SINR and the Poisson–Dirichlet(α, θ) process

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Paper dictionary

Our goal is to explain

KB Keeler–Blaszczyszyn 2014 —about the connection between the *SINR process* in wireless networks and the *two-parameter Poisson–Dirichlet process.*
For this, we refer to two other papers:

BKK Blaszczyszyn–Karray–Keeler 2013 —about the infinite Poisson model for wireless networks, a propagation invariance result (Lemma 1.), and introducing the SINR process.

PY Pitman–Yor 1997 —about the properties of the two-parameter Poisson–Dirichlet process.

We will use these abbreviations for the three main papers we quote —*see the exact citations in the end.*

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$\Rightarrow$ new properties of the SINR/STINR processes that had not been known before, applying results from the paper PY, e.g.:

- the ratios of consecutive STINR values have beta distributions,
- the factorial moment density of the STINR process has been computed.

Sketch of the talk

1. The SINR process
   - Infinite Poisson model. Lemma 1 of BKK
2. The two-parameter Poisson–Dirichlet process
   - Definition from size-biased permutations
   - Relation to stable (α) subordinators
   - STIR process is $\mathrm{PD}(\frac{2}{\beta}, 0)$
3. Consequences
   - Ratios of consecutive STINR values have beta distributions
   - Factorial moment measures

Infinite Poisson model of BKK for wireless networks

- Geographic locations of the base stations → homogeneous Poisson point process $\Phi = \{X_i\}_{i \in \mathbb{N}}$ with intensity λ on $\mathbb{R}^2$,
- the 'typical user' is in the origin (wlog, due to stationarity of $\{X_i\}$).

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- the 'typical user' is in the origin (wlog, due to stationarity of $\{X_i\}$).
- $l(X_i)$ is the *distance loss function* between X_i and the origin, with $l(x) = (K|x|)^\beta$ for $K, \beta > 0$.
- Adding fading/shadowing to the model, the *propagation loss* is defined as $L_{X_i} = \frac{l(X_i)}{S_{X_i}}$, where $\{S_x\}_{x\in\mathbb{R}^2}$ are i.i.d. positive random variables.
- The *power received at the origin* from the base station X_i with starting power P_{X_i} is $p_{X_i} = \frac{P_{X_i}}{L_{X_i}} = \frac{P_{X_i}S_{X_i}}{l(X_i)}$.

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  In BKK: constant power $P_{X_i} = P > 0$ (no power control) $\Rightarrow$ equivalent model: power included in the associated shadowing variables: $\tilde{S}_{X_i} = P_{X_i} S_{X_i}$, emitted power $\tilde{P}_{X_i} = 1$. (Such a model also exists if P_{X_i} are i.i.d.).

Propagation loss process. Lemma 1 of BKK.

Now let $\Theta = \{Y\} = \{Y_i\}_{i\in\mathbb{N}} := \{L_{X_i}\}_{i\in\mathbb{N}} = \{\frac{l(X_i)}{S_{X_i})}\}_{i\in\mathbb{N}}$ be the process of propagation losses experienced in the origin w.r.t. the stations of Φ, as a point process in $\mathbb{R}^+$.
The distribution of Θ determines all characteristics of the typical user that can be expressed in the terms of propagation losses. This motivates Lemma 1. of BKK.

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Lemma (Lemma 1.)

Assume infinite Poisson model with distance-loss $l(x) = (K|x|)^\beta$ *and generic shadowing variable* S *satisfying* $\mathbb{E}(S^{\frac{2}{\beta}}) < \infty$.
Then the process of propagation losses Θ *is a non-homogeneous Poisson point process on* $\mathbb{R}^+$ *with intensity measure*

$$\Lambda([0,T]) = \mathbb{E}(\Theta([0,t])) = at^{\frac{2}{\beta}},$$

where $a = \frac{\lambda\pi\mathbb{E}(S^{\frac{2}{\beta}})}{K^2}$.

Proof of Lemma 1. (Part I.)

Lemma (Lemma 1.)

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Proof.

The point process Θ is a transformation of the point process of base stations Φ by the probability transition kernel

$$p(x, A) = \mathbb{P}\left(\frac{l(x)}{S} \in A\right), \; x \in \mathbb{R}^2, A \in \mathcal{B}(\mathbb{R}^+).$$

The Displacement Theorem (see Baccelli-Blaszczyszyn 2009) implies that the point process Θ is Poisson in $\mathbb{R}^+$ with intensity measure

$$\Lambda([0,t)) = \lambda \int_{\mathbb{R}^2} \mathbb{P}\left(\frac{l(x)}{S} \in [0,t)\right) \mathrm{d}x.$$

Proof of Lemma 1. (Part II.)

Proof.

Doing the computations, we get

$$\begin{aligned}
\Lambda([0,t)) &= \lambda \int_{\mathbb{R}^2} \mathbb{P}\left(\frac{l(x)}{S} \in [0,t)\right) \mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^2 \times \mathbb{R}+} \mathbf{1}_{\{\frac{l(x)}{s} < t\}} \mathbb{P}_s(\mathrm{d}s)\mathrm{d}x \\
&\underbrace{=}_{\text{Fubini}} \lambda \int_{\mathbb{R}^2 \times \mathbb{R}+} \mathbf{1}_{\{\frac{l(x)}{s} < t\}} \mathrm{d}x\mathbb{P}_s(\mathrm{d}s) \\
&\underbrace{=}_{l(x)=(K|x|)^\beta} \lambda \int_{\mathbb{R}^2 \times \mathbb{R}+} \mathbf{1}_{\left\{|x| < \frac{(ts)^{\frac{1}{\beta}}}{K}\right\}} \mathrm{d}x\mathbb{P}_s(\mathrm{d}s) \\
&= \lambda \int_{\mathbb{R}+} \frac{\pi (st)^{\frac{2}{\beta}}}{K^2} \mathbb{P}_s(\mathrm{d}s) \\
&= \frac{\lambda \pi \mathbb{E}\left[S^{\frac{2}{\beta}}\right]}{K^2} t^{\frac{2}{\beta}}, \qquad (1)
\end{aligned}$$

which was the claim. □

Consequences of Lemma 1.

