a similar manner. \square

For the local modulus we need the following refinement of Lemma 4.6.

Lemma 4.8. *If $P \neq K$ then there is $0 < t_{\min} \leq T$ such that Assumption 2 is fulfilled on $I_0 := [0, t_{\min}]$. Consequently, for any $0 \leq t \leq t_{\min}$ it holds $u_0(t) > 0$.*

Proof. The function $t \mapsto |c(t)|$ is continuous, and under $P \neq K$ it holds $|c(0)| > 0$ from which the first assertion follows. \square

Theorem 4.5. *Suppose that $P \neq K$ and $u^\dagger = N(S^\dagger)$ for some $S^\dagger \in \mathcal{D}_+(N)$. For each $\varepsilon > 0$ there is $\delta(\varepsilon)$ such that*

$$\omega(N^{-1}, u^\dagger, \delta) \leq (1 + \varepsilon) \frac{\log^2(P/K)}{2 \log(1/\delta)}, \quad 0 < \delta \leq \delta(\varepsilon).$$

Proof. We are to apply Proposition 2.5. By Lemma 3.3 condition (10) is globally fulfilled. Condition (14) is fulfilled with $S_0 := \bar{u}$ by using (38). As discussed in Remark 3.1, the value $m(t_0)$ as required in Case 2 equals t_0 if $K < P$, whereas this equals zero if $K > P$ and $t_0 \leq \frac{1}{r} \log(K/P)$. Thus for any $t_0 \leq \min \{t_{\min}, \frac{1}{r} |\log(K/P)|\}$ from above we can apply Proposition 2.5 to obtain that for some $\delta_0 > 0$ and $\tilde{L} < \infty$ it holds

$$\omega(N^{-1}, u^\dagger, \delta) \leq \tilde{g}(m_0, \delta) + \tilde{L}\delta, \quad 0 < \delta \leq \delta_0.$$

Given $\varepsilon > 0$ we first choose $t_0 > 0$ so small that $c^2(m_0) \leq \sqrt{1 + \varepsilon} c^2(0)$. Now we apply the local analysis. Using the equivalence from Lemma 3.6, there is $0 < \delta_1 \leq \delta_0$ such that

$$\omega(N^{-1}, u^\dagger, \delta) \leq \sqrt{1 + \varepsilon} \eta(m_0, \delta), \quad 0 < \delta \leq \delta_1.$$

But $\eta(m_0, \delta) = c^2(m_0)/(2 \log(1/\delta))$, and we can complete the proof. \square

Again we may add the result in case that $P = K$ and $r = 0$. Then the kernel $\tilde{g}(t, u)$ is globally convex and we are in Case 1.

Theorem 4.6. *If $P = K$ and $r = 0$ then there is a constant $L < \infty$ such that*

$$\omega(N^{-1}, \mathcal{D}_{\bar{s}}(N^{-1}), \delta) \leq L\delta.$$

We could not treat the case when $P = K$ but $r > 0$, but for this case we are going to provide some lower bound, next.

4.3. Lower bounds. The asymptotics of the forward and inverse kernels \tilde{k} and \tilde{g} , respectively, revealed in Lemmas 3.5 and 3.6 a different behavior depending whether $c(t)$ equals zero or not. As it will turn out, for the lower bounds the situation when the kernel is concave is relevant.

Again we restrict the domains of definition to $\mathcal{D}_+(N)$ and the corresponding restriction of $\mathcal{D}_{\bar{s}}(N^{-1})$ to $\mathcal{D}_{+, \bar{s}}(N^{-1})$ for the inverse N^{-1} .

