THE PARABOLIC ANDERSON MODEL

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Abstract. This is a survey on the parabolic Anderson model, the Cauchy problem for the heat equation with random potential. This model and many variants is studied for decades by many authors, with a particularly high intensity since 1990. It has rich and deep connections with questions on random motions through random potential, trapping of random paths, branching processes in random medium, Anderson localisation, and more. It shows interesting behaviours like intermittency, concentration, ageing, Poisson process convergence of eigenvalues and eigenfunction localisation centres, and more. Furthermore, its mathematical treatment requires combinations of tools from various parts of probability and analysis, like spectral theory of random operators, large deviations, extreme value analysis.

In this survey, we introduce the model, present various aspects and the relevant questions, the mathematical tools for its general and special treatments of its various properties and give an account on the available results. We also explain connections with related models and objects like random directed polymers in random environment, functionals of local times of random motions.

We tried our best to write this text in a pedagogical way, such that it can be digested and taken as an inspiration by advanced undergratuate mathematics students. As far as it was possible, we drop all technical details and concentrate on the explanation of the effects and the description of the essence of the mathematical proofs.

Contents

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sec-Intro

1. Introduction

Random motions in random media are an important subject in probability theory since there are a lot of applications to real-world problems in the sciences, like astrophysics, magnetohydrodynamics, chemical reactions. For these reasons and also because of its mathematical interest, they have been studied a lot for decades, with a particular intensity in the last twenty years. There is a number of different models of random motions in random media, like random walk in random environment, the random conductance model, random walk in random scenery, random walk in a random potential. Because of the variety of models, there is also a variety of questions and of mathematical methods to answer them, like homogenisation, subadditive ergodic theorems, Lyapounov exponents, and more.

In this survey, we are interested in the model of a *random walk in a random potential*, often also called the *parabolic Anderson model (PAM)*. Alternately, the PAM is often introduced as the solution to the (Cauchy problem for) *heat equation with random potential*, a fundamental partial differential equation (PDE) with random coefficients. Here the walk has a strong tendency to be confined to a preferable part of the random medium, and these turn out to be very local and widely spread. Therefore, the global properties of the system are not determined by an average behaviour (like in situations where homogenisation works well), but by some local extreme behaviour. This makes the PAM an exciting field to study. Furthermore, for the study of the PAM, a significant number of mathematical tools had to be developed or to be adapted since 1990, which have later found useful also for the study of a number of other models in statistical physics. Furthermore, because of its relative simplicity and the extraordinary explicitness of representations of the solution, the PAM serves as one prime example of a PDE with highly irregular behaviour, for which a detailed rigorous analysis of its solution is possible, going far beyond questions like existence and uniqueness.

The PAM has become a popular model to study among probabilists and mathematical physicists, since there are a lot of interesting and fruitful connections to other interesting topics such as branching random walks with random branching rates, the spectrum of random Schrödinger operators, extreme value statistics, convergence of point processes and variational problems. The mathematical activity on the PAM is on a high level since about 1990, and many specific and deeper questions and variants were studied specially in the last few years, including, but not being limited to, time-dependent potentials, connections with Anderson localisation or transitions between quenched and annealed behaviour. For this reason, it seems rather appropriate to provide a survey that collects, in a pedagogical manner, most of the relevant investigations and their interrelations, and to put them into a unifying perspective.

We decided to devote most attention to the case of a *static* random potential, i.e., a potential that does not depend on time, although there are many good reasons to study also the case of a time-dependent potential, which one could call the dynamic case. For our decision, there are a couple of reasons, the most prominent of which are that (1) the static case has, in contrast to the dynamic case, many connections with the spectral properties of the Anderson Hamilton operator, and (2) the results that have been derived in the static case are much more explicit and more directly interpretable than in the dynamic case. Nevertheless, the set of time-dependent potentials that are interesting for the PAM is definitely much richer and comes from more different interesting applications and is still growing.

We also decided to put the main weight on the PAM in the *discrete* spatial setting, i.e., on $\mathbb{Z}^d$, even though the spatially continuous setting on $\mathbb{R}^d$ (i.e., with random walks replaced by Brownian motion) is equally interesting and mathematically challenging. The main reason for this mild self-restriction is the existence of the formidable monograph on the $\mathbb{R}^d$-case, [Szn98], which is developed from the viewpoint of random path measures for Brownian motion trapped in a Poisson field of obstacles and contains a lot of results that have a direct impact to the PAM. In these notes, we will by no means neglect the material of [Szn98], but rephrase it in a way that is relevant for the PAM; however, we take the freedom to refer sometimes to [Szn98] for deeper comments on proof methods or interpretations.

These notes partially rely on the older survey [GärKön05], but take the freedom to stress new aspects, give much more intuitive comments that might be helpful for the beginner, and of course collect and comment the latest developments of the work on the PAM.

In Section 1.1 we introduce the model and the relevant questions, explain the heuristics and survey the most important tools. One of the most fundamental questions, the asymptotics of the moments of the total mass of the model, is heuristically explained in Section 4: first we reveal the mechanism, then we bring and comment some detailed formulas. The almost sure asymptotics of the total mass is explained in Section 6. Again, we first clarify the mechanism and provide explicit formulas afterwards. Section 7 is devoted to the question where the total mass mainly comes from, that is, to concentration properties of the model. Here we also explain rigorous connections to Anderson localisation. Finally, in Section 8, we briefly summarize a number of directions that have been studied recently, like acceleration and deceleration, PAM in a random environment and ageing. In the final Section 10, we enter the big and largely unexplored world of time-dependent potentials and describe some of the most relevant examples, motivations, results and open questions.

sec-PAM 1.1 The parabolic Anderson model on $\mathbb{Z}^d$.

Let us introduce the model in the spatially discrete case. We refer the reader to [Mol94] and [CarMol94] for more background, to [GärMol90] for details of basic mathematical properties of the model, and to [GärKön05] for a survey on mathematical results up till 2005.

We consider the non-negative solution $u\colon [0,\infty)\times\mathbb{Z}^d \to [0,\infty)$ to the *Cauchy problem* for the *heat equation with random coefficients* and localised initial datum,

$$\frac{\partial}{\partial t}u(t,z) = \Delta^{\rm d} u(t,z) + \xi(z)u(t,z), \qquad \text{for } (t,z)\in(0,\infty)\times\mathbb{Z}^d, \tag{1.1}$$

$$u(0,z) = \delta_0(z), \qquad \text{for } z\in\mathbb{Z}^d. \tag{1.2}$$

Here $\xi=(\xi(z)\colon z\in\mathbb{Z}^d)$ is an i.i.d. random potential with values in $[-\infty,\infty)$, and $\Delta^{\rm d}$ is the discrete Laplacian,

$$\Delta^{\rm d} f(z) = \sum_{y\sim z}\big[f(y)-f(z)\big], \qquad \text{for } z\in\mathbb{Z}^d,\ f\colon\mathbb{Z}^d\to\mathbb{R}.$$

Certainly, $\Delta^{\rm d}$ applies only to the spatial dependence of u; we understand $\Delta^{\rm d} u(t,z)$ as $[\Delta^{\rm d} u(t,\cdot)](z)$. The parabolic problem (1.1) is called the *parabolic Anderson model (PAM)*. We denote expectation and probability with respect to the random potential by $\langle\cdot\rangle$ and Prob, respectively. The operator $\Delta^{\rm d}+\xi$ appearing on the right is called the *Anderson Hamiltonian*; its spectral properties are well-studied in mathematical physics, see Remark 2.3.3.

