Large Deviations of the Interference in a Wireless Communication Model

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Overview

Model Description

- $\{(X_k, P_k)\}_{k\geq 1}$ marked PP on the plane
- locations of the antennas: $\{X_k\}_{k\geq 1}$ homogenous PPP of intensity λ
- transmission power: $P_k \in (0,\infty)$ i. i. d., ind. of $\{X_k\}_{k\geq 1}$
- receiver location: w. l. o. g. in the origin
- noise power at the receiver: $w > 0$
- threshold: $\beta > 0$
- attenuation function $L\colon \mathbb{R}^2 \to (0,\infty)$ measurable positive
- random fading between node n and the origin: H_n i. i. d., ind. of $\{X_k\}_{k\geq 1}$

- Define $Y_n := P_n H_n$.
- So $\{Y_k\}_{k\geq 1}$ is i. i. d. and ind. of $\{X_k\}_{k\geq 1}$

Decoding of the signal of station n possible, if

$$\frac{Y_n L(X_n)}{w + \sum_{k\neq n} Y_k L(X_k)} \geq \beta.$$

Proportion ϕ considered as noise:

$$\frac{Y_n L(X_n)}{w + \phi \sum_{k\neq n} Y_k L(X_k)} \geq \beta.$$

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Decoding of the signal of station n impossible, if

$$\sum_{k \neq n} Y_k L(X_k) > \frac{1}{\phi} \left(\frac{Y_n L(X_n)}{\beta} - w \right).$$

Thus, the probability of the decoding failure is given by

$$\mathbb{P}\left(\sum_{k \neq n} Y_k L(X_k) > \frac{c}{\phi} \right), \quad \text{where } c = \frac{Y_n L(X_n)}{\beta} - w.$$

- We choose $L(x) = \ell(\|x\|) = \max(R, \|x\|)^{-\alpha}$ with $R > 0, \alpha > 2$.
- node located at $x \in \mathbb{R}^2$ with transmission power Y ind. of $\{(X_k, P_k)\}_{k \geq 1}$.
- Let P_x be the Palm probability of the poisson process at $x \in \mathbb{R}^2$.

We define

$$V := \sum_{k \geq 1} Y_k \ell(\|X_k\|),$$

where $\ell(x) = \max(R, x)^{-\alpha}$ with $R > 0, \alpha > 2$.

Due to *Slivnyak's theorem* and the independence between Y and $\{(X_k, P_k)\}_{k\geq 1}$ we have

$$P_x\left(\frac{Y\ell(\|x\|}{w+V-Y\ell(\|X_k\|}<\beta\middle| Y=y\right)$$
$$=\mathbb{P}\left(\frac{Y\ell(\|x\|}{w+V}<\beta\middle| Y=y\right)$$
$$=\mathbb{P}\left(V>\frac{y\ell(\|x\|}{\beta}-w\right).$$

Main aim: Provide large deviation principles for the total interference V at the origin and since $w>0$, this will yield large deviations for the SINR.

Notations

- Remark: $\alpha > 2 \implies \mathbb{E}[V] < \infty$ if $\mathbb{E}[Y_1] < \infty$.

Define $R_0 := 0,\ R_k := \sqrt{k}R,\ k \geq 1.$

$$V^k := \sum_{i=1}^{\infty} Y_i \ell(\|X_i\|) \mathbb{1}_{\{R_k \leq \|X_i\| < R_{k+1}\}}.$$

Then

$$V = \sum_{k \geq 0} V^k.$$

Different models for the law of Y_1

The distribution of the transmission power has

1) bounded support.
2) superexponential tails.
3) exponential tails.
4) sub exponential tails and it belongs to the domain attraction of the Gumbel distribution.
5) regularly varying tails.

Basic definitions in large deviations theory

Definition

A family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ on $(\mathbb{R}, \mathcal{B})$ *obeys a large deviation principle (LDP)* with rate function $I(\cdot)$ and speed $v(\cdot)$, if $I\colon \mathbb{R} \to [0, \infty]$ is a lower semicontinuous function, $v\colon (0, \infty) \to (0, \infty)$ is a measurable function which diverges to infinity at the origin, and the following inequalities hold for every Borel set $B \in \mathcal{B}$:

$$-\inf_{x \in B^\circ} I(x) \le \liminf_{\varepsilon \to 0} \frac{1}{v(\varepsilon)} \log \mu_\varepsilon(B) \le \limsup_{\varepsilon \to 0} \frac{1}{v(\varepsilon)} \log \mu_\varepsilon(B) \le -\inf_{x \in \overline{B}} I(x)$$

Definition

A family of $\mathbb{R}$-valued random variables $\{V_\varepsilon\}_{\varepsilon>0}$ *obeys an LDP*, if $\{\mu_\varepsilon\}_{\varepsilon>0}$ obeys an LDP and $\mu_\varepsilon(\cdot) = \mathbb{P}(V_\varepsilon \in \cdot)$.

