Power control policy on the SINR graph

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1 The model

1.1 Introduction

Let Φ a point process modelling the location of the nodes of a network. For any $t \in \mathbb{Z}_+$ let $\Phi_T(t) \subset \Phi$ be the set of nodes that are transmitting at time t and $\Phi_R(t) = \Phi \setminus \Phi_T(t)$ the ones that are receiving. We define the SINR from a node $x \in \Phi_T(t)$ to a node $y \in \Phi_R(t)$ as

$$SINR_{xy}(t) := \frac{P_x(t)h_t(x,y)\ell(x,y)}{\gamma I(t) + N}.$$

- $P_x(t)$ the transmitted power from x.
- $h_t(x,y)$ the space-time fading coefficients from x to y.
- $\ell(x,y)$ the path-loss function.
- $I(t) = \sum_{z \in \Phi_T(t) \setminus \{x,y\}} P_z(t)h_t(z,y)\ell(z,y)$ the interference coming from the other nodes.
- γ the interference suppression constant, N the noise.

We say that the transmission from $x \in \Phi_T(t)$ to $y \in \Phi_R(t)$ has been successful if $SINR_{xy} > \beta$ a positive constant. The goal is to propose a model where:

- The expected value of the delay time of successfully transmitting one package from one node to another is finite.
- The average velocity in which the package travels around the net is not zero.

1.2 The model

We will assume that:

- Φ is a homogeneous PPP with intensity λ in $\mathbb{R}^2$.
- $h_t(x,y), x, y \in \Phi, t = 0, 1, ...$ are independent $\sim exp(\mu)$.
- $\ell(x,y) = \ell(|x-y|) = |x-y|^{-\alpha} \wedge 1$, $\alpha > 2$.

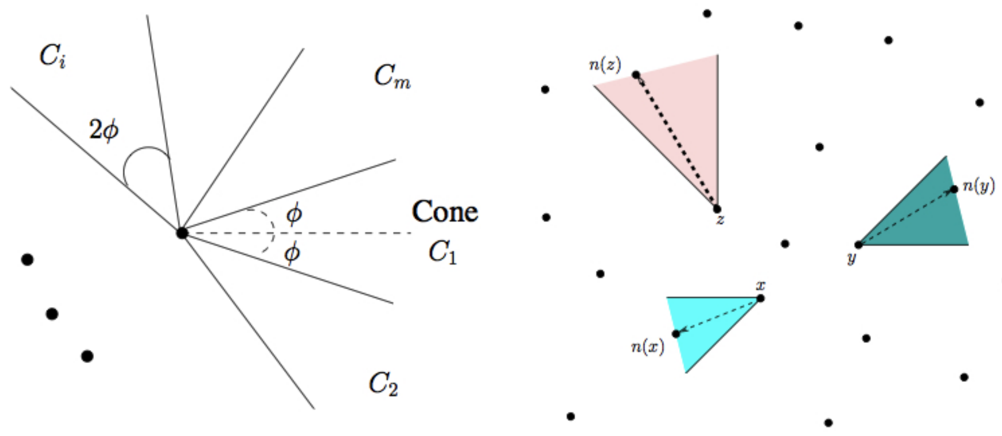

Figure 1: Definition of cones with angle 2ϕ and transmission of each node to its nearest neighbour in the destination cone.

- every node is a transmitter or a receiver following a Bernoulli random variable $\mathbf{1}_x(t)$, with transmission probability $P(1_x(t) = 1) = p_x(t)$,
- $0 < \gamma < 1$.

We will track a tagged package that traverse the network following a conic forwarding strategy for that let $C_1, ..., C_m$ be cones centred in the origin with angle $2\Phi < \frac{\pi}{2}$ s.t. $\cup_{i=1}^m C_i = \mathbb{R}^2$ and are disjoint, also lets assume that C_1 is symmetric with respect to the x axis. At time t the node x will transmit through the cone $x + C_d(x, t)$ that contains the final destination of the package to $n_t(x)$ the nearest node in that cone.
We will assume that

- If the node x is on at time t then it transmits with power $P_x(t) = c\ell(x, n_t(x))^{-1}$ where $c = M(1-\varepsilon)^{-1}$, $0 < \varepsilon < 1$.
- $M = P_x(t)p_x(t)$.