Due to Lemma 1., the distribution of the propagation loss process Θ is invariant w.r.t. the distribution of the shadowing/fading random variable S having the same given value of the moment $\mathbb{E}\left[S^{\frac{2}{\beta}}\right]$.

In particular: the infinite Poisson network with arbitrary shadowing variable S is perceived at the origin in the same manner as an equivalent infinite Poisson with *constant shadowing* $s_{const} = \mathbb{E}\left[S^{\frac{2}{\beta}}\right]^{\frac{\beta}{2}}$.

Constant shadowing is equivalent to the model with no shadowing (only distance loss, $S \equiv 1$) and the constant K replaced by $\tilde{K} = \frac{K}{\sqrt{\mathbb{E}\left[S^{\frac{2}{\beta}}\right]}}$.

Application of Lemma 1: the usual models for wireless networks (e.g. log-normal shadowing, Rayleigh fading) satisfy $\mathbb{E}\left[S^{\frac{2}{\beta}}\right] < \infty$.

Definitions of SIR, SINR, STIR, STINR processes

SINR process Ψ on $\mathbb{R}^+$: Signal to (interference plus noise) ratio.
STINR process Ψ' on $(0,1]$: Signal to (total received power plus noise) ratio.

$$\Psi = \{Z\} := \left\{ \frac{Y^{-1}}{W + (I - Y^{-1})} : Y \in \Theta \right\},$$

$$\Psi' = \{Z'\} = \left\{ \frac{Y^{-1}}{W + I} : Y \in \Theta \right\},$$

where $W \geq 0$ is the noise power (constant in our model),
$Y_i = \frac{l(X_i)}{S_{X_i}}$ are the points of propagation loss process and
$I = \sum_{Y \in \theta} Y^{-1}$ is the power received from the whole network.
Information on Ψ' $\Leftrightarrow$ information on Ψ, by $Z = \frac{Z'}{1-Z'}$ and $Z' = \frac{Z}{1+Z}$. (Working with Ψ' is algebraically simpler.)
STIR: STINR without noise, SIR: SINR without noise ($W = 0$).

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Size-biased permutations

In this section, we follow PY, the fundamental paper on the two-parameter Poisson–Dirichlet distribution, published in 1997. General setting: we consider probability distributions (V_n) with the properties:

$$(V_n) = (V_1, V_2, \ldots), \; V_1 > V_2 > \ldots > 0, \; \sum_{n=1}^{\infty} V_i = 1. \tag{2}$$

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Interpretation: a large population with a large number of species/types. As the population size tends to ∞, so does the number of species.

In the limiting "infinite idealized population", V_n is the proportion of the nth most common species.

A size-biased permutation is an enumeration of the proportions (V_n) in the order of a random sampling of the population.

We describe this more precisely as follows.

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For (V_n) as in (2), $\tilde{V}_1$ is called a *size-biased pick* from (V_n) if

$$\mathbb{P}(\tilde{V}_1 = V_n | V_1, V_2, \ldots) = V_n, \ \forall n \in \mathbb{N}. \tag{3}$$

Then, we call $(\tilde{V}_n)$ a *size-biased permutation* of (V_n) if $\tilde{V}_1$ is a size-biased pick from (V_n), and for each $n = 1, 2, \ldots, j = 1, 2, \ldots$ the following is true:

$$\mathbb{P}(\tilde{V}_{n+1} = V_j | \tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_n, V_1, V_2, \ldots) = \frac{V_j \mathbf{1}_{\{\forall 1 \leq i \leq n \ V_j \neq \tilde{V}_i\}}}{1 - \sum_{i=1}^{n} \tilde{V}_i}.$$

Invariance with respect to size-biased permutation

Let us consider the so-called stick-breaking model or residual allocation model $(\tilde{V}_n)$ for some sequence of independent real-valued random variables (U_n):

$$\tilde{V}_1 := U_1, \quad \tilde{V}_n := (1 - U_1) \cdots (1 - U_{n-1}) U_n \quad (n \geq 2). \tag{4}$$

Note that $(\tilde{V}_n)$ is a distribution.

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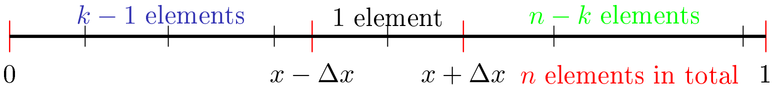
When is the distribution of $(\tilde{V}_n)$ invariant w.r.t. size-biased permutation (ISBP), i.e., when does it hold that $(\tilde{V}_n)$ equals its SBP in distribution?

Reminder on the Beta distributions

Recall that for $a, b > 0$ the Beta(a, b) distribution has density $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\mathbf{1}_{[0,1]}(x)$.
If $a = k \in \mathbb{N}$, $b = (n+1) - k$ with $k \leq n \in \mathbb{N}$, the Beta(a, b) distribution with density

$$\frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k}\mathbf{1}_{(0,1)}(x)$$



is the distribution of the kth smallest element of the order statistics of an i.i.d. sample of size n from the uniform distribution on $(0, 1)$.