Proposition 4.7. *Suppose that $r > 0$. There are $c > 0$ and $\delta_0 > 0$ such that*

$$\omega(N^{-1}, \mathcal{D}_{+, \bar{s}}(N^{-1}), \delta) \geq \frac{c}{\log(1/\delta)}, \quad 0 < \delta \leq \delta_0.$$

Proof. For $r > 0$ there is $0 < t_0 \in I$ with $c(t_0) \neq 0$. For $\delta > 0$ we can find $u_1, u_2 \in N_{+, s_{\max}}$ with $\|u_1 - u_2\|_{C(I)} \leq \delta$ and $u_1(t_0) = \delta$, $u_2(t_0) = h(t_0, \delta)$ for the auxilliary kernel h from § 3.2.1. Plainly it holds that

$$\omega(N^{-1}, \mathcal{D}_{+, \bar{s}}(N^{-1}), \delta) \geq g(t_0, \delta) - g(t_0, h(t_0, \delta)).$$

Since $\delta = o(\eta(t_0, \delta))$ we use Lemma A.1 to deduce that

$$g(t_0, \delta) - g(t_0, h(t_0, \delta)) \sim \eta(t_0, \delta),$$

and there is $\delta_0 > 0$ such that it holds

$$g(t_0, \delta) - g(t_0, h(t_0, \delta)) \geq \frac{1}{2}\eta(t_0, \delta) = \frac{c(t_0)^2}{4 \log(1/\delta)}, \quad 0 < \delta \leq \delta_0.$$

The proof is complete. \square

In particular this result shows that the bounds from Theorem 4.4 are order optimal.

The situation is more subtle for the forward operator, since in this case the concave part is met for t_0 with $c(t_0) = 0$. If $c(t) \neq 0$ for all $t \in I$, i. e., if Assumption 2 is met, then this cannot happen, and we provided the asymptotics in Theorem 4.1, which is order optimal by Lemma 2.2. This could further be strengthened in Theorem 4.2 to the case when $c(t) \neq 0$ for some interval $t \in (0, t_0]$. We are left with the case that $c(0) = 0$. This happens exactly when the option is *at the money*, i. e., $P = K$. In view of Theorem 4.3 the interesting situation is $P = K$ and $r > 0$. In this case we cannot argue as in the proof of Proposition 4.7. However, we conjecture that it holds

$$\omega(N, \mathcal{D}_+(N), \delta) \geq (1 - \varepsilon) \frac{P}{\sqrt{2\pi}} \sqrt{\delta}, \quad 0 < \delta \leq \delta(\varepsilon)$$

for $P = K$ and $r > 0$.

5. EXTENSION AND DISCUSSION

5.1. European put options. The above analysis was carried out for European call options. We find it interesting to note that the analysis extends to European puts, easily. Indeed, starting from the *put-call-parity*

$$(39) \quad u_{BS}^{put}(P, K, r, t, s) + P = u_{BS}^{call}(P, K, r, t, s) + Ke^{-rt},$$

we obtain, using the decomposition $P - Ke^{-rt} = (P - Ke^{-rt})_+ - (P - Ke^{-rt})_-$ into its positive and negative parts, that

$$\begin{aligned} u_{BS}^{put}(P, K, r, t, s) - (P - Ke^{-rt})_- &= u_{BS}^{call}(P, K, r, t, s) - (P - Ke^{-rt})_+ \\ &= \tilde{k}_t(s), \end{aligned}$$

i. e., the shifted kernel from (19). From this it is easy to derive that all subsequent results, and the theorem hold true for the Black-Scholes operator corresponding to a put. Actually, the representation (30) is the price of the European put (if $c > 0$) given by

$$u_{BS}^{put}(P, K, r, t, s) = Ke^{-rt}\Phi(-d) - P\Phi(-d - \sqrt{s}), \quad s > 0.$$

5.2. Discussion. We analyzed the moduli of continuity for several constellations of P, K and r , both for the forward and inverse Black-Scholes operators. In particular the assertions from Theorem 4.4 and Theorem 4.5 confirm the conjecture mentioned in § 1.2. As can be seen from the analysis, these well- or ill-conditioning effects are due to convexity or concavity in a neighborhood of zero, as this is expressed in Cases 1 and 2.

We stress that the lower bound from Proposition 4.7 extends to hold for any option pricing model (with corresponding non-linear operators \tilde{N} and \tilde{N}^{-1}) which includes the one with time dependent volatility. So, ill-conditioning must occur in other models, too.

APPENDIX A. LOCAL ANALYSIS OF INDEX FUNCTIONS WHICH ARE EQUIVALENT AT ZERO

Here we collect several results for index functions and the related equivalence relation, as introduced in Definitions 1 and 2, and which are used in the asymptotic analysis of this paper, but which may be of independent interest.