The fundamental starting point of the study of the PAM is provided by the following.

Thm-PAMexist **Theorem 1.1** (Existence and uniqueness)**.** *Almost surely, the equation* (1.1)–(1.2) *has precisely one solution $u(t,\cdot)$ if the potential satisfies the integrability condition*

$$\Big\langle\Big(\frac{\xi(0)_+}{(\log(\xi(0))_+}\Big)^d\Big\rangle < \infty, \tag{1.3}$$ existcond

where x_+ is the positive part of x.

See [GärMol90, Theorem 2.1] for a proof of this fact and the derivation of the *Feynman-Kac formula* for the solution, which we spell out only in Section 2.1.2. It is also shown there that the condition (1.3) is necessary in a certain sense. The main argument for the existence part is that the Feynman-Kac formula is shown to be finite (using a comparison of the speed of the underlying random walk and the asymptotic growth of the potential), and this implies that this formula is a solution to (1.1)–(1.2). The uniqueness part is done by showing that, for some sufficiently negative α, the set $\{z\in\mathbb{Z}^d\colon \xi(z)\le\alpha\}$ does not contain any unbounded component.

Henceforth, we assume that (1.3) is satisfied and denote by u the non-negative solution. See Remark 1.2 for other initial conditions instead of (1.2).

The PAM describes a random particle flow in $\mathbb{Z}^d$ through a random field of sinks and sources. Sites z with $\xi(z)<0$ are interpreted as sinks, traps or obstacles ("hard" for $\xi(z)=\infty$ and "soft" for $\xi(z)\in(-\infty,0)$), while sites z with $\xi(z)\in(0,\infty)$ are called sources. Two competing effects are present: the diffusion mechanism governed by the Laplacian, and the local growth governed by the potential. The diffusion tends to make the random field $u(t,\cdot)$ flat, whereas the random potential ξ has a tendency to make it irregular. This is understood best by considering the two separate equations

$\frac{\partial}{\partial t}u(t,z) = \Delta^{\rm d} u(t,z)$ and $\frac{\partial}{\partial t}u(t,z) = \xi(z)u(t,z)$ under the same initial condition. The first one is called the (Cauchy problem for the) *heat equation* and implies that the exponential growth rate of $u(t,z)$ at some point $z \in \mathbb{Z}^d$ is proportional to the sum $\sum_{y\sim z}[u(t,y) - u(t,z)]$. In particular, $u(t,z)$ grows if the average value of u in the neighbouring points is higher than $u(t,z)$ itself and decreases in the opposite case, which corresponds to heat spreading evenly over a surface. The second equation admits the simple solution $u(t,z) = {\rm e}^{t\xi(z)}$, $z \in \mathbb{Z}^d$, which does not admit any interaction between different lattice points and is extremely irregular for large t as we may have considerably different growth rates along the lattice points. In (1.1), both these effects interact, and it is highly non-trivial to separate them again in a meaningful way. There is an additional interpretation in terms of a branching process in a field of random branching rates, see Remark 2.1.1.

Remark 1.2. (Other initial conditions.) Instead of the localised initial condition $u(0,\cdot) = \delta_0(\cdot)$ in (1.2), certainly also other initial conditions $u(0,\cdot) = u_0(\cdot)$ may be considered, as long as the initial function u_0 is non-negative and satisfies

$$\limsup_{z\to\infty} \frac{(\log u_0(z))_+}{|z|\log|z|} < 1; \tag{1.4}$$

u0cond

see [GärMol90, Theorem 2.1]. Observe that the *superposition principle* holds: If $u(t,\cdot)$ and $\widetilde{u}(t,\cdot)$ are the solutions with initial condition u_0 and $\widetilde{u}_0$, respectively, then $(u+\widetilde{u})(t,\cdot)$ is the solution with initial condition $u_0 + \widetilde{u}_0$. The most-studied choice, apart from $u_0 = \delta_0$, is $u_0 \equiv 1$, in which case the random field $u(t,\cdot)$ is *stationary*, i.e., its distribution is shift-invariant for any t. Denoting the solution by $v\colon (0,\infty)\times\mathbb{Z}^d \to [0,\infty)$, the superposition principle implies that $v(t,0) = \sum_{z\in\mathbb{Z}^d} u(t,z)$ if u solves (1.1). ◇

OtherIniCond

Remark 1.3. (The PAM in boxes.) The PAM can also be considered in a given finite set $B \subset \mathbb{Z}^d$, but one has to specify the boundary conditions. The two mainly used boundary conditions are the *Dirichlet boundary conditions* (by which we mean *zero boundary conditions*) and the *periodic boundary conditions*, the latter only for the case that B is a rectangle (We will then take B always as a cube, often a centred cube). If B contains the origin, we denote by $u_B\colon (0,\infty)\times\mathbb{Z}^d \to [0,\infty)$ the localised solution with zero boundary condition, i.e., the solution to (1.1)–(1.2) on $(0,\infty)\times\mathbb{Z}^d$ such that $u(t,z) = 0$ for every $z \in \mathbb{Z}^d \setminus B$. Note that in the term $\Delta^{\rm d} u(t,\cdot)$ also bonds between B and $B^{\rm c}$ occur. If $B = (-R,R]^d \cap \mathbb{Z}^d$ with $R \in \mathbb{N}$ is a centred cube, then we denote by $u_B^{\rm (per)}\colon (0,\infty)\times B \to [0,\infty)$ the solution to (1.1)–(1.2) with periodic boundary condition. There are two ways to understand this definition. First, one conceives $u_B^{\rm (per)}$ as the solution to (1.1)–(1.2) on $(0,\infty)\times\mathbb{Z}^d$ with the extra condition of periodicity (i.e., $u(t,z+R{\rm e}_i) = u(z)$ for any $t \in (0,\infty)$, $z \in \mathbb{Z}^d$ and $i \in \{i,\dots,d\}$ (where ${\rm e}_i \in \mathbb{Z}^d$ is the i-th unit vector) and restricts this solution to $(0,\infty)\times B$, or one restricts (1.1) to $z \in B$ and replaces the Laplace operator $\Delta^{\rm d}$ by the one on $\ell^2(B)$ with periodic boundary condition. Alternatively, we consider B as the d-dimensional torus and take $\Delta^{\rm d}$ as the canonical Laplace operator on this torus.