I.e., let $Y_1, Y_2, \ldots, Y_n \sim U(0, 1)$ be i.i.d. Let X_k be the kth smallest one of $Y_1, \ldots, Y_n$, then $0 < X_1 < \ldots < X_n < 1$. Then $X_k \sim$Beta$(k, (n+1-k))$.

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Proposition (Pitman 1996)

For independent (U_n), in the stick-breaking model (5), $(\tilde{V}_n)$ is ISBP if and only if U_n has Beta$(1 - \alpha, \theta + n\alpha)$ distribution $\forall n$, where $0 \leq \alpha < 1$ and $\theta > -\alpha$.

In this case, the distribution of $(\tilde{V}_n)$ is called two-parameter Poisson–Dirichlet distribution with parameter (α, θ), denoted as PD(α, θ).

Definition of the PD(α, θ) process

Equivalent definition of the PD(α, θ) distribution in PY:

Definition

For $0 \leq \alpha < 1$, $\theta > -\alpha$ and (U_n) with $U_n \sim$Beta$(1-\alpha, \theta + n\alpha)$ independent, consider the $(\tilde{V}_n)$ stick-breaking model as in (5), and let $V_1 \geq V_2 \geq \dots$ be the decreasing order statistics of $(\tilde{V}_n)$. Then we call the distribution of (V_n) *Poisson–Dirichlet distribution with parameters* (α, θ), abbreviated as PD(α, θ).

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In this case, (U_n), (V_n) and $(\tilde{V}_n)$ have the same joint distribution (as a consequence of the ISBP property), and $(\tilde{V}_i)$ is a size-biased permutation of (V_i). See PY, Section 1.1.

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In this case, (U_n), (V_n) and $(\tilde{V}_n)$ have the same joint distribution (as a consequence of the ISBP property), and $(\tilde{V}_i)$ is a size-biased permutation of (V_i). See PY, Section 1.1.
We associate a point process $\xi = \sum_i \delta_{V_i}$ to (V_n) on $(0, \infty)$, this is the PD(α, θ) point process. See: Handa 2009.
E.g. $\xi([t, \infty)) = 0$ means that $V_1 < t$. Thus, (V_n) equals in distribution with the point process ξ.

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Subordinators without drift components

Subordinators $\rightarrow$ description of the relation between the PD and the STIR process.

Subordinators without drift components

A *subordinator* $(\tau_s)_{s\geq 0}$ is an almost surely increasing process with stationary independent increments and cadlag (right-continuous, left limits) paths. W.l.o.g., $\tau_0 = 0$.

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Assume now: the subordinator has no drift component (i.e., it has only jumps. The jumps are positive, they may accumulate).
The jumps of the subordinator: $(\tau_s - \tau_{s-})_{s\geq 0}$ form a Poisson point process on $(0, \infty)$, let $s\Lambda(\mathrm{d}x)$ be its intensity measure.
Denote the jumps of the subordinator on $(0, s)$ in decreasing order by $V_1(\tau_s) \geq V_2(\tau_s) \geq \dots$ Clearly, $\tau_s = \sum_{i=1}^{\infty} V_i(\tau_s)$.

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Recall: Laplace functional of a Poisson p.p. Φ with intensity $\Lambda(\mathrm{d}x)$:

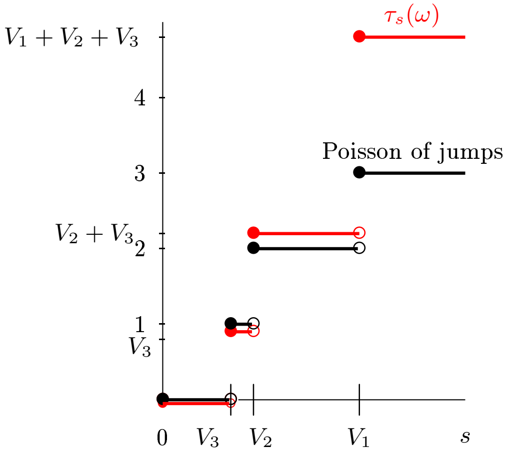
$$\mathcal{L}_\Phi(f) = \mathbb{E}\left[e^{-\int_{\mathbb{R}} f(x) \mathrm{d}\Phi(x)}\right] = \exp\left(-\int_{\mathbb{R}} (1 - e^{-f(x)}) \Lambda(\mathrm{d}x)\right)$$

Thus, the Laplace transform of the subordinator can be given as:

$$\mathbb{E}[\exp(-\lambda \tau_s)] = \exp\left(-s \int_0^\infty \exp(-\lambda x) \Lambda(\mathrm{d}x)\right).$$

(Ito 1942)

Illustration of the subordinator and the Poisson point process of its jumps



Locations of jumps → Poisson process of jumps $(\tau_s - \tau_{s-})$.
Sum of these locations up to time s equals τ_s.

Stable (α) subordinators. Relation to the PD process

Let $0 < \alpha < 1$. Then we call (τ_s) a stable (α) subordinator if $\Lambda = \Lambda_\alpha$, where Λ_α with $\Lambda_\alpha(x, \infty) = Cx^{-\alpha}$ $(x > 0)$, thus $\Lambda_\alpha(\mathrm{d}x) = Dx^{-\alpha-1}$. Then we have that

$$\mathbb{E}[\exp(-\lambda\tau_s)] = \exp(-sK\lambda^\alpha),$$

where $K = C\Gamma(1-\alpha)$.

A crucial observation (in PY, referring to several other papers) is the following.

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Proposition (PY, Proposition 6.)