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Remark

The semicontinuity of $I(\cdot)$ means that its level sets

$$\{x \in \mathbb{R} | I(x) \leq a\}, \ a \geq 0$$

are closed.

Definition

When the level sets are compact, the rate function is said to be *good*.

Theorem (LDP I)

Suppose that Y_1 has bounded support with supremum $b > 0$. Then, the family of random variables $\{V_\varepsilon^0\}_{\varepsilon>0}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)$ and good rate function

$$I_1(x) = \frac{R^\alpha x}{b}.$$

Lemma (LDUB)

Suppose that there exist constants $\tilde{c} > 0$ and $\beta > 0$ such that $\log(\varphi(\theta)) \sim \tilde{c}\theta^{\beta}$ as $\theta \to \infty$. Then

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon}{\log^{\frac{1}{\beta}}\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_{\varepsilon}^{0} \geq x\right) \leq -\tilde{c}^{-\frac{1}{\beta}} R^{\alpha} x, \;\; x \geq 0.$$

Lemma (LDLB)

If Y_1 has compact support whose supremum, denoted b, is strictly positive (i. e. Y_1 is not identically zero), then

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 > x\right) \geq -\frac{R^\alpha x}{b},\ x \geq 0.$$

Proof of Lemma (LDLB)

Since Y_1 has compact support with supremum $b > 0$, for arbitrarily small $\delta > 0$, there is a $p = p_\delta > 0$ such that $\mathbb{P}(Y_1 > (1-\delta)b) = p$.
Recall that the independent thinning with retention probability p of a PPP with intensity μ is a PPP with intensity $p\mu$.
Therefore, if we define

$$\tilde{N}_0 := \sum_{i=1}^{N_0} \mathbb{1}_{\{Y_i > (1-\delta)b\}},$$

then $\tilde{N}_0$ is a Poisson random variable with mean $p\lambda_0$.

Proof of Lemma (LDLB)

We now have

$$V_\varepsilon^0 \geq \varepsilon R^{-\alpha}(1-\delta)b \sum_{i=1}^{N_0} \mathbb{1}_{\{Y_i > (1-\delta)b\}} = \varepsilon R^{-\alpha}(1-\delta)b\tilde{N}_0.$$

Thus

$$\mathbb{P}\left(V_\varepsilon^0 > x\right) \geq \mathbb{P}\left(\tilde{N}_0 > \frac{R^\alpha x}{\varepsilon(1-\delta)b}\right)$$

from which we deduce that, for $x \geq 0$

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 > x\right) \geq \liminf_{\varepsilon \to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(\tilde{N}_0 > \frac{R^\alpha x}{\varepsilon(1-\delta)b}\right)$$

Proof of Lemma (LDLB)

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 > x\right) \\
&\geq \liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(\tilde{N}_0 > \frac{R^\alpha x}{\varepsilon(1-\delta)b}\right) \\
&= -\frac{R^\alpha x}{b}. \qquad (1)
\end{aligned}$$

Letting δ decrease to zero, we obtain, the claim of the lemma. □

Theorem (LDP I)

Suppose that Y_1 has bounded support with supremum $b > 0$ (strictly positive). Then, the family of random variables $\{V_\varepsilon^0\}_{\varepsilon>0}$ obeys an LDP on $[0, \infty)$ with speed $\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)$ and good rate function

$$I_1(x) = \frac{R^\alpha x}{b}.$$

Proof of Theorem (LDP I)

$I_1(x) = \frac{R^\alpha x}{b}$ is continuous on $[0, \infty)$ and has compact level sets $\implies$ good rate function.
Y_1 has compact support with supremum b, $\varphi(\cdot)$ be the moment generating function of Y_1: It can be easily shown that $\log \varphi(\theta) \sim b\theta$ as $\theta \to \infty$.
Hence by *Lemma (LDUB)*:

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 \geq x\right) \leq -\frac{R^\alpha x}{b} = -I_1(x), \;\; x \geq 0. \quad (2)$$

This upper bound matches the corresponding lower bound in *Lemma (LDLB)*.

