Evolutionary Γ-convergence for multiscale problems

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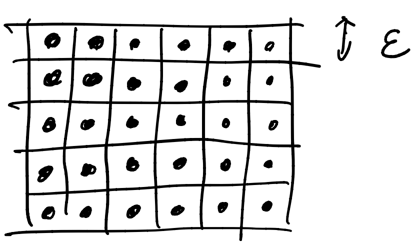
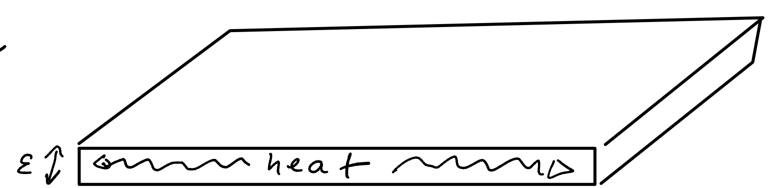
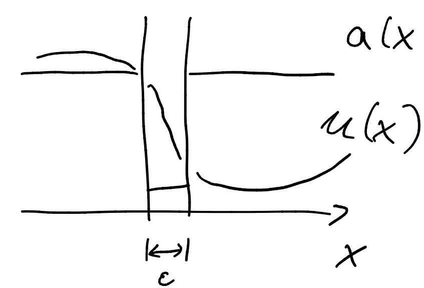
1. Introduction

Consider ODEs/PDEs of form: $\boxed{\dot{u} = V_\varepsilon(u(t)), \quad u(0) = \overset{\circ}{u}_\varepsilon \quad (*)}$

$\varepsilon > 0$ small parameter which typically describes a microscopic length scale, e.g. periodic microstructure, thin layers, singular perturbation, numerical approximation ($\varepsilon = h$)







Let $t \mapsto u_\varepsilon(t)$ denote the solution of $(*)$

General aim: Find effective model / equation

$$\boxed{\dot{u}(t) = V_{\mathrm{eff}}(u(t)), \quad u(0) = u_0^\circ}$$

s. t. any suitable limit $t \mapsto u_*(t)$ of u_ε is solution

Often: $\varepsilon \mapsto V(\varepsilon, u) = V_\varepsilon(u)$ is singular in $\varepsilon = 0$

Example: Parabolic PDE with rapidly oscillating coefficients

$$c\left(\tfrac{x}{\varepsilon}\right)\dot{u}(t,x) = \partial_x\left(a\left(\tfrac{x}{\varepsilon}\right)\partial_x u(t,x)\right) - b\left(\tfrac{x}{\varepsilon}\right)u(t,x) \quad \text{in } (0,1)$$

$$u(t,0) = u(t,1) = 0 \quad \forall t > 0 \quad + \text{ initial condition}$$

$a, b, c \in L^\infty(0,1)$ 1-periodic, i.e. $a(x+1) = a(x)$

Find effective constants $a_{\mathrm{eff}}, b_{\mathrm{eff}}, c_{\mathrm{eff}}$

Assumption: $(*)$ has a (generalized) gradient structure, i.e., there exists $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ where

- X is state space (reflexive Banach space)
- $\mathcal{E}_\varepsilon : X \longrightarrow \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ driving functional (free energy)
- $\mathcal{R}_\varepsilon : X \times X \longrightarrow [0, \infty]$ dissipation potential
  (i.e., $\mathcal{R}(u, \cdot)$ is convex, lsc, proper, $\mathcal{R}_\varepsilon(u,0) = 0$)

(convex subdifferential)

$$\dot u_\varepsilon(t) = V_\varepsilon(u(t)) \quad \Longleftrightarrow \quad \underbrace{\partial_{\dot u} \mathcal{R}_\varepsilon(u(t), \dot u(t))}_{\text{viscous forces}} + \underbrace{D\mathcal{E}_\varepsilon(u(t))}_{\substack{\text{potential}\\ \text{restoring forces}}} \ni 0$$

If $\mathcal{R}_\varepsilon(u, \dot u) = \frac{1}{2}\langle \mathbb{G}_\varepsilon(u)\dot u, \dot u\rangle$ then

$$\underbrace{\partial_{\dot u}\mathcal{R}_\varepsilon(u,\dot u) = \mathbb{G}_\varepsilon(u)\dot u = -D\mathcal{E}_\varepsilon(u)}_{\text{Force balance}} \Rightarrow \underbrace{\dot u = -\mathbb{G}_\varepsilon(u)^{-1} D\mathcal{E}_\varepsilon(u)}_{\text{Rate equation}}$$