Which implies that $p_x(t) = (1-\varepsilon)\ell(x, n_t(x))$. This is what is called power control strategy. Then we have that the SINR from node x to node y at time t is given by

$$SINR_{xy}(t) := \frac{P_x(t)h_t(x,y)\ell(x,y)\mathbf{1}_x(t)(1-\mathbf{1}_y(t))}{\gamma I(t) + N}.$$

The next indicator function tell us if the transmission has been successful

$$e_{xy}(t) := \begin{cases} 1 & 1 \text{ if } SINR_{xy} > \beta \\ 0 & \text{otherwise.} \end{cases}$$

2 Main results

2.1 Finited expected exit time

Definition 1. *Let the minimum exit time taken by any packet to be successfully transmitted from node x to its nearest neighbour $n(x)$ in the destination cone of the packet be:*

$$T(x) = \min\{t > 0 : e_{x,n_t(x)}(t) = 1\}.$$

Theorem 2. *Suppose $\beta\gamma < 1$ then the SINR graph with power control policy satisfies that $E(T(x)) < \infty$ for any $x \in \Phi$.*

Proof. (Sketch) Without loss of generality, we will suppose that the package is being transmitted by the origin $o \in \Phi$, let C_d the destination cone of this package and $n(o)$ the nearest neighbour of o in C_d. We have that

$$P(T(o) > k \mid \Phi) = E\{\prod_{t=1}^{k} P(A(t) \cup B(t) \mid \mathcal{G}_k)\mathbf{1}_F|\Phi\}.$$

- $F := \cap_{j=2}^{k}\{p_o(j) = p_o(1)\}$.
- $\mathcal{G}_k$ is the σ-algebra generated by Φ and the choice of the cones made at all nodes of Φ up to time k.
- $A(t) := \{o \in \Phi_R(t)\}$.
- $B(t) := \{o \in \Phi_T(t), n(o) \in \Phi_R(t), SINR_{o,n(o)}(t) \leq \beta\}$.

From the fact that $h_t(o, n(o)) \sim exp(\mu)$ and the properties of the power control strategy on the event F for $a = \frac{\mu\beta\gamma}{c}$

$$P(A(t) \cup B(t)|\mathcal{G}_k) \leq 1 - p_o(1)\varepsilon e^{-\frac{\mu\beta N}{c}} E\{e^{-aI^*(1)}|\Phi\}.$$

Where $I^*(1) = \sum_{z\in\Phi\setminus\{o,n(o)\}} \mathbf{1}^*_z P^*_z h_1(z, n(o))\ell(z, n(o))$ and p^*_z, P^*_z are fixed values for each z. Then for $J := p_o(1)\varepsilon e^{-\frac{\mu\beta N}{c}} E\{e^{-aI^*(1)}|\Phi\}$ we get that

$$P(T(o) > k|\Phi) \leq (1 - J)^k$$

and since $0 < 1 - J < 1$

$$E(T(o)) = \sum_{k\geq 0} P(T(o) > k) = E(\sum_{k\geq 0} P(T(o) > k|\Phi)) \leq E(J^{-1}).$$

By Cauchy-Schwartz on J^{-1}

$$E(T(o)) \leq \frac{e^{\frac{\mu\beta N}{c}}}{\varepsilon}(E\{p_o(1)^{-2}\}E\{\frac{1}{(E\{e^{-aI^*(1)}|\Phi\})^2}\})^{\frac{1}{2}}.$$

On the one hand from the definition of the transmission probability $p_o(t)$,

$$E\{p_o(1)^{-2}\} \leq E\{(\frac{c}{M})^2(|n(o)|^{2\alpha} \vee 1)\} < \infty.$$

On the other hand if we let Φ_0 be a PPP independent of the other nodes and intensity $\lambda \mathbf{1}_{\{(o+C_d)\cap B(o,|n(o)|\}}$ again as a consequence of the power control strategy and the Campbell‘s theorem

$$\begin{aligned} E\{\frac{1}{(E\{e^{-aI^*(1)}|\Phi\})^2}\} &\leq E\{\prod_{z\in\Phi\setminus\{o,n(o)\}\cup\Phi_0} e^{-2\log(1-c_1\ell(|z|))}\} \\ &\leq \exp(\lambda \int_{\mathbb{R}^2} (e^{-2log(1-c_1\ell(|z|))} - 1))dz) \\ &\leq \exp(\frac{2\lambda c_1}{(1-c_1)^2}\int_{\mathbb{R}^2} \ell(|z|)dz) < \infty. \end{aligned}$$

□

2.2 Information velocity strictly positive

Now we want to measure how fast the package moves in time from the origin to its destination.