A.1. Initial calculus. We start with the following result, which relates the equivalence relation \sim introduced in Definition 2 to some other local relation $= o$. Precisely, given two index functions φ, ψ we shall write $\varphi = o(\psi)$ if $\varphi(s)/\psi(s) \rightarrow 0$ as $s \rightarrow 0+$, where we again refer to [3, Chapt. V]. There it is seen that $\varphi \sim \psi$ is equivalent to $\psi - \varphi = o(\psi)$ as $s \rightarrow 0+$. Moreover, if $\psi = o(\varphi)$ then $\varphi \pm \psi \sim \varphi$. We add the following technical and more involved result.

Lemma A.1. *Suppose that φ, ψ is a pair of mutually inverse index functions and that the index function f obeys $f \sim \psi$. If $\text{id}(s) = o(\psi)$ then*

$$(40) \quad f(s) - f(\varphi(s)) \sim \psi(s), \quad \text{as } s \rightarrow 0.$$

Proof. We first show that $f(\varphi(s)) = o(f)$. Indeed, we let $\varepsilon = 1/3$ and find $\delta_0 > 0$ such that

$$\frac{f(\varphi(s))}{f(s)} \leq \frac{(1 + \varepsilon)\psi(\varphi(s))}{(1 - \varepsilon)\psi(s)} = 2\frac{\text{id}(s)}{\psi(s)} \rightarrow 0 \text{ as } \delta_0 \geq s \rightarrow 0.$$

Now the proof can be completed from

$$f(s) - f(\varphi(s)) \sim f(s) \sim \psi(s), \quad \text{as } s \rightarrow 0.$$

□

Finally we mention the following immediate consequence of the equivalence relation: If $\varphi \sim \psi$, then for each $\varepsilon > 0$ there is $\delta > 0$ such that $\varphi(s) \leq (1 + \varepsilon)\psi(s)$, $0 < s \leq \delta$.

A.2. Coordinate transformations. In our analysis it will be important to know, how the equivalence at zero behaves with respect to coordinate transformations. The answer is easy for transformations *from the right*.

Lemma A.2. *Let $s = \varrho(u)$ be a coordinate transformation with an index function ϱ . Then $\varphi \circ \varrho \sim \psi \circ \varrho$ holds if and only if $\varphi \sim \psi$.*

Proof. The “if” part holds in general, see [3, Chapt. V, § 1.3]. But since along with ϱ the inverse ϱ^{-1} is also an index function, this allows to accomplish the proof. □

Remark A.1. We do not pay attention to the domains of definition, since the equivalence is a local property. Restricting the domain of definition to a smaller neighborhood will resolve problems. One might instead study the equivalence as introduced above in Definition 2, for *germs of index functions* rather than for index functions.

As a consequence we state

Corollary A.1. *If $\varphi^{-1} \circ \psi \sim \text{id}$ then $\varphi^{-1} \sim \psi^{-1}$.*

This follows by applying the coordinate transformation ψ^{-1} from the right on both sides .

Applying such transformation in the domain space, i.e., from the left, may have different impact, and we exhibit this at the pair $\varrho_1(s) := e^{-1/s}$ and $\varrho_2(s) := 1/\log(1/s)$ of mutually inverse index functions.

Example A.1 (Destroying equivalence). Plainly, the functions $\varphi(s) = s$ and $\psi(s) = s + s^{3/2}$ are equivalent. However,

$$\frac{\varrho_1(\psi(s))}{\varrho_1(\varphi(s))} = e^{1/\varphi^{-1}/\psi} = e^{\frac{s^{3/2}}{s(s+s^{3/2})}} \rightarrow \infty, \quad \text{as } s \rightarrow 0,$$

hence $\varrho_1 \circ \varphi \not\sim \varrho_1 \circ \psi$.