Example: $X = H^1_0(0,1)$

$$\mathcal{E}_\varepsilon(u) = \int_0^1 \frac{a(x/\varepsilon)}{2} |\partial_x u|^2 + b(x/\varepsilon)\, u^2 \, dx$$

$$\mathcal{R}_\varepsilon(\dot u) = \int_0^1 \frac{c(x/\varepsilon)}{2} \dot u^2 \, dx$$

$$\Longrightarrow D\mathcal{R}_\varepsilon(\dot u) = c(\tfrac{x}{\varepsilon})\, \dot u = \partial_x\big(a(\tfrac{x}{\varepsilon})\partial_x u\big) - b(x/\varepsilon)\, u = -D\mathcal{E}_\varepsilon(u)$$

<u>Questions:</u> (1) Has (**) generalized gradient structure, $(X, \mathcal{E}_{eff}, \mathcal{R}_{eff})$?

(2) Find conditions on $\mathcal{R}_\varepsilon$ and $\mathcal{E}_\varepsilon$ that guarantee that limits of u_ε are solutions of effective system?

(3) In which sense are $\mathcal{E}_{eff}$ and $\mathcal{R}_{eff}$ limits of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$?

(4) Is there "interaction" between $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ in the limit?

Crucial point: Work with functionals. Variational methods!
(connection to large deviation principles $\to$ Michiel's talk)

2. Evol. Γ-convergence of gradient systems (GS)

Def. (evol. Γ-convergence of GS) We write $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow[\Gamma]{evol} (X, \mathcal{E}_{eff}, \mathcal{R}_{eff})$

iff
- $u_\varepsilon : [0,T] \to X$ is sol. for $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$
- $u^\circ_\varepsilon \to u^\circ_0$, $\mathcal{E}_\varepsilon(u^\circ_\varepsilon) \to \mathcal{E}_{eff}(u^\circ_0)$

$\Longrightarrow$

- $\exists\, u_* : [0,T] \to X$ sol. of $(X, \mathcal{E}_{eff}, \mathcal{R}_{eff})$, $u_*(0) = u^\circ_0$
- $u_\varepsilon(t) \to u_*(t)$
- $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_{eff}(u_*(t))$

$\Big\}\ \forall t \in [0,T]$

First step: Rewrite $\partial_{\dot u}\mathcal{R}_\varepsilon(u,\dot u) + D\mathcal{E}_\varepsilon(u) \ni 0$ (force balance eqn.) in a more suitable form using

Legendre transform gives **dual dissipation potential**

$\mathcal{R}^*_\varepsilon : X \times X^* \to [0,\infty]$, $\quad \mathcal{R}^*_\varepsilon(u, \xi) = \sup_{v\in X} \{ \langle \xi, v\rangle - \mathcal{R}_\varepsilon(u, v)\}$

ξ: dual variable (force) $\in X^*$

v: rate, velocity $\in X$

Legendre – Fenchel equivalences

(LF) $\underbrace{\xi \in \partial_v \tilde{\mathcal{R}}(v)}_{\text{eqn. in } X^*} \overset{(I)}{\Longleftrightarrow} \underbrace{v \in \partial_\xi \tilde{\mathcal{R}}^*(\xi)}_{\text{eqn. in } X} \overset{(II)}{\Longleftrightarrow} \underbrace{\tilde{\mathcal{R}}(v) + \tilde{\mathcal{R}}^*(\xi) = \langle \xi, v\rangle}_{\text{eqn. in } \mathbb{R}\,!!!}$

Using (LF)(I) we get (FB) $\Longleftrightarrow$ $\underbrace{\dot{u} \in \partial_\xi \mathcal{R}^*_\varepsilon(u; -D\mathcal{E}_\varepsilon(u))}_{\text{rate equation (RE)}} = V_\varepsilon(u)$

with (LF)(II) we obtain

$$(FB) \Longleftrightarrow (RE) \Longleftrightarrow \underbrace{\mathcal{R}_\varepsilon(u,\dot{u}) + \mathcal{R}^*_\varepsilon(u, -D\mathcal{E}_\varepsilon(u))}_{\text{power balance (PB)}} \underbrace{=}_{\text{"}\geq\text{" always true}} -\langle D\mathcal{E}_\varepsilon(u), \dot{u}\rangle$$

Integrate (PB) over time $t \in [0,T]$

$$\int_0^T \left\{ \mathcal{R}_\varepsilon(u,\dot{u}) + \mathcal{R}^*_\varepsilon(u, -D\mathcal{E}_\varepsilon(u)) \right\} dt = - \int_0^T - \underbrace{\langle D\mathcal{E}_\varepsilon(u), \dot{u}\rangle}_{\text{Can we write } \frac{d}{dt}\mathcal{E}_\varepsilon(u)\,?} dt$$

We assume additionally that a chain rule holds, e.g.