Definition 3. *Let T_0 be the time taken by this tagged package starting at $X_0 = o \in \Phi$ to successfully reach its nearest neighbour $X_1 = n(o)$ in the destination cone C_1. More generally let T_{i-1} be the time taken for the packet to successfully reach the nearest neighbour X_i of X_{i-1} in the destination cone $X_{i-1} + C_1$.*

Definition 4. *The information velocity of SINR network is defined as*

$$v = \liminf_{t\to\infty} \frac{d(t)}{t}$$

where $d(t)$ is the distance of the tagged packet from the origin at time t.

Theorem 5. *Under the conditions of Theorem 2. the information velocity $v > 0$ a.s.*

Proof. (Sketch) For all $i \geq 0$, let $R_i := |X_{i+1} - X_i|$,$\theta_i := \arcsin(\frac{X_{i+1,2}-X_{i,2}}{R_i})$ where $X_i = (X_{i,1}, X_{i,2})$. Since Φ is an homogeneous PPP with intensity λ, we have that $\{(R_i, \theta_i), i \geq 0\}$ is an i.i.d. sequence of random vectors where R_i has density and θ_i is uniformly distributed on $(-\phi, \phi)$. Our goal is to construct an stationary sequence of stopping times such that for all $i \geq 0$, $T_i' \geq T_i$.
Let $\{(R_{-i}, \theta_{-i}, i \geq 1)\}$ an i.i.d sequence of random vectors with distribution (R_0, θ_0). Define $\widetilde{\Phi} = \{X_{-i}, i \geq 1\}$ starting from X_{-1} to satisfy: $R_{-i} = |X_{-i} - X_{-i+1}|$,$\theta_{-i} = \arcsin(\frac{X_{-i+1,2}-X_{-i,2}}{R_{-i}})$.
For $i \geq 0$ let Φ_i be an PPP of intensity $\lambda \mathbf{1}_{\{(X_i+C_1)\cap B(X_i,R_i)\}}$ independent of everything else, T_i' be the delay experienced by the packet in going from X_i to X_{i+1} when the interference is coming from the nodes in $(\Phi \setminus \{X_i, X_{i+1}\}) \cup \widetilde{\Phi} \cup_{j=0}^{i-1} \Phi_j$. $(T_i', i \geq 0)$ is a stationary sequence with $T_i' \geq T_i$. We want to proof that $E(T_0') < \infty$ and then use the Birkoff´s ergodic theorem.
Let $\widetilde{I}(t) = \sum_{z\in\widetilde{\Phi}} \mathbf{1}_z P_z(t) h_t(z, n(o))\ell(z, n(o))$. Analogously to the first theorem we have that

$$E(T_0') \leq \frac{e^{\frac{\mu\beta N}{c}}}{\varepsilon}(E\{p_o(1)^{-2}\}E\{\frac{1}{(E\{e^{-a(I^*(1)+\widetilde{I}^*(1))}|\Phi \cup \widetilde{\Phi}\})^2}\})^{\frac{1}{2}}.$$

Since $I^*(1)$ and $\widetilde{I}^*(1)$ are independent by Cauchy-Schwartz

$$E\{\frac{1}{(E\{e^{-a(I^*(1)+\widetilde{I}^*(1))}|\Phi \cup \widetilde{\Phi}\})^2}\} \leq E\{\frac{1}{(E\{e^{-aI^*(1))}|\Phi\})^4}\}E\{\frac{1}{(E\{e^{-a\widetilde{I}^*(1)}|\widetilde{\Phi} \cup \{n(o)\}\})^4}\}$$