If (τ_s) is a stable (α) subordinator for some $0 < \alpha < 1$, then $\forall s > 0$, $\left(\frac{V_1(\tau_s)}{\tau_s}, \frac{V_2(\tau_s)}{\tau_s}, \dots\right)$ has $\mathrm{PD}(\alpha, 0)$ distribution.

Also for every fixed $t > 0$, $\left(\frac{V_1(t)}{t}, \frac{V_2(t)}{t}, \dots\right)$ has $\mathrm{PD}(\alpha, 0)$ distribution.

This relates the STIR process to the PD process.

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Subordinator representation with $\alpha := \frac{2}{\beta}$

Now turn back to the STINR process.

By Lemma 1 of BKK, the signals from all the base stations, or, equivalently, the *inverse values* Y_i^{-1} of the propagation process

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form an inhomogeneous Poisson process with intensity measure $\frac{2a}{\beta}t^{-1-2/\beta}\mathrm{d}t$. (Here $a = \frac{\lambda\pi\mathbb{E}(S^{\frac{2}{\beta}})}{K^2}$, as in Lemma 1.)

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If $\{Y_{(i)}\}$ denote the *increasing* order statistics of $\{Y\}$, and $\{Z_{(i)}'\}$ denote the decreasing order statistics of $\{Z'\}$, then we have:

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Then the sequence of elements of the STIR process $\{Z'_{(i)}\}$ equals $\{V_i\}$ in distribution for $\alpha = \frac{2}{\beta}, \theta = 0$. I.e, the STIR process Ψ' is a PD($\frac{2}{\beta}$, 0) *point process.*

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This is now clear by $\tau_1 = I = \sum_{n=1}^{\infty} Y_{(n)}^{-1} = \sum_{n=1}^{\infty} V_n(\tau_1)$ and

$\left(Z'_{(i)}\right) = \left(Y_{(i)}^{-1} / \left(\sum_{n=1}^{\infty} Y_{(n)}^{-1}\right)\right)$, which has, by PY, PD($\frac{2}{\beta}$, 0) distr.

Illustration: the randomized access policy

The fact that $\{\tilde{V}_i\}(\stackrel{d}{=}\{Z'_{(i)}\})$ form a size-biased permutation of $\{V_i\}$, can be also interpreted regarding the STIR process $\rightarrow$ *randomized access policy*.

1. The typical user chooses a base station randomly, picking a station i with a bias proportional to Y_i^{-1} (signal strenght). Then its STIR, w.r.t. chosen station, has the distribution of $\tilde{V}_1$, i.e., Beta$(1-\frac{2}{\beta},\frac{2}{\beta})$. /See soon why this is true!/
2. Second user in the same position chooses according to the same rule, but excluding the station already chosen by the first user. Then the joint distribution of the STIR's experienced by these two users is equal to the distribution of the random pair $(\tilde{V}_1,\tilde{V}_2)$.

... and so on, the model can be extended to arbitrarily many users.

This and the corresponding evaluation of the STIR values is of potential interest for managing user hotspots.

1. The SINR process
   - Infinite Poisson model. Lemma 1 of BKK
2. The two-parameter Poisson–Dirichlet process
   - Definition from size-biased permutations
   - Relation to stable (α) subordinators
   - STIR process is PD($\frac{2}{\beta}$, 0)
3. Consequences
   - Ratios of consecutive STINR values have beta distributions
   - Factorial moment measures

Plan: prove results for the PD process (proofs obtained from PY) $\rightarrow$ apply the results directly to the S(T)I(N)R process (as in KB).

Ratios of consecutive STINR values

First goal: the ratios of consecutive STINR values $\frac{Z'_{(i+1)}}{Z'_{(i)}}$ have beta distributions. For this, we refer to PY and first prove the following:

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Suppose (V_n) *has* $\mathrm{PD}(\alpha, 0)$ *distribution for some* $0 < \alpha < 1$.

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(iii) *Let* $X_n := \Lambda_\alpha(\delta_n, \infty) = C\delta_n^{-\alpha} = LV_n^{-\alpha}$.
Then the $X_1 < X_2 < \ldots$ *are the points of a homogeneous Poi(1) point process on* $(0, \infty)$, *i.e.,* $X_i = \varepsilon_1 + \ldots + \varepsilon_i$, $\forall i \geq 1$, *where* (ε_i) *are independent* $\mathrm{Exponential}(1)$ *random variables, and* (V_n) *can be represented in the terms of* (X_n) *as* $V_n = \frac{(X_n)^{-\frac{1}{\alpha}}}{\sum_m X_m^{-\frac{1}{\alpha}}}$.

Proof of Proposition 10. Part I.

Proof.

It is enough to prove the assertions (i)–(iii) for any particular sequence (V_n) with $\mathrm{PD}(\alpha, 0)$ distribution.

Hence

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Let $V_n = \frac{V_n(\tau_1)}{\tau_1}$ for a stable (α) subordinator. First, let us prove the assertions (i)–(iii) modified, with the definitions of Σ, L, δ_n replaced by

$$L := C\tau_1^{-\alpha}, \Sigma = \tau_1, \delta_n := V_n(\tau_1). \tag{6}$$

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Then the modified form of (ii):

$\Sigma \overset{d}{=} \tau_1$, δ_n are the ranked points of a Poisson process with intensity $\Lambda_\alpha(x, \infty) = Cx^{-\alpha}$ for $x > 0$, and $V_n = \frac{\delta_n}{\Sigma}$, where $\Sigma = \sum_k \delta_k$,

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Indeed: $\mathbb{E}[\sharp\{Y_i^{-\alpha} \le s\}] = \mathbb{E}[\sharp\{Y_i \ge s^{-1/\alpha}\}] = \int_{s^{-1/\alpha}}^{\infty} C\alpha t^{-\alpha-1}\mathrm{d}t = s.$

Proof of Proposition 10. Part II.