Both u_B and $u_B^{\rm (per)}$ are important for the study of the PAM, as they will turn out to serve as lower, respectively upper, bounds for u, see Remark 2.1.3. ◇

Rem-PAMBox

Remark 1.4. (The PAM on $\mathbb{R}^d$.) The spatially continuous version of the parabolic Anderson model is given by

$$\frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + V(x)u(t,x), \qquad \text{for } (t,x) \in (0,\infty)\times\mathbb{R}^d, \tag{1.5}$$

$$u(0,x) = \delta_0(x), \qquad \text{for } x \in \mathbb{R}^d, \tag{1.6}$$

where Δ is now the usual Laplace operator, and $V\colon \mathbb{R}^d \to [-\infty,\infty)$ is a random field, which we assume to be sufficiently regular and integrable. Much of the preceding has an analogue; we are not going to spell out all this explicitly. If V is stationary (i.e., if its distribution is invariant under shift by any vector in $\mathbb{R}^d$), then also $u(t,\cdot)$ is a stationary field for any t. One possible choice is to take V constant on the unit boxes $z+(-\frac{1}{2},\frac{1}{2}]^d$ for any $z\in\mathbb{Z}^d$ with value $\xi(z)$ for some i.i.d. field $(\xi(z))_{z\in\mathbb{Z}^d}$; which mimicks the discrete case above, and this potential is not stationary with respect to all shifts. One interesting choice of V is a white noise, i.e., an i.i.d. potential on $\mathbb{R}^d$, but its analysis is widely open yet. This is truely a potential with correlation length zero, while i.i.d. potentials ξ on $\mathbb{Z}^d$ should be conceived as belonging to the class of potentials on $\mathbb{R}^d$ that have a fixed positive correlation length, via the above device. (We use the term *correlation length* in the sense of the infimum over $R>0$ that potential values are independent if their distance is $>R$.) However, practically all interesting potentials on $\mathbb{R}^d$ have a positive correlation length, some even infinitely large ones. $\diamond$

Rem-PAMcont

Remark 1.5. (The PAM on graphs.) It makes perfect sense to consider the PAM on an arbitrary graph G instead of $\mathbb{Z}^d$, replacing Δ^{d} by the standard graph Laplacian

$$\Delta\varphi(g)=\sum_{h\in G\colon (g,h)\text{ is an edge}}(\varphi(g)-\varphi(h))^2.$$

One interesting choice is the graph $G=\{0,1\}^N$ for some $N\in\mathbb{N}$, which models the set of all gene sequences of length N (where we simplify the presence of four alleles to just two, called 0 and 1). For this choice, the branching process picture that we explain in Remark 2.1.1 makes good sense for biological applications, since it models the random occurrences of mutants in a large population. The state space G is not interpreted as the region in which the population lives, but the gene pool that the individuals may have. The potential $\xi\colon G\to\mathbb{R}$ is the 'fitness landscape', which attaches to each gene sequence g its fitness $\xi(g)$. Choosing the $\xi(g)$ as independent and identically distributed random variables is in accordance with the current state of understanding of biological systems.

This model is considered in [AveGünHes15], where the question is answered how much time (in dependence of the length of the gene sequences , N) the system needs to reach the 'fittest' site with the main bulk of the population. This question was answered for the complete graph $\{1,2,\dots,N\}$, where all bonds $\{i,j\}$ are edges with $i\neq j$, in [FleMol90]. $\diamond$

Rem-PAMgraph

ainquestions

1.2 Main questions

As is common in statistical mechanics, we distinguish between the so-called *quenched* setting, where we consider $u(t,\cdot)$ almost surely with respect to the medium ξ, and the *annealed* one, where we average with respect to ξ. It is clear that the quantitative properties of the solution strongly depend on the distribution of the field ξ (more precisely, as we we will see, on the upper tail of the distribution of the random variable $\xi(0)$), and that different phenomena occur in the quenched and the annealed settings.

Our main purpose is the description of the solution $u(t,\cdot)$ asymptotically as $t\to\infty$. One of the main objects of interest is the *total mass* of the solution,

$$U(t)=\sum_{z\in\mathbb{Z}^d}u(t,z),\qquad\text{for } t>0. \tag{1.7}$$ Utdef

We ask the following questions:

(i) What is the asymptotic behavior of $U(t)$ as $t\to\infty$ in the annealed and in the quenched setting?
(ii) Where does the main mass of $u(t,\cdot)$ stem from? What are the regions that contribute most to $U(t)$? What are these regions determined by? How many of them are there and how far away are they from each other?

(iii) What do the typical shapes of the potential $\xi(\cdot)$ and of the solution $u(t,\cdot)$ look like in these regions?
(iv) What is the behaviour of the entire process $(u(t,\cdot))_{t\in[0,\infty)}$ of the mass flow? Does it exhibit ageing properties?

Remark 1.6. (Intermittency.) The long-time behaviour of the parabolic Anderson problem is well-studied in the mathematics and mathematical physics literature because it is an important example of a model exhibiting an *intermittency effect*. This means, loosely speaking, that most of the total mass $U(t)$ defined in (1.7) is concentrated on a small number of remote islands, called the *intermittent islands*. Much of the investigations of the PAM was motivated by a desire to understand this effect in detail.

However, this definition of intermittency is on one hand too detailed to be formulated concisely and on the other hand too little rigorous, as a precise definition depends on details of the potential. Hence, less detailed, but rigorous definitions of intermittency are helpful. One of the most often used defintions is in terms of the moments of $U(t)$: intermittency is often defined by the requirement

$$\limsup_{t\to\infty}\frac{1}{t}\log\frac{\langle U(t)^p\rangle^{1/p}}{\langle U(t)^q\rangle^{1/q}}<0,\qquad \text{for } 0<p<q, \tag{1.8}$$ Intermitt

where $\langle\,\cdot\,\rangle$ denotes expectation with respect to ξ. The left-hand side is non-positive by Hölder's inequality; the requirement is that the quotient of the p-norm and the q-norm decays even exponentially fast in t.