Example A.2 (Enforcing equivalence). Let $\varphi(s) := e^{-1/s}$ and $\psi(s) := s^\alpha e^{-1/s}$ for some real constant α . Then

$$\frac{\psi(s)}{\varphi(s)} = s^\alpha \rightarrow \begin{cases} 0, & \alpha > 0, \\ \infty, & \alpha < 0, \end{cases} \quad \text{as } s \rightarrow 0,$$

such that $\varphi \sim \psi$ if and only if $\alpha = 0$. However, for coordinate transformation ϱ_2 from above, we have that

$$\frac{\varrho_2(\varphi(s))}{\varrho_2(\psi(s))} = \frac{-\alpha \log s + 1/s}{1/s} = 1 - \alpha s \log s \rightarrow 1, \quad \text{as } s \rightarrow 0,$$

showing that $\varrho_2 \circ \varphi \sim \varrho_2 \circ \psi$ even for $\alpha \neq 0$.

Since $\varphi^{-1} = \varrho_2$ we infer from Corollary A.1 that $\psi^{-1} \sim \varrho_2 = \varphi^{-1}$, regardless of the value of α .

Therefore we restrict the class of coordinate transformations for which equivalence will be preserved.

Property (R). For each $0 < \varepsilon < 1$ there are $0 < \delta < 1$ and $0 < s_0 \leq a/2$ such that

$$(41) \quad \frac{\varrho((1 + \delta)s)}{\varrho(s)} \leq 1 + \varepsilon, \quad 0 < s \leq s_0.$$

First, it is easily seen that this property is compatible with the equivalence relation, i.e., if $\varrho \sim \varrho'$ and ϱ has this property, then this holds also true for ϱ' .

Lemma A.3. *If $\varphi \sim \psi$, and if ϱ has Property (R), then $\varrho \circ \varphi \sim \varrho \circ \psi$.*

Proof. Using Property (R), given $\varepsilon > 0$ we can find $\delta > 0$ and s_0 such that (41) holds. We let $\bar{\delta} := \delta/(1 + \delta)$. Using the equivalence of φ and ψ at zero there is $0 < \bar{s}_0 \leq s_0$ for which

$$(42) \quad \frac{1}{1 + \delta} = 1 - \bar{\delta} \leq \frac{\varphi(s)}{\psi(s)} \leq 1 + \bar{\delta} \leq 1 + \delta, \quad 0 < s \leq \bar{s}_0.$$

In particular it holds $\varphi(s) \leq (1 + \delta)\psi(s)$ for $0 < s \leq \bar{s}_0$, and property (R) implies that

$$\frac{\varrho(\varphi(s))}{\varrho(\psi(s))} \leq \frac{\varrho((1 + \delta)\psi(s))}{\varrho(\psi(s))} \leq 1 + \varepsilon,$$

in this case. Taking reciprocals in (42) and interchanging the roles of φ and ψ we also infer that $\varrho(\varphi(s))/\varrho(\psi(s)) \geq 1/(1 + \varepsilon)$, provided that $0 < s \leq \bar{s}_0$, which allows to complete the proof. \square

Corollary A.2. *Suppose that $\varphi \sim \psi$. If the inverse function φ^{-1} has Property (R) then $\varphi^{-1} \sim \psi^{-1}$.*

Proof. By Lemma A.3 we conclude that $\varphi^{-1} \circ \psi \sim \varphi^{-1} \circ \varphi \sim \text{id}$, and Corollary A.1 applies. \square

We return to the pair ϱ_1, ϱ_2 of mutually inverse index functions from above.

Example A.3. The index function ϱ_1 cannot have property (R), as shows Example A.1.

In contrast, the index function ϱ_2 has Property (R). Indeed, we may let $\delta = 1/2$, because then

$$\frac{\varrho((1 + \delta)s)}{\varrho(s)} = \frac{\log(1/s)}{\log(1/(1 + \delta)) + \log(1/s)} \rightarrow 1, \quad \text{as } s \rightarrow 0.$$

Thus for each $0 < \varepsilon < 1$ there is $s_0 > 0$ such that (41) is satisfied for $s \leq s_0$. In particular, for the functions from Example A.1 we see that

$$\frac{\varrho_2(\varphi(s))}{\varrho_2(\psi(s))} = \frac{\log(1/(x + x^{3/2}))}{\log(1/x)} = \frac{\log(1/x) + \log(1/(1 + x^{1/2}))}{\log(1/x)} \rightarrow 1,$$

as $s \rightarrow 0$, hence equivalence at zero is preserved.

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