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Proof of Theorem (LDP I)

Next step: Extend upper and lower bounds from half intervals $[x,\infty)$ and (x,∞) to arbitrary closed and open sets.
Let F be a closed subset of $[0,\infty)$ and let x denote the infimum of F. Since $I_1(\cdot)$ is increasing, $I_1(x) = \inf_{y\in F} I_1(y)$.
Now $F \subseteq [x,\infty)$, and so we obtain by (2) that

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 \in F\right) \\
&\leq \limsup_{\varepsilon\to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 \geq x\right) \\
&\leq -I_1(x) = -\inf_{y\in F} I_1(y).
\end{aligned}
$$

Proof of Theorem (LDP I)

Now, let G be an open subset of $[0, \infty)$.
Suppose first that $0 \notin G$ and set $\nu := \inf_{y \in G} I_1(y)$. Then, $\nu < \infty$ and, for arbitrary $\delta > 0$, we can find $x \in G$ such that $I_1(x) \leq \nu + \delta$. Since G is open, we can also find $\eta > 0$ such that $(x - \eta, x + \eta) \subseteq G$. Now

$$\begin{aligned} \mathbb{P}\left(V_\varepsilon^0 \in G\right) &\geq \mathbb{P}\left(V_\varepsilon^0 \in (x - \eta, x + \eta)\right) \\ &= \mathbb{P}\left(V_\varepsilon^0 > x - \eta\right) - \mathbb{P}\left(V_\varepsilon^0 \geq x + \eta\right). \end{aligned} \tag{3}$$

Proof of Theorem (LDP I)

Moreover

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 > x - \eta\right) \geq -I_1(x-\eta)$$

by *Lemma (LDUB)*, whereas

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 \geq x + \eta\right) \leq -I_1(x+\eta)$$

by (2).

Proof of Theorem (LDP I)

Lemma (variant of the principle of the largest term)

Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive numbers such that $a_n > b_n$ for all $n \geq 1$. Assume that

$$\liminf_{n\to\infty} c_n \log a_n \geq -a \quad \text{and} \quad \limsup_{n\to\infty} c_n \log b_n \leq -b$$

where $\{c_n\}_{n\geq 1}$ is a sequence of positive numbers converging to 0, and $0 < a < b$. Then

$$\liminf_{n\to\infty} c_n \log(a_n - b_n) \geq -a.$$

Proof of Theorem (LDP I)

Since $I_1(x-\eta) < I_1(x+\eta)$, we obtain using (3) and the *Lemma* that

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 \in G\right) \geq -I_1(x-\eta).$$

Since $I(\cdot)$ is continuous, by letting η decrease to zero, we get

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 \in G\right) \geq -I_1(x) \geq \inf_{y\in G} I_1(y) - \delta.$$

Proof of Theorem (LDP I)

If $0 \in G$, then, since G is open, there is an $\eta > 0$ such that $[0, \eta) \subseteq G$. Hence

$$\mathbb{P}\left(V_\varepsilon^0 \in G\right) \geq 1 - \mathbb{P}\left(V_\varepsilon^0 \geq \eta\right).$$

By similar arguments to the above, we can show, that

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)} \log \mathbb{P}\left(V_\varepsilon^0 \in G\right) \geq 0.$$

Since $\inf_{y \in G} I_1(y) = I_1(0) = 0$ as $I(\cdot)$ is increasing, this establishes the large deviation lower bound if $0 \in G$, and completes the proof of the theorem. $\square$

Theorem (LDP II)

Suppose that Y_1 has bounded support with supremum $b > 0$. Then, the family of random variables $\{V_\varepsilon\}_{\varepsilon>0}$ obeys an LDP on $[0,\infty)$ with speed $\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)$ and good rate function

$$I_1(x) = \frac{R^\alpha x}{b}.$$

Lemma (LDP III)

Suppose that the family of random variables $\{X_\varepsilon\}_{\varepsilon>0}$ obeys an LDP on $[0,\infty)$ with speed $v(\cdot)$ and rate function $I(x)=\gamma x$ for some $\gamma>0$. Let $\{Y_\varepsilon\}_{\varepsilon>0}$ be a family of nonnegative random variables independent of $\{X_\varepsilon\}_{\varepsilon>0}$, satisfying

$$\mathbb{P}\left(Y_\varepsilon \geq x\right) \leq \exp(-v(\varepsilon)\gamma' x)$$

for all $\varepsilon>0$ and $x\geq 0$, and for some $\gamma'\geq\gamma$. Define $Z_\varepsilon := X_\varepsilon + Y_\varepsilon$. Then, $\{Z_\varepsilon\}_{\varepsilon>0}$ obeys and LDP on $[0,\infty)$ with speed $v(\cdot)$ and rate function $I(\cdot)$.