Let us briefly illustrate what (1.8) implies for the large-time behaviour of the solution, see [GärMol90, Section 1]. Write $-a$ for the left hand side of (1.8), pick some $\varepsilon\in(0,a)$, and consider the event

$$E_t=\{U(t)>\mathrm{e}^{t\varepsilon}\langle U(t)^p\rangle^{1/p}\}.$$

With Prob denoting the probability w.r.t. the random potential ξ, we may estimate

$$\mathrm{Prob}(E_t)=\mathrm{Prob}\Big(\frac{U(t)^p}{\langle U(t)^p\rangle}\geq \mathrm{e}^{tp\varepsilon}\Big)\leq \mathrm{e}^{-tp\varepsilon}, \tag{1.9}$$ probEtsmall

with the help of Markov's inequality, so the E_t are exponentially rare events. On the other hand, we see that the main contribution of the q^{th} moment comes from these rare events as follows. We have

$$\frac{\langle U(t)^q 1\!\!1_{E_t^c}\rangle}{\langle U(t)^q\rangle}=\frac{\langle U(t)^q 1\!\!1\{U(t)^q\leq \mathrm{e}^{tq\varepsilon}\langle U(t)^p\rangle^{q/p}\}\rangle}{\langle U(t)^q\rangle}\leq\frac{\mathrm{e}^{tq\varepsilon}\langle U(t)^p\rangle^{q/p}}{\langle U(t)^q\rangle}. \tag{1.10}$$ Intermitt1

Hence, combining with (1.8), we see that its exponential rate is negative:

$$\limsup_{t\to\infty}\frac{1}{t}\log\frac{\langle U(t)^q 1\!\!1_{E_t^c}\rangle}{\langle U(t)^q\rangle}\leq q\varepsilon-qa<0.$$

This means that the left-hand side of (1.10) decays exponentially fast towards 0, which implies that $\langle U(t)^q\rangle\sim\langle U(t)^q 1\!\!1_{E_t}\rangle$. Summarizing, the main contribution of the q^{th} moment comes from an event whose probability decays exponentially fast.

Strictly speaking, (1.8) does not say anything about the spatial structure of the solution $u(t,\cdot)$. However, if one recalls from Remark 1.2 that $U(t)=v(t,0)$ with v the solution of (1.1) with homogeneous initial condition $v(0,\cdot)\equiv 1$, then we see from (1.9), using the ergodic theorem, that the set $\{z\in\mathbb{Z}^d\colon v(t,z)>\mathrm{e}^{t\varepsilon}\langle U(t)^p\rangle^{1/p}\}$ of highest exceedances of the field $v(t,\cdot)$ has an exponentially small density. What is not clear at the moment (and whose formulation needs also some more care) is that it is this set that gives the main contribution to the total mass $U(t)$, more precisely, to its q-th moments. Much of this following, in particular Sections 4 and 7, is devoted to a thorough explanation of intermittency in this spatial sense.

One of the starting points of the interest in the PAM and of the research on the PAM is the following fact, see [GärMol90, Theorem 3.2].

Theorem 1.7. *Whenever ξ is truly random, the parabolic Anderson model is intermittent in the sense that* (1.8) *holds.*

This fact is one of the leading sources of motivation and has been severely extended into various directions; much of this text is devoted to this. $\diamond$

em-Intermitt

sec-Tools

2. Tools and examples of potentials

sec-IntMot

2.1 Probabilistic aspects

The PAM has a lot of relations to other questions and models, which explains the great interest that the PAM receives. We briefly survey the most important ones. In this section we concentrate on probabilistic aspects; see Section 2.3 for spectral theoretic aspects.

em-Branching

2.1.1. *Branching process with random branching rates.* The solution u to (1.1) also admits an interpretation that arises from branching particle dynamics, see [GärMol90]. The following model is one important representative of a class of models called *branching random walk in random environment (BRWRE).*

Imagine that initially, at time $t = 0$, there is a single particle at the origin, and all other sites are vacant. This particle moves according to a continuous-time symmetric random walk with generator $\Delta^{\rm d}$. When present at site x, the particle is split into two particles with rate $\xi_+(x)$ and is killed with rate $\xi_-(x)$, where $\xi_+ = (\xi_+(x))_{x\in\mathbb{Z}^d}$ and $\xi_- = (\xi_-(x))_{x\in\mathbb{Z}^d}$ are independent random i.i.d. fields. ($\xi_-(x)$ may attain the value ∞.) Every particle continues from its birth site in the same way as the parent particle, and their movements are independent. Put $\xi(x) = \xi_+(x) - \xi_-(x)$. Then, given ξ_- and ξ_+, the expected number of particles present at the site x at time t, as a function of $(t, x) \in [0,\infty) \times \mathbb{Z}^d$, solves the equation (1.1) and is therefore, by uniqueness, equal to $u(t, x)$ [GärMol90]. Here the expectation is taken over the particle motion and over the splitting resp. killing mechanism, but not over the random medium (ξ_-, ξ_+). The fact that the expected particle number solves (1.1), is standard in the study of branching processes; see [Hol00] for an elementary derivation.

The successful work on the PAM since 1990 has fertilized also the study of the BRWRE, but to a surprisingly little extent yet. In Section 9.2 below, we survey some heuristics and results on the BRWRE that are influenced by the PAM.

Rem-FKform

2.1.2. *Feynman-Kac formula.* Some of the most interesting applications of the PAM are best explained in terms of an explicit formula for the solution in terms of random walks. A very useful standard tool for the probabilistic investigation of (1.1) is the well-known *Feynman-Kac formula* for the solution u, which reads

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,{\rm d}s\Big\}\delta_z(X(t))\Big], \qquad (t,z)\in[0,\infty)\times\mathbb{Z}^d. \tag{2.1}$$

FKform

Here $(X(s))_{s\in[0,\infty)}$ is a continuous-time random walk on $\mathbb{Z}^d$ with generator $\Delta^{\rm d}$ starting at $z\in\mathbb{Z}^d$ under $\mathbb{E}_z$. By summing up over all $z\in\mathbb{Z}^d$, we see that the total mass $U(t)$ admits the Feynman-Kac representation

$$U(t) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,{\rm d}s\Big\}\Big], \qquad t\in[0,\infty). \tag{2.2}$$

FKformTotal

We refer the reader to [GärMol90, Theorem 2.1] for a proof of (2.2) or (2.5), which is intimately connected with the almost sure existence and uniqueness of a solution to problem (1.1). Actually, the restriction to a finite box is technically much easier to handle (see Section 2.1.3); in the infinite-space version (2.1) one has to control the decay of the potential ξ at infinity for proving the finiteness, and some percolation arguments are necessary for uniqueness, see our remarks after Theorem 1.1.

For the sake of better understanding, we give an explanation why the Feynman-Kac representation given in (2.1) actually solves problem (1.1). Consider the family of operators $(\mathcal{P}_t)_{t\geq 0}$ acting on the bounded functions on the lattice as

$$\mathcal{P}_t f(z) = \mathbb{E}_z\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\} f(X(t))\Big], \qquad (t,z)\in[0,\infty)\times\mathbb{Z}^d. \tag{2.3}$$

By time reversal, we see that (2.1) is tantamount to

$$u(t,z) = \mathcal{P}_t\delta_0(z). \tag{2.4}$$

An application of the Markov property shows that the family $(\mathcal{P}_t)_{t\geq 0}$ is a semigroup. Elementary calculations reveal that the corresponding generator is equal to $\Delta^{\mathrm{d}}+\xi$ showing up on the right hand side of (1.1). At least formally, we obtain the forward equation $\frac{\partial}{\partial t}\mathcal{P}_t f = (\Delta^{\mathrm{d}}+\xi)\mathcal{P}_t f$, which means that (2.4) solves the parabolic Anderson problem (1.1). The initial condition $u(0,z)=\delta_0(z)$ is trivially satisfied as $\mathcal{P}_0 = \mathrm{Id}$. From the derivation above, we see that the Feynman-Kac formula is true for other initial conditions as well.