- $u \in W^{1,p}(0,T;X)$, $\xi \in L^{p^*}(0,T;X^*)$ with $p \geq 1$ and
  $\xi(t) \in \tilde{\partial}\mathcal{E}_\varepsilon(u(t))$ a.e. in $[0,T]$

  some notion of (sub) differential, e.g. Fréchet subdifferential

$\Rightarrow$
- $t \mapsto \mathcal{E}_\varepsilon(u(t))$ is abs. continuous (time derivative in L^1)
- $\frac{d}{dt}\mathcal{E}_\varepsilon(u(t)) = \langle \xi(t), \dot{u}(t) \rangle$ a.e. in $[0,T]$.

see e.g.
- Brezis, Opérateurs maximaux monotone ..., 1973
  Lemma 3.3 (Hilbert space, convex $\mathcal{E}$)
- Rossi-Savaré, GFs of non-convex functionals, 2006)
- Mielke-Rossi-Savaré, Nonsmooth Analysis of doubly nonlinear evolution. eqn., 2013,

"Thm." (Energy-Dissipation principle, De Giorgi principle) 8

Assume $\mathcal{E}_\varepsilon$ satisfies a chain rule on X, then $u_\varepsilon \in W^{1,1}([0,T], X)$ solves (FB), (RE) iff it solves (EDE)

$$\underbrace{\mathcal{E}_\varepsilon(u_\varepsilon(T))}_{\text{energy at } t=T} + \underbrace{\int_0^T \left[\mathcal{R}_\varepsilon(u_\varepsilon; \dot{u}_\varepsilon) + \mathcal{R}_\varepsilon^*(u_\varepsilon, -D\mathcal{E}_\varepsilon(u_\varepsilon))\right] dt}_{\text{dissipated energy over } [0,T]} \overset{(\leq)}{=} \underbrace{\mathcal{E}_\varepsilon(u_\varepsilon^\circ)}_{\text{initial energy}}$$

Proof: Only show (EDE) => (PB)

$$\int_0^T -\langle D\mathcal{E}_\varepsilon(u_\varepsilon), \dot{u}_\varepsilon\rangle dt \overset{\text{definition of } \mathcal{R}_\varepsilon^*}{\leq} \int_0^T \mathcal{R}_\varepsilon(u_\varepsilon, \dot{u}_\varepsilon) + \mathcal{R}_\varepsilon^*(u_\varepsilon, -D\mathcal{E}_\varepsilon(u_\varepsilon))\, dt$$

$$\overset{EDE}{\leq} \mathcal{E}_\varepsilon(u_\varepsilon(0)) - \mathcal{E}_\varepsilon(u_\varepsilon(T)) \underset{\text{rule}}{\overset{\text{Chain}}{=}} \int_0^T -\langle D\mathcal{E}_\varepsilon(u_\varepsilon), \dot{u}_\varepsilon\rangle dt$$

-> all inequalities are equalities, (PB) holds a.e. □

Naïve idea: Use Γ-convergence of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ to pass to the limit in (EDE), (EDI),

2.1 Brief overview over static Γ-convergence

handbook PDF

Books by Andrea Braides, Γ-convergence for beginners, 2002
Gianni Dal Maso, Introduction to Γ-convergence, 1993

$J_\varepsilon : X \to \mathbb{R}_\infty$, X reflexive Banach space,

$$\boxed{\begin{array}{l} \min J_\varepsilon \to ? \\ \operatorname{argmin} J_\varepsilon \to ? \end{array}}$$

Definition: (i) $J_\varepsilon \xrightarrow{\Gamma} J_0$ in X iff

(G1w) $\forall u_\varepsilon \rightharpoonup u_0$
(G1s) $\forall u_\varepsilon \to u_0$
$\Big\}$: $\liminf_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) \geq J_0(u_0)$

"asymptotic lower bound"...