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Then, since $X_n = \varepsilon_1 + \ldots + \varepsilon_n$ with $\varepsilon_i \sim Exp(1)$ i.i.d., $\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} 1$ a.s. by the strong law of large numbers, we see that a.s. convergence in the modified version of (i) holds:

$$\boxed{nV_n^\alpha \underset{n\to\infty}{\longrightarrow} L,\ \mathbb{P}\text{-a.s.},}$$

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Now consider again the random variables defined in (6):

$$L := C\tau_1^{-a}, \Sigma = \tau_1, \delta_n := V_n(\tau_1).$$

They can be recovered from L via the original form of the definitions of (ii): $\Sigma = \left(\frac{L}{C}\right)^{-\frac{1}{\alpha}}$, $\delta_n = V_n\Sigma$. This implies that (i)–(iii) hold for any sequence (V_n) with $\mathrm{PD}(\alpha, 0)$ distribution

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Ratios of consecutive values of the PD$(\alpha, 0)$ distribution

Using Prop. 10, it is easy to prove that the ratios of consecutive values of the PD$(\alpha, 0)$ distribution have beta distributions.

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Suppose that (V_n) *has* $\mathrm{PD}(\alpha, 0)$ *distribution for some* $0 < \alpha < 1$.
Let $R_n := \frac{V_{n+1}}{V_n}$.
Then the R_n *are mutually independent, and* R_n *has* $Beta(n\alpha, 1)$ *distribution, i.e.* $\mathbb{P}(R_n \leq r) = r^{n\alpha} \quad (0 \leq r \leq 1)$.

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With the notation of Proposition 10, we have:
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Change of variables $\Rightarrow$ the proposition holds because the ratios of consecutive values X_n/X_{n+1} of a homogeneous Poi(1) process on $[0, \infty)$ are independent and Beta$(n, 1)$ distributed. $\square$

A useful corollary

Since (V_n) can be recovered from (R_n):

$$V_1 = \frac{1}{1 + R_1 + R_1 R_2 + R_1 R_2 R_3 + \dots}, V_{n+1} = V_1 R_1 \dots R_n \text{ for } n \geq 1,$$

the next corollary is immediate from proposition 8 of PY:

Corollary

Suppose (R_n) is a sequence of independent random variables with $R_n \sim$ Beta $(n\alpha, 1)$ for all $n \geq 1$ and for some $0 < \alpha < 1$. Then (V_n) defined as above has $\mathrm{PD}(\alpha, 0)$ distribution.

Proof of Proposition 10. Part III. Convergence in L^p.

Now we prove the L^p-convergence in Proposition 10. (i). This proof takes part in the talk optionally.

For $X_1 < X_2 < \ldots$, the points of a homogeneous Poisson process with parameter 1 on $[0, \infty)$, we have:

$$\mathbb{E}\left[f(X_{m+1}, X_{m+2}, \ldots)\right] = \frac{1}{m!}\mathbb{E}\left[X_1^m f(X_1, X_2, \ldots)\right]$$

for any $m \in \mathbb{N}^+$ and f non-negative and measurable function.

First: this claim holds for any m with $f = f(X_{m+1})$ because X_{m+1} has density $\frac{x^m}{m!}e^{-x}$, while X_1 has density e^{-x} (on $[0, \infty)$).

Then, using the independence and stationarity of the increments of the Poisson(1) process, denoting the cumulative distribution function of the random variable Z by F_Z, we have:

$$F_{X_{m+1}, X_{m+2}}(x, y) = F_{X_{m+1}, X_{m+2}-X_{m+1}}(x, y-x)$$

$$= F_{X_{m+1}}(x) F_{X_{m+2}-X_{m+1}}(y-x) = F_{X_1}(x)\frac{x^m}{m!}F_{X_2-X_1}(y-x) = \frac{x^m}{m!}F_{X_1,X_2}(x,y).$$

Generalize this proof $\Rightarrow$ the claim holds, also for f with infinitely many arguments.

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$$R_n = \frac{V_{n+1}}{V_n} = \frac{\delta_{n+1}}{\delta_n} = \left(\frac{X_n}{X_{n+1}}\right)^{\frac{1}{\alpha}}$$

(as in Proposition 8 of PY), we have by a change of variables:

$$\mathbb{E}\left[f(R_{m+1}, R_{m+2}, \ldots)\right] = \frac{1}{m!}\mathbb{E}\left[X_1^m f(R_1, R_2, \ldots)\right],$$

where $X_1 = \lim_{n\to\infty} n(R_1 \ldots R_n)^\alpha$ a.s.

This is a paraphrase of the a.s. convergence from Proposition 10. (i), which we already know.

We show that this convergence also takes place in L^p for all $p \geq 1 \Rightarrow$ same for the convergence in Proposition 10. (i).

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$X_1 = \lim_{n\to\infty} n(R_1 \ldots R_n)^\alpha$ a.s.
By a direct calculation of the density ratio using proposition 8 (R_n are mutually independent Beta$(n\alpha, 1)$ random variables), we get:

$$\mathbb{E}\left[f(R_{m+1}, \ldots, R_{m+n})\right] = \binom{n+m}{m}\mathbb{E}\left[(R_1 R_2 \cdots R_n)^{m\alpha} f(R_1, \ldots, R_n)\right].$$

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Thus, $\mathbb{E}\left[X_1^m | R_1, \ldots, R_n\right] = \frac{(n+m)!}{n!}(R_1 \ldots R_n)^{m\alpha}$.