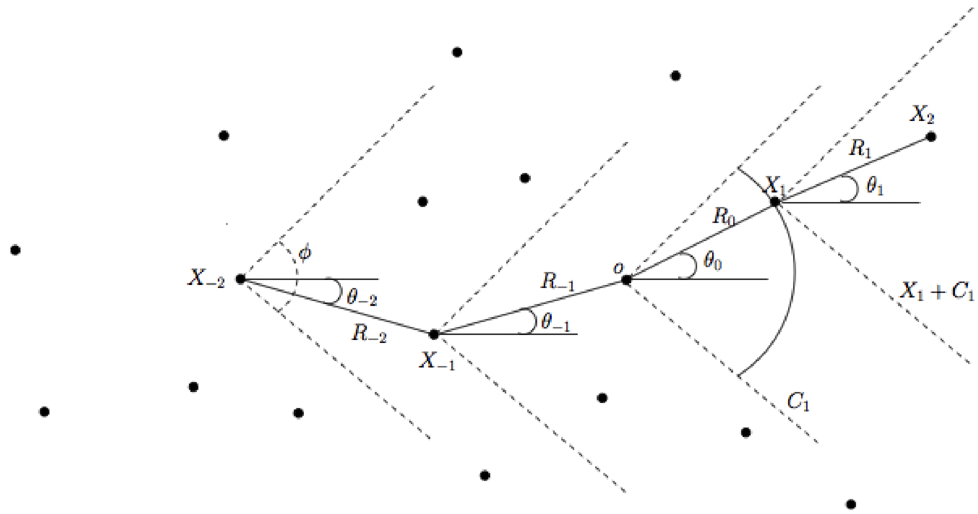

Figure 2: Addition on infinite sequence of points to make T_i' stationary.

Again by Campbell's theorem

$$E\{\frac{1}{(E\{e^{-aI^*(1)}|\Phi\})^4}\} \leq \exp(\frac{\lambda}{(1-c_1)^4}\int_{\mathbb{R}^2}(1-(1-c_1\ell(|z|))^4)dz) < \infty.$$

Since for all $i \in \mathbb{N}$, $\sum_{j=0}^{i} R_{-j}\cos(\theta_{-j}) \leq |X_{-i} - n(o)|$ and ℓ is decreasing

$$\begin{aligned} E\{\frac{1}{(E\{e^{-a\widetilde{I}^*(1)}|\widetilde{\Phi}\cup\{n(o)\}\})^4}\} &\leq E\{\prod_{i=1}^{\infty} e^{-4\log(1-c_1\ell(X_{-i},n(o)))}\} \\ &\leq E\{\prod_{i=1}^{\infty} e^{-4\log(1-c_1\ell(\sum_{j=0}^{i} R_{-j}\cos(\theta_{-j}))}\} \\ &= E\{e^{\sum_{n=1}^{\infty} g(S_{n+1})}\} \end{aligned}$$

where $S_n = \sum_{j=0}^{n-1} R_{-j}\cos(\theta_{-j})$ and $g(x) = -4\log(1-c_1\ell(x))$.
Since g is non-increasing, we get

$$E\{e^{\sum_{n=1}^{\infty} g(S_n)}\} = E\{e^{\sum_{n=1}^{N} g(S_n)+\sum_{n=N+1}^{\infty} g(S_n)}\} \leq E\{e^{\sum_{n=1}^{N} g(0)+\sum_{n=N+1}^{\infty} g(n\delta)}\} \leq e^{\sum_{n=1}^{\infty} g(n\delta)}E\{e^{g(0)N}\}.$$

Let $0 < \delta < E\{R\cos(\theta)\}$, by the Chernoff bound

$$P(\frac{S_n}{n} < \delta) \leq e^{-\zeta(\delta)n},$$

with $\zeta(\delta) = \sup_{v\leq 0}\{\nu\delta - \log(E(e^{\nu R\cos(\theta)}))\}$. Then by the Borel Cantelli lemma exists $N(\omega)$, and $c_2 > 0$ s.t.

$$P(N \geq m) = P(S_n < n\delta \text{ for some } n \geq m) \leq \sum_{n=m}^{\infty} e^{-\zeta(\delta)n} \leq c_2 e^{-\zeta(\delta)m}.$$

On the one hand by the comparison test $\sum_{n=1}^{\infty} g(n\delta) < \infty$. On the other hand since $R\cos(\theta) > 0$ then $\zeta(\delta) \uparrow \infty$ as $\delta \downarrow 0$, and we can choose δ s.t. $\zeta(\delta) > g(0)$ so it follows that $E\{e^{g(0)N}\} < \infty$. Then by Birkoffs ergodic theorem exist a r.v. $T^{'} \geq 1$ s.t.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T_k^{'} = T^{'}.$$

Finally from the fact that $\mathbf{1}_{T^{n-1} \leq t < T^n} d(t) \geq \sum_{k=1}^{n-1} R_k \cos(\theta_k)$, we conclude that

$$\liminf_{t\to\infty} \frac{d(t)}{t} \geq \lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} R_k \cos(\theta_k)}{\sum_{k=1}^{n-1} T_k^{'}} = \frac{E(R\cos(\theta))}{T^{'}} > 0,$$

as we wanted. $\square$

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