(G2w) $\forall u_0 \, \exists \, \hat{u}_\varepsilon \rightharpoonup u_0$:
(G2s) $\forall u_0 \, \exists \, \hat{u}_\varepsilon \to u_0$:
$\Big\}$ $\limsup_{\varepsilon \to 0} J_\varepsilon(\hat{u}_\varepsilon) \leq J_0(u_0)$

..."that can be attained"

We write: $J_\varepsilon \xrightarrow{M} J_0$ (Mosco convegence) iff (G1w) + (G2s) hold.

Fundamental theorem of Γ-convergence

Assume $J_\varepsilon \xrightarrow{\Gamma} J_0$ + equicompactness of J_ε
(selection of converging subseq.)

then

$$\min J_\varepsilon \longrightarrow \min J_0 \quad \text{and} \quad \operatorname{argmin} J_\varepsilon \to \operatorname{argmin} J_0$$

Properties; 1) Assume $Z \Subset X$ (compactly) [Proof in Mielke 16 Prop. 2.5]

$\forall J > 0 \ \exists R > 0 \ \forall \varepsilon > 0, u \in X, \ J_\varepsilon(u) \le J \Rightarrow \|u\|_Z \le R$

$\Rightarrow J_\varepsilon \xrightarrow{\Gamma} J_0$ in Z is equivalent to $J_\varepsilon \xrightarrow{M} J_0$ in X.

2.) Continuity of Legendre transform w.r.t. Γ-convergence

$J_\varepsilon : X \to [0, \infty]$ convex, lsc., and $J_\varepsilon(0) = 0$, then

$$J_\varepsilon \xrightarrow{\Gamma} J_0 \text{ in } X \iff J_\varepsilon^* \xrightarrow{\Gamma} J_0^* \text{ in } X^*.$$

Example: $X = H^1_0(0,1)$

$$\mathcal{E}_\varepsilon(u) = \int_0^1 \frac{a(x/\varepsilon)}{2}|\partial_x u|^2 + b(x/\varepsilon)\, u^2 \,dx$$

$$\Rightarrow \quad \mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0 = u \longmapsto \int_0^1 \frac{a_{harm}}{2}|\partial_x u|^2 + b_{arith}\, u^2 \,dx$$

$$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \tilde{\mathcal{E}}_0 = u \longmapsto \int_0^1 \frac{a_{arith}}{2}|\partial_x u|^2 + b_{arith}\, u^2 \,dx$$

where $\quad a_{harm} = \left(\int_0^1 \frac{1}{a(y)}\,dy \right)^{-1}, \quad a_{arith} = \int_0^1 a(y)\,dy$

Similarly, for $Y = L^2(0,1)$

$$\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0 = v \longmapsto \int_0^1 \frac{c_{harm}}{2}|v|^2 \,dx$$

$$\mathcal{R}_\varepsilon \longrightarrow \tilde{\mathcal{R}}_0 = v \longmapsto \int_0^1 \frac{c_{arith}}{2}|v|^2 \,dx$$

Moreover, $\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$ in Y, $\mathcal{R}_\varepsilon|_X \xrightarrow{\Gamma} \tilde{\mathcal{R}}_0|_X$ in X

Poll: Which gradient system gives effective equation?

a) $(X, \mathcal{E}_0, \mathcal{R}_0)$ weak H^1 / weak L^2

b) $(X, \tilde{\mathcal{E}}_0, \mathcal{R}_0)$ strong H^1 / weak L^2

c) $(X, \mathcal{E}_0, \tilde{\mathcal{R}}_0)$ weak H^1 / strong L^2

d) $(X, \tilde{\mathcal{E}}_0, \tilde{\mathcal{R}}_0)$ strong H^1 / strong L^2

a) $c_{harm}\dot{u} = a_{harm}\partial_{xx}^2 u - b_{arith} u$?

b) $c_{harm}\dot{u} = a_{arith}\partial_{xx}^2 u - b_{arith} u$?

c) $c_{arith}\dot{u} = a_{harm}\partial_{xx}^2 u - b_{arith} u$?

d) $c_{arith}\dot{u} = a_{arith}\partial_{xx}^2 u - b_{arith} u$?