FK FORMel ON $\mathbb{R}^d$ FORMULIEREN?

2.1.3. *Finite-space Feynman-Kac formulas.* If we equip the Anderson operator $\mathcal{H}$ with zero boundary condition in some finite set $B\subset\mathbb{Z}^d$, then the corresponding solution u_B (see Remark 1.3) may be represented as

$$u_B(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X([0,t])\subset B\}\mathbb{1}\{X(t)=z\}\Big], \tag{2.5}$$

i.e., the zero boundary condition is translated into the condition that that random walk does not leave B by time t. More precisely, the Laplace operator with zero boundary condition in B generates the simple random walk before it exits B, i.e., restricted to not leaving B. Then it is clear that $u_B\leq u$ and that the total mass of u_B,

$$U_B(t) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X([0,t])\subset B\}\Big], \tag{2.6}$$

satisfies $U_B\leq U$. Now let $B=(-R,R]^d\cap\mathbb{Z}^d$ with $R\in\mathbb{N}$ be a centred box and consider the Anderson operator $\mathcal{H}$ with periodic boundary condition. We obtain a Feynman-Kac formula by noting that the Laplace operator with periodic boundary condition generates the periodised simple random walk, $X^{(R)} = (X^{(R)}(s))_{s\in[0,\infty)}$, which can be pathwise realised as $X^{(R)}(s) = X(s)\,\mathrm{mod}B$. This walk never leaves B. Hence, we obtain

$$u_B^{(\mathrm{per})}(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X^{(R)}(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X(t)=z\}\Big], \tag{2.7}$$

and hence for its total mass:

$$U_B^{(\mathrm{per})}(t) = \sum_{z\in B} u_B^{(\mathrm{per})}(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X^{(R)}(s))\,\mathrm{d}s\Big\}\Big]. \tag{2.8}$$

We will see in Section 5.2 that, after taking expectation with respect to ξ, $U_B^{(\mathrm{per})}(t)$ turns out to be an upper bound for $U(t)$, i.e., $\langle U(t)\rangle \leq \langle U_B^{(\mathrm{per})}(t)\rangle$.

Rem-LocTim

2.1.4. *Local times and moments.* The functional in the exponent in the above Feynman-Kac formulas, $\int_0^t \xi(X(s))\,\mathrm{d}s$, is indeed a functional of the *local times* of the walk,

$$\ell_t(z) = \int_0^t \delta_z(X_s)\,\mathrm{d}s, \qquad t > 0,\, z \in \mathbb{Z}^d. \tag{2.9}$$

loctimdef

The family $(\ell_t(z))_{z\in\mathbb{Z}^d}$ is a random measure on $\mathbb{Z}^d$ with total mass equal to t, which registers the amount of time that the random walk spends in z up to time t. The *occupation times formula* says that

$$\int_0^t \xi(X_s)\,\mathrm{d}s = \sum_{z\in\mathbb{Z}^d} \xi(z)\ell_t(z).$$

Taking into account that the potential ξ is i.i.d., we may easily calculate the expectation of the main term in the Feynman-Kac formula:

$$\langle \mathrm{e}^{\int_0^t \xi(X_s)\,\mathrm{d}s}\rangle = \Big\langle \prod_{z\in\mathbb{Z}^d} \mathrm{e}^{\xi(z)\ell_t(z)} \Big\rangle = \prod_{z\in\mathbb{Z}^d} \langle \mathrm{e}^{\xi(0)\ell_t(z)}\rangle = \prod_{z\in\mathbb{Z}^d} \mathrm{e}^{H(\ell_t(z))} = \exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\}, \tag{2.10}$$

FKcalc

where

$$H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle, \qquad t\in\mathbb{R}, \tag{2.11}$$

Hdef

is the so-called *cumulant generating function* of $\xi(0)$, the logarithm of the moment generating function. Certainly, for this calculation we have to assume that $H(t)$ is finite for all positive t, i.e., that all positive exponential moments of $\xi(0)$ are finite. Using Fubini's theorem for interchanging the two expectations, we arrive at

$$\langle U(t)\rangle = \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\}\Big], \tag{2.12}$$

U(t)ExpFK

and similar formulas for the expectations of u, also for zero and periodic boundary conditions in some box B.

Remark 2.1. (Random walk in random scenery.) The exponent in the Feynman-Kac formula in (2.1), the process $\int_0^t \xi(X(s))\,\mathrm{d}s$, is sometimes called the *random walk in random scenery (RWRSc)*. This is an interesting object to study on its own, also in discrete time and for Brownian motion instead of random walks. In recent years, several authors got interested in the description of its extreme behaviour, which, on a technical level, has much to do with the analysis of the PAM. A rich phenemonology of asymptotic behaviours arises, depending on the upper tails of the scenery ξ, the dimension d and the degree of extremality. See Section 8.5 for a survey on results on the upper tails of RWRSc. $\diamond$

Rem-RWRSc
Rem-Pathmeas

2.1.5. *Quenched and annealed transformed path measures.* The intermittency effect may also be studied from the point of view of *typical paths* $X(s)$, $s\in[0,t]$, giving the main contribution to the expectation in the Feynman-Kac formula (2.1). This leads to the question where the so called *quenched path measures*

$$Q_{\xi,t}(\mathrm{d}X) = \frac{\mathrm{e}^{\int_0^t \xi(X_s)\,\mathrm{d}s}}{U(t)}\mathbb{P}_0(\mathrm{d}X), \qquad t\ge 0, \tag{2.13}$$

eq-PathMeasu

put their main mass. We consider $Q_{\xi,t}$ and $\mathbb{P}_0$ as probability measures on the set of paths $[0,t]\to\mathbb{Z}^d$. These quenched path measures obviously depend on the realisation of the potential ξ and do not necessarily constitute a consistent family of measures. For a fixed time t, the random walker X under $Q_{\xi,t}$ should be likely to move quickly and as far as possible through the potential landscape to reach a region of exceptionally high potential and then stay there up to time t. This would make the integral in the enumerator on the right of (2.13) large. On the other hand, the probability (under $\mathbb{P}_0$) to reach such a distant potential peak up to t may be rather small. Hence, the main mass in $Q_{\xi,t}$ comes from paths that find a good compromise between the high potential values and the far distance, and so does the main contribution to $U(t)$. This contribution is given by the height of the peak. The second order contribution to $U(t)$ is determined by the precise manner in which the optimal walker moves within the potential peak, and this depends on the geometric properties of the potential in that peak.