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Thus, $\mathbb{E}\left[X_1^m | R_1, \ldots, R_n\right] = \frac{(n+m)!}{n!}(R_1 \ldots R_n)^{m\alpha}$.
As a corollary of Prop. 8., the random vector $(R_1, \ldots, R_n)$ is independent of the random sequence $(X_{n+1}, X_{n+2}, \ldots)$, $\forall n \geq 1$.
Then, $X_1/(R_1 \ldots R_n)^\alpha = X_{n+1}$.
$X_{n+1} \sim$Gamma$(n+1)$ and it is independent of $\sigma(R_1, \ldots, R_n)$.
(This holds by Proposition 10. (iii) X_{n+1} is the $(n+1)$th smallest point of a homogeneous Poisson(1) process.)
Thus, with $m = 1$: $\mathbb{E}\left[X_1 | \sigma(R_1, \ldots, R_n)\right] = (n+1)X_{n+1}$.

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$X_1 = \lim_{n\to\infty} n(R_1 \ldots R_n)^\alpha$ a.s.

$$\mathbb{E}\left[f(R_{m+1}, \ldots, R_{m+n})\right] = \binom{n+m}{m}\mathbb{E}(R_1 R_2 \cdots R_n)^{m\alpha} f(R_1, \ldots, R_n).$$

Thus, $\mathbb{E}\left[X_1^m | R_1, \ldots, R_n\right] = \frac{(n+m)!}{n!}(R_1 \ldots R_n)^{m\alpha}$.
As a corollary of Prop. 8., the random vector $(R_1, \ldots, R_n)$ is independent of the random sequence $(X_{n+1}, X_{n+2}, \ldots)$, $\forall n \geq 1$.
Then, $X_1/(R_1 \ldots R_n)^\alpha = X_{n+1}$.
$X_{n+1} \sim$Gamma$(n+1)$ and it is independent of $\sigma(R_1, \ldots, R_n)$.
(This holds by Proposition 10. (iii) X_{n+1} is the $(n+1)$th smallest point of a homogeneous Poisson(1) process.)
Thus, with $m = 1$: $\mathbb{E}\left[X_1 | \sigma(R_1, \ldots, R_n)\right] = (n+1)X_{n+1}$.
Hence $(nX_n)_{n=1,2,\ldots}$ is an L^p-bounded martingale for all $p \geq 1$.

Proof of Proposition 10. Part III. Convergence in L^p.

$\mathbb{E}\left[f(R_{m+1}, R_{m+2}, \ldots)\right] = \frac{1}{m!}\mathbb{E}\left[X_1^m f(R_1, R_2, \ldots)\right]$, where

$X_1 = \lim_{n\to\infty} n(R_1 \ldots R_n)^\alpha$ a.s.

$$\mathbb{E}\left[f(R_{m+1}, \ldots, R_{m+n})\right] = \binom{n+m}{m}\mathbb{E}(R_1 R_2 \cdots R_n)^{m\alpha} f(R_1, \ldots, R_n).$$

Thus, $\mathbb{E}\left[X_1^m | R_1, \ldots, R_n\right] = \frac{(n+m)!}{n!}(R_1 \ldots R_n)^{m\alpha}$.

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Then, $X_1/(R_1 \ldots R_n)^\alpha = X_{n+1}$.

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Thus, with $m = 1$: $\mathbb{E}\left[X_1 | \sigma(R_1, \ldots, R_n)\right] = (n+1)X_{n+1}$.

Hence $(nX_n)_{n=1,2,\ldots}$ is an L^p-bounded martingale for all $p \geq 1$.

By Doob's martingale convergence theorem,

$\lim_{n\to\infty} n(R_1 \ldots R_n)^\alpha = X_1$ in L^p, $\forall p \geq 1$.

Proof of the L^p convergence in Prop.10.(i) is analogous.

Ratios of consecutive elements from the STINR process 1.

Now, we apply the previous propositions of PY for the PD$(\alpha, 0)$ process – with $\alpha = \frac{2}{\beta}$ – to the STIR process.
Recall: $Y_{(i)}$ are the increasing order statistics of the propagation process $\{Y\}$, $Z'_{(i)}$ are the decreasing order statistics of the STINR process $\{Z'\}$. For any $n \geq 1$, $\frac{Z'_{(i+1)}}{Z'_{(i)}} = \frac{Y_{(i)}}{Y_{(i+1)}}$ holds.

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Proposition (KB, Proposition 5.)

For the STINR process Ψ' $(W \geq 0)$, the random variables $R_i := \frac{Z'_{(i+1)}}{Z'_{(i)}} = \frac{Y_{(i)}}{Y_{(i+1)}}$ have, respectively, Beta$(\frac{2i}{\beta}, 1)$ distributions, furthermore R_i are mutually independent.

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Proof.

Proposition 8. of PY implies the claim in the case $W = 0$ (the STIR process is PD$(\frac{2}{\beta}, 0)$). Since R_i is a ratio of $Y_{(i)}$ values, this result is invariant under the noise term W. $\square$

Ratios of consecutive elements from the STINR process 2.

Proposition (KB, Proposition 9.)

For the STINR process Ψ' ($W \geq 0$), we have
$\frac{W}{I} = \left(\sum_{i=1}^{\infty} Z'_{(i)}\right)^{-1} - 1$ *and* $W + I = \left(\frac{L}{\alpha}\right)^{\frac{-\beta}{2}}$*, where the limit*
$L := \lim_{i\to\infty} i(Z'_{(i)})^{\frac{2}{\beta}}$ *both exists almost surely and in* L^p, $\forall p \geq 1$.