2.2 A first result

Aim: Pass to the in (EDE)

$$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \left\{ \mathcal{R}_\varepsilon(u_\varepsilon; \dot{u}_\varepsilon) + \mathcal{R}_\varepsilon^*(u_\varepsilon, -D\mathcal{E}_\varepsilon(u_\varepsilon)) \right\} dt = \mathcal{E}_\varepsilon(u_\varepsilon^0)$$

Thm: (Modified Sandier & Serfaty result 2004) $\boxed{\mathcal{R}_\varepsilon(u,v) = \mathcal{R}_\varepsilon(v)}$

(i) X, Y reflexive Banach spaces with $X \subset\subset Y$

(ii) $u_\varepsilon^0 \to u_0^0$ in Y and $\underbrace{\mathcal{E}_\varepsilon(u_\varepsilon^0) \to \mathcal{E}_0(u_0^0) < \infty}_{\text{well-prepared initial conditions}}$

(iii) $\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$ in Y and $\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0$ in Y

(iv) $\exists\, C > 0,\ p > 1,\ \forall \varepsilon > 0,\ v \in Y$ $\quad \frac{1}{C}\|v\|_Y^p - C \le \mathcal{R}_\varepsilon(v) \le C\|v\|_Y^p + C$

(v) $\exists\, C>0\ \forall \varepsilon>0,\ u\in X$: $\mathcal{E}_\varepsilon(u) \geq \frac{1}{C}\|u\|_X - C$

(vi) $\exists\, \Lambda\in\mathbb{R}\ \forall\varepsilon>0$: $u \mapsto \mathcal{E}_\varepsilon(u) + \Lambda\|u\|_Y^2$ is convex

(note $\Lambda \geq 0 \Rightarrow \mathcal{E}_\varepsilon$ can be non-convex)

Then, we have $(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow[\Gamma]{\text{evol}} (X, \mathcal{E}_0, \mathcal{R}_0)$

(Sketch of) Proof: (see Liero-Reichelt, 2018 for details)
Homogenization of Cahn-Hilliard eqn.

1. A priori bounds:

Well-preparedness (ii) and (iv) give uniform bounds

$$\|\dot u_\varepsilon\|_{L^p(0,T;Y)} \leq C \quad\text{and}\quad \|D\mathcal{E}_\varepsilon(u_\varepsilon)\|_{L^{p'}(0,T;Y^*)} \leq C$$

(v) yields $\|u_\varepsilon\|_{L^\infty(0,T;X)} \leq C$

2. Converging subsequences (not relabeled)

$u_\varepsilon \to u_*$ in $W^{1,p}(0,T;Y)$, $\xi_\varepsilon = -D\mathcal{E}_\varepsilon(u_\varepsilon) \to \xi_*$ in $L^{p'}(0,T;Y^*)$

$u_\varepsilon \overset{*}{\rightharpoonup} u_* \; L^\infty(0,T;X)$

Arzelà-Ascoli improves convergence s.t. $u_\varepsilon(t) \to u_*(t)$ in X $\forall t \in [0,T]$ (and hence $u_\varepsilon(t) \to u_*(t)$ in Y)

3. Passing to the limit in (EDP)

$$\underbrace{\mathcal{E}_\varepsilon(u_\varepsilon(T))}_{(\mathrm{I})} + \int_0^T \underbrace{\mathcal{R}_\varepsilon(\dot u_\varepsilon)}_{(\mathrm{II})} + \underbrace{\mathcal{R}^*_\varepsilon(\xi_\varepsilon)}_{\mathrm{III}} \, dt = \underbrace{\mathcal{E}_\varepsilon(u^\circ_\varepsilon)}_{\checkmark}$$

ad (I): Mosco convergence in Y gives $\liminf\limits_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon(T)) \geq \mathcal{E}_0(u_*(T))$

ad (III): Verson of Ioffe's lsc result for $\mathcal{J}(\varepsilon, \xi) = \int_0^T \mathcal{R}^*_\varepsilon(\xi)\, dt$

- integrand is seq. weakly lsc (see Mielke-Rossi-Savaré 2013)
- convex in ξ $\Longrightarrow \liminf \int_0^T \mathcal{R}^*_\varepsilon(\xi_\varepsilon)\, dt \geq \int_0^T \mathcal{R}^*_0(\xi_*)\, dt$

ad (II) Ioffe's theorem is not applicable since

$$\dot u_\varepsilon \rightharpoonup \dot u_* \text{ in } W^{1,p}(0,T;Y) \quad \text{and} \quad \mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0 \text{ in } Y$$

but, we have $\boxed{u_\varepsilon(t) \rightarrow u_*(t) \text{ in } Y}$ $\forall t \in [0,T]$.