In analogy, the *annealed path measures* are defined as

$$Q_t(\,\mathrm{d}X) = \frac{\langle \mathrm{e}^{\int_0^t \xi(X_s)\,\mathrm{d}s}\rangle}{\langle U(t)\rangle}\,\mathbb{P}_0(\,\mathrm{d}X) = \frac{\mathrm{e}^{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))}}{\mathbb{E}_0[\mathrm{e}^{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))}]}\,\mathbb{P}_0(\,\mathrm{d}X), \qquad t\geq 0. \tag{2.14}$$

eq-PathMeasur

where we recall the local times and the cumulant generating function from Remark 2.1.4. In (5.5) below we will see that this density has an attractive effect on the path, as the functional $\mu\mapsto \exp\{\sum_z H(t\mu(z))\}$, seen as a map on probability measures on $\mathbb{Z}^d$, is convex. Hence, one may already here expect that the walk will, under Q_t, spread out on a smaller area than the free random walk, i.e., we may expect that $X_t \ll \sqrt{t}$ as $t\to\infty$, typically under Q_t.

Remark 2.2. (Random motions in random potential.) As we have seen right now, the PAM is a model of a random motion in a random surrounding, and it is one of the fundamental models of this kind. Other are *random walk in random scenery*, see Remark 2.1, *random walk in random environment (RWRE)*, where the environment consists of step distributions (on $\mathbb{Z}^d$) or drifts (on $\mathbb{R}^d$), attached to the sites, and the random walk uses them locally for its movement decisions, and *random directed polymers in random environment*, see Section 9.1. To separate from these models, the PAM is often also called *random walk in random potential*, and one uses the term *potential* for the random field ξ respectively V. $\diamond$

Rem-RMRS

2.2 Examples of potentials

c-examples

2.2.1. *Discrete case.* In principle, any i.i.d. potential $\xi=(\xi(z))_{z\in\mathbb{Z}^d}$ that satisfies the condition (1.3) is interesting for a consideration of the PAM, but some are more interesting, as they induce some particular interpretation, or they turn out to lead to interesting limiting behaviours of the PAM, as we will see later. Here we list some important single-site distributions and give some remarks about the main properties of the PAM with the respective distribution.

Example 2.3. (White noise.) MISSING. $\diamond$

-whitenoise

Recall from Section 1.1 the classification of sites z as a hard trap if $\xi(z)=-\infty$, a soft trap if $\xi(z)\in(-\infty,0)$, neutral if $\xi(z)=0$ and a sink if $\xi(z)\in(0,\infty)$. If $\xi\leq 0$, then the PAM induces, via the density $\mathrm{e}^{\int_0^t \xi(X(s))\,\mathrm{d}s}$, an exponential killing of the particle mass, as is seen from the Feynman-Kac formula (2.2).

Example 2.4. (Bernoulli traps.) The case when the field ξ assumes the values $-\infty$ and 0 only has a nice interpretation in terms of a *survival probability* and is therefore of particular importance. It is called *simple random walk among Bernoulli traps* and may be seen as the survival probability of the walk. Indeed, let

$$\mathcal{O}=\{z\in\mathbb{Z}^d\colon \xi(z)=-\infty\}$$

be the set of obstacles or traps, then it is clear that the exponent $\int_0^t \xi(X(s))\,\mathrm{d}s$ in the Feynman-Kac formula is equal to $-\infty$ as soon as the path $X([0,t])$ contains any trap. This implies that

$$u(t,z)=\mathbb{P}_0(X([0,t])\subset\mathcal{O}^{\mathrm{c}}, X(t)=z)$$

is the probability that the path does not hit any trap by time t and ends up at the site z, and $U(t)$ is the survival probability. Introducing the stopping time $T_{\mathcal{O}}=\inf\{t>0\colon X(t)\in\mathcal{O}\}$ of the first visit to the obstacles, we may also write $u(t,z) = \mathbb{P}_0(T_{\mathcal{O}} > t, X(t) = z)$, and $U(t) = \mathbb{P}_0(T_{\mathcal{O}} > t)$ is the upper tail of $T_{\mathcal{O}}$. Hence, the measure $Q_{\xi,t}$ defined in (2.13) has the density $\mathbb{1}\{T_{\mathcal{O}} > t\}/\mathbb{P}_0(T_{\mathcal{O}} > t)$.

The logarithmic moment generating function is $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle = \log p$ for $t > 0$, where p is the probability that a given site is neutral, and $H(0) = 0$. The density of the annealed measure, Q_t, can easily be calculated from (2.10), since

$$\sum_{z\in\mathbb{Z}^d} H(\ell_t(z)) = \log p \sum_{z\in\mathbb{Z}^d} \mathbb{1}\{\ell_t(z) = 0\} = R_t \log p,$$

where $R_t = |\{X(s)\colon s \in [0,t]\}|$ is the *range* of the walk by time t, the number of visited sites. Hence, the density of Q_t with respect to the simple random walk measure is equal to $\mathrm{e}^{-\nu R_t}/\mathbb{E}_0[\mathrm{e}^{-\nu R_t}]$, where $\nu = -\log(p)$. Hence, the expected total mass of the solution of the PAM, $\langle U(t)\rangle = \mathbb{E}_0[\mathrm{e}^{-\nu R_t}]$, is equal to a negative exponential moment of the range. The large-t study of the latter has been called the *range problem* by some authors.

The intermittent islands are the ones where $u(t,\cdot)$ achieves its maximum, which is zero. It will turn out that these islands depend on t and are rather large; in fact, in the annealed setting their radius is of order $t^{1/(d+2)}$, and in the quenched setting they are of order $(\log t)^{1/(d+2)}$.

Let us mention that a discussion of general trapping problems from a physicist's and a chemist's point of view, including a survey on related mathematical models and a collection of open problems, is provided in [HolWei94]. ◇

Rem-Survival

Example 2.5. (Other bounded potentials.) As we will later see, it is of interest to extend the scope of bounded potentials, by which we actually mean potentials that are bounded *from above*. For such potentials, it is no restriction to assume that the essential supremum $\operatorname{esssup}(\xi(0))$ is equal to zero, as the addition of a constant C leads to a multiplication of $u(t,x)$ with e^{Ct}. When we now want to determine the single-site distribution from the view point of the large-t asymptotics of the PAM, it will be only relevant to specify the tails of $\xi(0)$ at its essential supremum zero. The relevant choice of parameters is the following. For some $D \in (0,\infty)$ and $\gamma \in (0,1)$,

$$\operatorname{Prob}(\xi(0) > -x) \approx \exp\big\{-Dx^{-\frac{\gamma}{1-\gamma}}\big\}, \qquad x \downarrow 0. \tag{2.15}$$ bounded