Notation

Next define

$$U^j := \sum_{i=1}^{\infty} Y_i \mathbb{1}_{\{R_j \leq \|X_i\| < R_{j+1}\}}, \ j \geq 1$$

$$W^k := V^0 + \sum_{j=1}^{k} R_j^{-\alpha} U^j, \ \ k \geq 1$$

and

$$W := \lim_{k \to \infty} W^k$$

where $R_j = \sqrt{j} R$.

Remark

- limit exists since random variable $U^j > 0$.
- So $\{W_k\}_{k \geq 1}$ is increasing
- choice of $R_j \implies$ areas of successive annuli $A_{j-1} := \{x \in \mathbb{R}^2 | R_{j-1} \leq \|x\| < R_j\}$ are equal.
- $R_j^{-\alpha} U^j$ is upper bound for interference due to nodes in A_j
- U_j i. i. d., since sum of marks of a homogenous marked PP over disjoint intervals of equal area
- $W < \infty$ a. s.:

$$\mathbb{E}[W] = \lambda_0 R^{-\alpha} \left(1 + \sum_{j \geq 1} j^{-\frac{\alpha}{2}}\right) \underbrace{\mathbb{E}[Y_1]}_{<\infty} < \infty,$$

since $\alpha > 2$.

Lemma (LDP IV)

Define $W_\varepsilon := \varepsilon W$ and note that $V_\varepsilon^0 \leq V_\varepsilon \leq W_\varepsilon$.
Suppose that the assumptions of Theorem (LDP) are satisfied.
Then, the family $\{W_\varepsilon\}_{\varepsilon>0}$ obeys an LDP on $[0,\infty)$ with the same speed and rate function as stated for $\{V_\varepsilon\}_{\varepsilon>0}$ in Theorem (LDP II).

Proof of Theorem (LDP II)

Under the assumptions, we established the LDP for $\{V_\varepsilon^0\}_{\varepsilon>0}$ in *Theorem (LDP I)*.
Lemma (LDP IV) establishes then the LDP for $\{W_\varepsilon\}_{\varepsilon>0}$ with the same speed and rate function.
Since $V_\varepsilon^0 \leq V_\varepsilon \leq W_\varepsilon$ for all $\varepsilon > 0$, the large deviation upper and lower bounds on half intervals also holds for V_ε.
These bounds can be extended to a full LDP as in the proof of *Theorem (LDP I)*. $\square$

Thank you for your attention!

Theorem

Suppose that there exist constants $c > 0$ and $\gamma > 1$ such that $-\log \mathbb{P}(Y_1 > y) \sim cy^{\gamma}$. Define $\eta := 1 - \frac{1}{\gamma}$. Then, the family of random variables $\{V_\varepsilon\}_{\varepsilon>0}$ obeys an LDP on $[0, \infty)$ with speed $\frac{1}{\varepsilon} \log^{\eta}\left(\frac{1}{\varepsilon}\right)$ and good rate function

$$I_2(x) = \gamma(\gamma - 1)^{-\eta} c^{\frac{1}{\gamma}} R^{\alpha} x.$$

Theorem

Suppose that there exists a constant $c > 0$ *such that* $-\log \mathbb{P}(Y_1 > y) \sim cy$*. Then, the family of random variables* $\{V_\varepsilon\}_{\varepsilon>0}$ *obeys an LDP on* $[0, \infty)$ *with speed* $\frac{1}{\varepsilon}$ *and good rate function*

$$I_3(x) = cR^\alpha x.$$

Theorem

Suppose that Y_1 is subexponential and that there exist constants $c > 0$ and $0 < \gamma < 1$ such that $-\log \mathbb{P}(Y_1 > y) \sim cy^\gamma$. Then, the family of random variables $\{V_\varepsilon\}_{\varepsilon>0}$ obeys an LDP on $[0,\infty)$ with speed $\left(\frac{1}{\varepsilon}\right)^\gamma$ and good rate function

$$I_4(x) = cR^{\alpha\gamma}x^\gamma.$$

Theorem

Suppose that $\mathbb{P}(Y_1 > y) \sim y^{-c}S(y)$, *for some constant* $c > 1$ *and slowly varying function* $S(\cdot)$. *Then, the family of random variables* $\{V_\varepsilon\}_{\varepsilon>0}$ *obeys an LDP on* $[0,\infty)$ *with speed* $\left(\frac{1}{\varepsilon}\right)^\gamma$ *and rate function*

$$I_5(x) = \begin{cases} 0\ , & \text{if } x = 0 \\ c\ , & \text{if } x > 0. \end{cases}$$

Reference

- Ayalvadi J. Ganesh and Giovanni Luca Torrisi. Large Deviations of the Interference in a Wireless Communication Model, *IEEE Transactions on information theory, vol. 54, No. 8*, August 2008