Reichelt's trick: Use time-discretization: $t_k = k\tau$, $k = 0, \dots, \underset{\shortparallel}{N} \frac{T}{\tau}$

with Jensen's inequality ($\mathcal{R}_\varepsilon$ convex) we get

$$\int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon)\,dt \geq \sum_{k=0}^{N-1} \tau \mathcal{R}_\varepsilon\left(\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} \dot u_\varepsilon \,dt\right) = \sum_{k=0}^{N-1} \tau \mathcal{R}_\varepsilon\Bigg(\underbrace{\frac{u_\varepsilon(t_{k+1}) - u_\varepsilon(t_k)}{\tau}}_{\substack{\xrightarrow[\text{in } Y]{\varepsilon \searrow 0} \frac{u_*(t_{k+1}) - u_*(t_k)}{\tau}}}\Bigg)$$

Thus we have

piecewise affine interpolant

$$\liminf_{\varepsilon \to 0} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon)\,dt \geq \int_0^T \mathcal{R}_0(\dot{\hat u}_*^\tau)\,dt$$

Pass to the limit for $\tau \to 0$ using that $\dot{\hat u}_*^\tau \rightharpoonup u_*$ in $W^{1,p}(0,T;Y)$ (Ioffe)

We arrive at

$$\mathcal{E}_0(u_*(T)) + \int_0^T \mathcal{R}_0(\dot{u}_*) + \mathcal{R}_0^*(\xi_*)\,dt \le \mathcal{E}_0(u_0^\circ)$$

Remains to show that $\xi_* = -D\mathcal{E}_0(u_*(t))$ f.a.a. $t \in [0,T]$

4. Identification of limit ξ_*

We use strong-weak closedness of $D\mathcal{E}_\varepsilon$

$$u_\varepsilon \longrightarrow u \text{ in } Y \quad \text{and} \quad D\mathcal{E}_\varepsilon(u_\varepsilon) =: \xi_\varepsilon \rightharpoonup \xi_* \text{ in } Y^*$$

$$\Longrightarrow \xi_* = D\mathcal{E}_0(u)$$

This follows from Mosco convergence and Λ-convexity (cf. Attouch, 1984, Thm. 3.66)

static definition, for time-dependent version use Banach-space-valued Young measures

$\square$

Original version of Sandier-Serfaty doesn't assume strong-weak closure of $D\mathcal{E}_\varepsilon$ nor Mosco convergence $\mathcal{R}_\varepsilon \xrightarrow{M} \mathcal{R}_0$ in Y

Instead

(i) $v_\varepsilon \rightharpoonup v$ in Y $\Rightarrow$ $\liminf_{\varepsilon \to 0} \mathcal{R}_\varepsilon(v_\varepsilon) \geq \mathcal{R}_0(v)$

(ii) $u_\varepsilon \to u$ in Y $\Rightarrow$ $\liminf_{\varepsilon \to 0} \mathcal{R}_\varepsilon^*(-D\mathcal{E}_\varepsilon(u_\varepsilon)) \geq \mathcal{R}_0^*(-D\mathcal{E}_0(u))$

$\Rightarrow$ more general (our conditions yield (i) & (ii) but not vice versa)

two separate liminfs.

Example: 1D homogenization

$$\mathcal{E}_\varepsilon(u) = \int_0^1 \frac{a(x/\varepsilon)}{2} |\partial_x u|^2 + \frac{b(x/\varepsilon)}{2} u^2 \, dx$$

$$\mathcal{R}_\varepsilon(v) = \int_0^1 \frac{c(x/\varepsilon)}{2} |v|^2 \, dx \qquad X \subset\subset Y$$

We have $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ in $H^1_0(0,1) \Rightarrow \mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$ in L^2

($\mathcal{E}_0$ given via a_{harm}, b_{arith})

$\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0$ in L^2 ($\mathcal{R}_0$ given via c_{arith})

$\mathcal{E}_\varepsilon$ is uniformly Λ-convex (take $\Lambda = \frac{\|b\|_{L^\infty}}{2}$)

$\Rightarrow$ effective PDE $\quad c_{arith}\, \dot{u} = a_{harm}\, \partial^2_{xx} u - b_{arith}\, u$.

