Mathematical Institute
University Leipzig
Summer term 2005

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ODE for Physicists - Homework 8

Due: June 7, 2005

21. (3 pts.) (a) Find the general solution to $y'' + 4y = x^2$ using ansatz of type of the right-hand side.

    (b) Find the general solution to $y'' + 4y = x^2$ using variation of constants.

    (c) Find the general solution to $y'' + 4y = x^2 + 5\cos 2x$ using any method.

22. (4 pts.) Use the method of power series to find the general solution to $(1 + x^2)y'' + xy' - y = 0$. Prove that the solution series converges for $|x| < 1$.

23. (5 pts.) Let $f\colon [0, \infty) \to \mathbb{R}$ be a function such that, for s in some non-trivial interval $I \subset \mathbb{R}$, the Laplace transform of f, i.e., the function $F(s) = \mathcal{L}f(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt$, is well-defined.

    (a) Fix $n \in \mathbb{N}$ and assume that also the Laplace transform of the map $t \mapsto t^n f(t)$ exists on I. Identify its Laplace transform in terms of F. *Hint:* Use induction over n. You may use *Lebesgue's theorem*: If a sequence of functions f_n converges pointwise to a function f and if there is an integrable function g such that $|f_n| \leq g$ for any $n \in \mathbb{N}$, then $\lim_{n\to\infty} \int f_n(x)\, dx = \int f(x)\, dx$.

    (b) Assume that the Laplace transform of the map $t \mapsto \frac{1}{t} f(t)$ also exists on I. Identify its Laplace transform in terms of F.

24. (4 pts.) Compute the Laplace transform (with explicit identification of its domain) of

    (a) the polynomial $t \mapsto t^n$ for $n \in \mathbb{N}$,

    (b) the map $t \mapsto t\mathrm{e}^{2t}$,

    (c) the map $t \mapsto \frac{1}{t} \sin(\omega t)$ for $\omega \in \mathbb{R} \setminus \{0\}$.
    Hint: You may use that the Laplace transform of the map $t \mapsto \sin(\omega t)$ is the map $s \mapsto \frac{\omega}{s^2+\omega^2}$.