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Here the first claim is trivial. Proposition 10 (part (i)) of PY yields the latter claim for STIR process, and noise invariance in the previous proposition implies that the claim is also true for the STINR process.

Application: received power and noise values can be recovered from SINR measurements!

Factorial moment measures

Factorial moment measures

For any point process $\xi = \{U_i\}$ taking values in $\mathbb{R}$, the nth factorial measure *(a.k.a. nth correlation measure)* of ξ, is a σ-finite measure μ_n such that for any non-negative measurable functions f on $\mathbb{R}$, if this measure exists (see Handa 2009):

$$\mathbb{E}\left[\sum_{\substack{i_1,\ldots,i_n \\ \text{distinct}}} f(X_{i_1},\ldots,X_{i_n})\right] = \int_{\mathbb{R}^n} f(x_1,\ldots,x_n)\mu_n(\mathrm{d}x_1\ldots\mathrm{d}x_n).$$

For the STINR process $\{Z'\}$, $n \geq 1$, we define equivalently the *nth factorial moment measure*
$M'^{(n)}(t'_1,\ldots,t'_n) = M'^{(n)}((t'_1,1]\times\ldots\times(t'_n,1])$ as follows:

$$M'^{(n)}(t'_1,\ldots,t'_n) = \mathbb{E}\left(\sum_{\substack{(Z'_1,\ldots,Z'_n)\in(\Psi')^{\times n} \\ \text{distinct}}} \prod_{j=1}^{n} \mathbf{1}_{\{Z'_j > t'_j\}}\right).$$

The equivalent measure for the SINR process $\Psi = \{Z\}$ is defined analogously, but on $(t_1,\infty]\times\ldots\times(t_n,\infty]$.

The factorial moment measure of the STINR process I.

The factorial measure moments of both the SINR and the STINR processes require two integrals. For $n \geq 1, x, x_i \geq 0$ we define the following two integrals:

$$\mathcal{I}_{n,\beta}(x) = \frac{2^n \int\limits_0^\infty u^{2n-1} e^{-u^2 - u^\beta x \Gamma(1-\frac{2}{\beta})^{-\beta/2}} \mathrm{d}u}{\beta^{n-1}(C'(\beta))^n (n-1)!}, \text{ with } C'(\beta) = \Gamma(1+\frac{2}{\beta})\Gamma(1-\frac{2}{\beta}),$$

$$\mathfrak{I}_{n,\beta}(x_1,\dots,x_n) = \frac{1+\sum\limits_{j=1}^n x_j}{n} \int\limits_{[0,1]^{n-1}} \frac{\prod\limits_{i=1}^{n-1} v_i^{i(\frac{2}{\beta}+1)-1}(1-v_i)^{\frac{2}{\beta}}}{\prod\limits_{i=1}^n (x_i+\eta_i)} \mathrm{d}v_1 \dots \mathrm{d}v_{n-1},$$

where $\eta_1 = v_1 \dots v_{n-1}, \; \eta_2 = (1-v_1)v_2 \dots v_{n-1},$
$\eta_3 = (1-v_2)v_3 \dots v_{n-1}, \; \dots, \; \eta_n = 1 - v_{n-1}.$

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$\eta_3 = (1-v_2)v_3 \ldots v_{n-1}$, $\ldots$, $\eta_n = 1 - v_{n-1}$.
Closed forms of the integrals $\mathcal{I}_{n,\beta}(x)$ and $\mathfrak{I}_{n,\beta}(x_1,\ldots,x_n)$ are in general not known, but for $n \leq 20$ the integrals are numerically tractable.

The factorial moment measure of the STINR process II.

Proposition

For $t'_i \in [0,1]$*, the factorial moment measure of order* $n \geq 1$ *of the STINR process* Ψ' *can be written as*

$$M'^{(n)}(t'_1,\dots,t'_n) = \left(\prod_{i=1}^{n} \hat{t}_i^{-\frac{2}{\beta}}\right) \mathcal{I}_{n,\beta}(Wa^{-\frac{\beta}{2}}) \mathfrak{J}_{n,\beta}(\hat{t}_1,\dots,\hat{t}_n) \mathbf{1}_{\Delta_n}(t'_1,\dots,t'_n).$$

Here $\Delta_n = \{(x_1,\dots,x_n) \in \mathbb{R}^n |\ x_1,\dots,x_n \geq 0, \sum_{i=1}^n x_i = 1\}$ *is the closed unit simplex in* $\mathbb{R}^n$ *and* $\hat{t}_i = \hat{t}_i(t'_1,\dots,t'_n) = \frac{t'_i}{\sum\limits_{j=1}^{n} t'_j}$.

Moreover, for $t_i \in (0,\infty)$ *the SINR process* Ψ *has the factorial moment measure* $M^{(n)}(t_1,\dots,t_n) = M'^{(n)}(t'_1,\dots,t'_n)$*, where* $t'_i = \frac{t_i}{1+t_i}$, $t_i = \frac{t'_i}{1-t'_i}$.

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Proof: using coordinate changes and the memoryless property of the exponential distribution, see Blaszczyszyn–Keeler, 2014.

Observe factorization of the noise contribution!

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The factorial moment measure of the PD process I.