2.3 A note on well-preparedness

Example by Luc Tartar, 1989/90

Initial conditions: $u^0_\varepsilon \to u^0_0$ in Y & $\mathcal{E}_\varepsilon(u^0_\varepsilon) \to \mathcal{E}_0(u^0_0)$

Note: $\dot{u} = V_\varepsilon(u)$ may have different gradient structures

$$(X, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \quad \text{and} \quad (X, \tilde{\mathcal{E}}_\varepsilon, \tilde{\mathcal{R}}_\varepsilon)$$

Well-preparedness may lead to different effective equations in the limit.

$$\dot{u} = V_0(u) \quad \text{and} \quad \dot{u} = \tilde{V}_0(u)$$

with $V_0 \neq \tilde{V}_0$

Example: $\dot{u}(t,x) = -a(\frac{x}{\varepsilon})u(t,x)$, $u(0,x) = u^0_\varepsilon(x)$

$x \in (0,\ell)$

Solution: $u_\varepsilon(t,x) = u^0_\varepsilon(x)\exp\left(-a(\frac{x}{\varepsilon})t\right)$

ODE in each $x \in (0,\ell)$

Two gradient systems: $X = L^2(\Omega)$

Ⓐ $\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_0^\ell a(\frac{x}{\varepsilon})u(x)^2\,dx$

$\mathcal{R}_\varepsilon(v) = \frac{1}{2}\int_0^\ell v^2\,dx$

$\downarrow$

$\dot{u} = -a_{harm}\,u$

Ⓑ $\tilde{\mathcal{E}}_\varepsilon(u) = \frac{1}{2}\int_0^\ell u^2\,dx$

$\tilde{\mathcal{R}}_\varepsilon(v) = \frac{1}{2}\int_0^\ell \frac{1}{a(x/\varepsilon)}\,v^2\,dx$

$\dot{u} = -a_{arith}\,u$

$\boxed{\text{which is the right one?}}$

For neither gradient system, we have evolutionary Γ-convergence!

Two other gradient structures on the space of nonnegative Radon measures $X = M_{\geq 0}([0,\ell])$

(C) $\hat{\mathcal{E}}_\varepsilon(u) = \int_0^\ell a(x/\varepsilon)\, u(x)\, dx$, $\hat{\mathcal{R}}_\varepsilon(u,v) = \int_0^\ell \frac{v(x)^2}{2u(x)} dx$

$$\Rightarrow \quad D_{\dot u}\hat{\mathcal{R}}_\varepsilon(u,\dot u) = \frac{\dot u}{u} = -\, a(x/\varepsilon) = -D\hat{\mathcal{E}}_\varepsilon(u)$$

(D) $\overline{\mathcal{E}}_\varepsilon(u) = \int_0^\ell \frac{1}{a(x/\varepsilon)}\, u\, dx$, $\overline{\mathcal{R}}_\varepsilon(u,v) = \int_0^\ell \frac{v(x)^2}{a(x/\varepsilon)^2 u} dx$

$$\Rightarrow \quad D_{\dot u}\overline{\mathcal{R}}_\varepsilon(u,\dot u) = \frac{\dot u}{a(x/\varepsilon)^2 u} = -\frac{1}{a(x/\varepsilon)} = -D\overline{\mathcal{E}}_\varepsilon(u)$$

Effective functionals

(C) $\hat{\mathcal{E}}_0(u) = \int_0^\ell a_{min}\, u(x)\,dx$, $\hat{\mathcal{R}}_0(u,v) = \int_0^\ell \frac{v(x)^2}{2u(x)}\,dx$

(D) $\overline{\mathcal{E}}_0(u) = \int_0^\ell \frac{1}{a_{max}}\, u(x)\,dx$, $\overline{\mathcal{R}}_0(u,v) = \int_0^\ell \frac{v(x)^2}{2\, a_{max}\, u(x)}\,dx$

Thm: We have evol. Γ-convergence

$$(X, \hat{\mathcal{E}}_\varepsilon, \hat{\mathcal{R}}_\varepsilon) \xrightarrow[\Gamma]{evol} (X, \hat{\mathcal{E}}_0, \hat{\mathcal{R}}_0)$$

$$(X, \overline{\mathcal{E}}_\varepsilon, \overline{\mathcal{R}}_\varepsilon) \xrightarrow[\Gamma]{evol} (X, \overline{\mathcal{E}}_0, \overline{\mathcal{R}}_0) .$$

Well-prepared initial conditions decide which effective equation is the right one.