Then $H(t) \approx -Ct^{\gamma}$ for some $C = C(D,\gamma)$. The strange way in which we incorporated γ in the power of x is motivated by an embedding in a larger class of potential distributions that we will discuss in Section 4.4 below. The boundary case $\gamma = 0$ contains the Bernoulli trap case of Example 2.4, but also more. The boundary case $\gamma = 1$ is phenemonologically contained in the almost bounded potentials of Example 2.7. ◇

Otherbounded

Example 2.6. (The double-exponential distribution.) Of high interest is also the single-site distribution given by

$$\operatorname{Prob}(\xi(0) > r) = \exp\Big\{-\mathrm{e}^{r/\rho}\Big\}, \qquad r \in \mathbb{R}, \tag{2.16}$$ double-exp

with parameter $\rho \in (0,\infty)$. The name refers to the right-hand side, but actually this distribution is just a reflected Gumbel distribution. The logarithmic moment generating function is $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle = \rho t \log t + \rho t + o(t)$ for large t (see [GärMol98], e.g.). The importance of this distribution for the PAM comes from the fact that the intermittent islands turn out to be discrete, i.e., not depending on t, in particular not growing with time, but still showing an interesting spatial shape. This makes it a nice to study distribution, since a great source of technical difficulties is absent. This potential is unbounded to infinity and produces high peaks in the solution $u(t,\cdot)$.

The parameter ρ describes the thickness of the tails, i.e., the tendency of the potential to assume very high values: the larger ρ is, the easier it is for the potential to assume large values. This is reflected in the fact that the size of the intermittent islands is decreasing with ρ, as we will later see. We will also see later that the two boundary cases $\rho = 0$ and $\rho = \infty$ correspond to the almost bounded case of Example 2.7 and to the heavy-tailed case of Example 2.8, respectively. $\diamond$

x-DoubleExp

Example 2.7. (Almost bounded potentials.) This is a class of single-site potential, which can be seen as interpolating between the bounded distributions of Example 2.5 for $\gamma = 1$ and the double-exponential distribution of Example 2.6 with $\rho = 0$. Indeed, one obtains examples of potentials (unbounded from above) by replacing ϱ in (2.16) by a sufficiently regular function $\varrho(r)$ that tends to 0 as $r \to \infty$, and other examples (bounded from above) by replacing γ in (2.15) by a sufficiently regular function $\gamma(x)$ tending to 1 as $x \downarrow 0$. It turns out in [HofKönMör06] (see Section 4.4) that the radius of the intermittent islands of the solution $u(t, \cdot)$ for the PAM with this potential diverges with $t \to \infty$ on a scale that interpolates between the bounded and the double-exponential case, as one may expect. Despite a somewhat tenacious introduction of examples of this class of distributions, they have the very nice property that the shape of the potential and the solution in the intermittent islands can be described in a rather explicit and clean way, see Remark 4.12. $\diamond$

ostBounded

Example 2.8. (Heavy-tailed potentials.) In the case that $\xi(0)$ is unbounded to ∞, the heavier the tails at ∞ are, the smaller the relevant islands are. For the double-exponential distribution with $\rho = \infty$ (defined in a suitable sense), it turned out in [GärKön00] (see Section 4.4) that the islands are singletons. However, for even more heavily tailed potentials, this concentration effect is even stronger pronounced and also, on the technical side, more easily proved. Therefore, very heavily tailed potentials are of high interest, since they admit mathematical proofs of highly detailed results, see Section 7.2. This class contains the Weibull distribution $\mathrm{Prob}(\xi(0) > r) = \mathrm{e}^{-Cr^\alpha}$ with $\alpha > 0$ and in particular the Gaussian distribution, and the Pareto distribution $\mathrm{Prob}(\xi(0)_r) = Cr^{-\beta}$ with $\beta > 0$. (Note that one has to assume that $\beta > d$, in order that the condition (1.3) is satisfied).

The Weibull distribution with $\alpha \leq 1$ and the Pareto distribution do not have finite exponential moments, i.e., the function $H(t)$ defined in (2.11) is not finite for $t > 0$. Accordingly, all the moments of the solution $u(t, z)$ are infinite, and the annealed setting does not exist. However, starting with [HofMörSid08, KönLacMörSid09, MörOrtSid11], distributional properties of $u(t, \cdot)$ and limit theorems in probability were derived, and the most detailed pictures that can currently be proved for the PAM were first derived for Pareto-distributed potentials; see for example the mass concentration property in Section 7.2 and ageing properties in Section 8.2. $\diamond$

heavytailed

2.2.2. *Continuous case.* Practically all examples of random potentials V on $\mathbb{R}^d$ for which the PAM is studied in the literature have a quite high degree of regularity and have positive and sometimes infinite correlation length. We will list below a number of examples. However, it is also of high interest to study the fully uncorrelated case, that is, a white-noise field, but the theory of existence and regularity of solutions to the heat equation with white-noise potential is currently not sufficiently far developed to begin a detailed analysis of the long-time behaviour of the solution. However, see [GubImkPer12] and [Hai13] for the introduction of novel methods from the recently coined theories of rough paths and regularity structures to the study of the PAM. It will be exciting to witness the future development of these methods to a deeper study of large-time properties of the PAM, in particular intermittency.

Example 2.9. (Poisson traps.) One of the most-studied case is when V is given in the form

$$V(x) = -\sum_i W(x - x_i), \tag{2.17}$$

VPoissontraps

where $(x_i)_i$ is a Poisson point process in $\mathbb{R}^d$ with constant intensity $\nu \in (0, \infty)$, and $W \colon \mathbb{R}^d \to [0, \infty]$ is a fixed given nonnegative function, called *cloud*. Canonical choices are $W = C\mathbb{1}_K$ for some compact set $K \subset \mathbb{R}^d$ (say, a centred ball) containing the origin and for some $C \in (0, \infty]$, or W some nonnegative continuous function with compact support, or $W(x) = C|x|^{-\alpha}$ for some $C \in (0, \infty)$ and $\alpha \in (d, \infty)$ (for $\alpha \leq d$, the potential V is not finite almost sure almost everywhere REFERENZ?). For all these choices, the model is called *Brownian motion in a Poisson field of traps*. The solution $u(t, \cdot)$ has its highest values (i.e., close to zero) where little or no Poisson points are present, and the relevant islands are large empty regions.