Reverted setting of the PD distribution (see PY):
For $0 \leq \alpha < 1$ and $\theta > -\alpha$, if $\tilde{V}_1$ is a size-biased pick (largest element of the size-biased permutation) from (V_n), where (V_n) has $\mathrm{PD}(\alpha, \theta)$ distribution, then $\tilde{V}_1 \sim \mathrm{Beta}(1-\alpha, \theta+\alpha)$.
As a consequence of the ISBP property, it becomes easy to calculate the factorial moment measures of the PD process!
For the first moment measure $\mathrm{M}_{\mathrm{PD}}(\mathrm{d}t)$ we have:

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For the first moment measure $\mathrm{M}_{\mathrm{PD}}(\mathrm{d}t)$ we have:

$$\int_0^1 f(V_i)\mathrm{M}_{\mathrm{PD}}(\mathrm{d}t) := \mathbb{E}\left[\sum_{i=1}^{\infty} f(V_i)\right] = \mathbb{E}\left[\sum_{i=1}^{\infty} V_i \frac{f(V_i)}{V_i}\right]$$

$$\underset{\text{ISBP}}{=} \mathbb{E}\left[\sum_{i=1}^{\infty} \frac{f(V_i)}{V_i}\mathbb{P}\left(\tilde{V}_1 = V_i\right)\right] = \mathbb{E}\left[\frac{f(\tilde{V}_1)}{\tilde{V}_1}\right]$$

$$= \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)\Gamma(1-\alpha)} \int_0^1 f(u)\frac{(1-u)^{\alpha+\theta-1}}{u^{\alpha+1}}\mathrm{d}u. \quad (7)$$

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$$= \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)\Gamma(1-\alpha)} \int_0^1 f(u)\frac{(1-u)^{\alpha+\theta-1}}{u^{\alpha+1}}\mathrm{d}u. \quad (7)$$

I.e., $\mathrm{M}_{\mathrm{PD}}(\mathrm{d}t) = \frac{1}{t}F_{\tilde{V}_i}(\mathrm{d}t)$, where $\tilde{V}_i \sim \mathrm{Beta}(1-\alpha, \theta+\alpha)$.

The factorial moment measure of the PD process II.

By induction, we can compute the density of the factorial moment measures of the PD(α, θ) processes. If $\alpha = \frac{2}{\beta}$, $\theta = 0$:

Proposition

The nth factorial moment density (a.k.a. the nth correlation function) of the PD$(\frac{2}{\beta}, 0)$ *process* Ψ' *($W \geq 0$) is given by*

$$\mu^{(n)}(t'_1, \dots, t'_n) := (-1)^n \frac{\partial^n M'^{(n)}(t'_1, \dots, t'_n)}{\partial t'_1 \dots \partial t'_n}$$

$$= c_{n, \frac{2}{\beta}, 0} \left(\prod_{i=1}^{n} t_i'^{-(\frac{2}{\beta}+1)} \right) \left(1 - \sum_{j=1}^{n} t'_j \right)^{\frac{2n}{\beta} - 1}$$

for $(t'_1, \dots, t'_n) \in \Delta_n$ *and zero elsewhere.*

Using the beta distributions of the independent $\{U_n\}$ for the stick-breaking model, see Handa 2009.

Factorial moment density of the STINR process

Using that the STIR process is $\mathrm{PD}(\frac{2}{\beta}, 0)$ and the noise factorization in the factorial moment measures of the STINR process, we get:

Proposition (KB, Proposition 10.)

The nth factorial moment density of the STINR process Ψ' $(W \geq 0)$ is given by

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$$= c_{n,\frac{2}{\beta},0} \frac{\mathcal{I}_{n,\beta}(W a^{-\beta/2})}{\mathcal{I}_{n,\beta}(0)} \left(\prod_{i=1}^{n} t_i'^{-(\frac{2}{\beta}+1)} \right) \left(1 - \sum_{j=1}^{n} t'_j \right)^{\frac{2n}{\beta}-1}$$

for $(t'_1, \ldots, t'_n) \in \Delta_n$ and zero elsewhere.

Using the factorial moment measures of the STINR

The expectation of a general function of the STINR process can be computed as follows (see e.g. Blaszczyszyn–Keeler 2014):

$$\mathbb{E}[\phi(\Psi')] = \phi(\emptyset) + \sum_{n=1}^{\infty} \int_{(0,1)^n} \phi_{t'_1,\dots,t'_n} \mu^{(n)}(t'_1,\dots,t'_n)\mathrm{d}t'_1 \dots \mathrm{d}t'_n,$$

where $\phi_{t'_1,\dots,t'_n}$ are given by the inclusion-exclusion principle, i.e.

$$\phi_{t'_1} = \phi(\{t'_1\}) - \phi(\emptyset),$$

$$\phi_{t'_1,t'_2} = \frac{1}{2}\phi(\{t'_1,t'_2\}) - \phi(\{t'_1\}) - \phi(\{t'_2\}) + \phi(\emptyset), \dots$$

$$\phi_{t'_1,\dots,t'_n} = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k} \sum_{\substack{t'_{i_1},t'_{i_2},\dots t'_{i_k} \\ \text{distinct}}} \phi(\{t_{i_1},\dots,t_{i_k}\}). \quad (8)$$

Remark: another result of Handa 2009 $\Rightarrow$ KB finds the joint density of $(Z_{(1)},\dots,Z_{(m)})$. Here not only $\mathrm{PD}(\alpha,0)$, but $\mathrm{PD}(\alpha,\theta)$ needed for the computations!

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$\Rightarrow$ new properties of the S(T)I(N)R processes that had not been known before, applying results from the paper PY. Among these we have seen:

- the ratios of consecutive STINR values have beta distr.

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In this content, we defined the SI(N)R, STI(N)R processes.

The paper KB establishes the relation between the SINR process of wireless networks and the two-parameter Poisson–Dirichlet process.

$\Rightarrow$ new properties of the S(T)I(N)R processes that had not been known before, applying results from the paper PY. Among these we have seen:

- the ratios of consecutive STINR values have beta distr.
- the factorial moment density of the STINR process.

Citations

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