For the special choice $W = \infty 1\!\!1_{B_a(0)}$ (where $B_a(x)$ is the ball of radius a centred at x), the solution u to (1.5) is equal to the *Brownian survival probability* among a trap field that consists of the union of a-balls around all the Poisson points. Indeed, let $\mathcal{O} = \bigcup_i B_a(x_i)$ be that union and consider the stopping time $T_{\mathcal{O}} = \inf\{t > 0 \colon B_t \in \mathcal{O}\}$, the first entry time into the obstacle set $\mathcal{O}$, then the Feynman-Kac representation reads

$$u(t, x) = \mathbb{P}_0\left(T_{\mathcal{O}} > t,\, X(t) \in \mathrm{d}x\right) / \,\mathrm{d}x,$$

i.e., $u(t, x)$ is equal to the sub-probability density of $X(t)$ on survival in the Poisson field of traps by time t for a Brownian motion $(X(s))_{s\in[0,t]}$ starting from the origin. The total mass $U(t) = \mathbb{P}_0(T_{\mathcal{O}} > t)$ is the survival probability by time t. The analogue of the path measure $Q_{\xi,t}$ is the conditional distribution given the event $\{T_{\mathcal{O}} > t\}$, i.e., it transforms with the Radon-Nikodym density $1\!\!1\{T_{\mathcal{O}} > t\}/U(t)$.

It is easily seen that the first moment of $U(t)$ coincides with a negative exponential moment of the volume of the *Wiener sausage* $S_a(t) = \bigcup_{s\in[0,t]} B_a(X(s))$, i.e.,

$$\langle U(t)\rangle = \mathbb{E}_0\big[\langle 1\!\!1\{X([0, t]) \cap \mathcal{O} = \emptyset\}\rangle\big] = \mathbb{E}_0\big[\langle 1\!\!1\big\{\#\{i \colon x_i \in S_a(t)\} = 0\big\}\big] = \mathbb{E}_0[\mathrm{e}^{-\nu|S_a(t)|}], \qquad (2.18)$$

where ν is the intensity of the Poisson process, and $|\cdot|$ denotes Lebesgue measure. For this reason, the analysis of the annealed transformed path measure Q_t is sometimes called the *Wiener sausage problem*; it was historically the first special case of a PAM for which substantial asymptotic results were derived [DonVar75].

In [DonVar75] it turned out that the PAM with the cloud $W(x) = C|x|^{-\alpha}$ with $\alpha \in (d + 2, \infty)$ and for indicator functions on compact polar sets possesses the same asymptotic properties, at least as it concerns the leading asymptotics of the total mass and the picture of intermittency. In [Fuk11], the interesting case $\alpha \in (d, d + 2)$ was shown to show a different behaviour, see Example 8.3 below. The case $\alpha \in (0, d)$ requires a normalisation and exhibits further new phenomena [Che12a, **?**], see ???. $\diamond$

Example 2.10. (Poisson shot-noise potential.) It makes perfect sense to chose the Poisson cloud in (2.17) with the other sign, in which case we want to use the notation $\varphi \colon \mathbb{R}^d \to [0, \infty)$ instead of $-W$ and write

$$V(x) = \sum_i \varphi(x - x_i), \qquad (2.19)$$

where $(x_i)_i$ is a Poisson point process in $\mathbb{R}^d$ with constant intensity $\nu \in (0, \infty)$. Such a potential is sometimes called a *Poisson shot-noise potential*. The solution $u(t, \cdot)$ can easily achieve very high values in areas where many Poisson points stand close together; then many copies of the cloud φ are superposed. Hence, it is not surprising that the intermittent islands turn out in [GärKön00, GärKönMol00] to be extremely small and the exceedances of the solution $u(t, \cdot)$ extremely high. $\diamond$

Example 2.11. (Gaussian potentials.) Another interesting choice is to take V as a Gaussian field with sufficiently good regularity properties. A canonical assumption is twice continuous differentiability of the covariance function [GärKön00, GärKönMol00], in which case the potential has a modification that is Hölder continuous with any parameter in $(0, 1)$. Here the potential has high peaks on small islands, such that the relevant islands depend on t and have a diameter of order $t^{-1/4}$, and the potential approaches a parabolic shape in the peaks. However, recently there was also some efforts to study the

PAM under much less regularity assumptions [Che12b], see Sections 4.5.2 and 6.2.2. ◇

Rem-Gauss

Example 2.12. (Alloy-type potentials.) One of the most-studied random potentials in the community of Anderson localisation is of the form

$$V(x)=\sum_{z\in\mathbb{Z}^d}\xi_z v(x-z),\qquad x\in\mathbb{R}^d, \tag{2.20}$$

where, as above, $\xi=(\xi_z)_{z\in\mathbb{Z}^d}$ is a random i.i.d. field of random variables, and $v\colon\mathbb{R}^d\to[0,\infty)$ is bounded, compactly supported cloud function. Also the term *generalised Anderson model* is used.

MEHR DARUEBER! ◇

-alloy-type

Example 2.13. (Perturbed-lattice potential (random displacement model).) Another interesting choice is

$$V(x)=-\sum_{z\in\mathbb{Z}^d}W(x-z-\eta_z),$$

where $W\colon\mathbb{R}^d\to[0,\infty)$ is a single-site potential, and $(\eta_z)_{z\in\mathbb{Z}^d}$ is a sequence of centred $\mathbb{R}^d$-valued random variables. The interpretation is that a copy of W is intended to sit at each lattice point, but actually has been randomly shifted by its individual amount. In the random Schrödinger operator community, this type of potentials is called the *random displacement model*. The PAM with this potential is analysed in [Fuk09a, FukUek10, FukUek11]. Most natural is to assume the η_q as i.i.d., but also just an ergodicity assumption is of interest. Let us remark that there is an interesting relation between the distribution of the set $\{z+\eta_z\colon z\in\mathbb{Z}^d\}$ with $(\eta_z)_{z\in\mathbb{Z}^d}$ a particular ergodic sequence) and the set of zeros of a certain complex power series with i.i.d. Gaussian coefficients, see ???. ◇

-Potentials

ec-QuesHeur

2.3 Functional analytic considerations

It belongs to the standard knowledge of functional analysis that the solution to the heat equation with potential ξ in a finite box can be represented in terms of an *eigenvalue expansion* (also called *Fourier expansion*), i.e., an expansion with respect to the spectrum of the operator on the right-hand side of (1.1), the *Anderson Hamiltonian*

$$\mathcal{H}=\Delta^{\mathrm{d}}+\xi. \tag{2.21}$$ AndHam

This is one of the most important and fruitful connections of the heat equation with analytic theory; let us introduce the relevant notions and recall the most important facts. Reall that we do not put a minus sign in front of the Laplace operator, unlike the mathematical physics community. In particular, we do not speak of the 'bottom of the spectrum' but of the 'top', and 'deep valleys' of the potential are here 'high exceedances'

em-EigenExp

2.3.1. *Eigenvalue expansion.* Let us neglect for a while that the potential ξ is random. We introduce Dirichlet (i.e., zero) boundary condition in a finite set $B\subset\mathbb{Z}^d$ and denote the Hamilton operator in (2.21) by $\mathcal{H}_B$. We consider the solution u_B of (1.1) in B; see Remark 1.3. It admits the spectral representation (sometimes also called *Fourier expansion* or *spectral decomposition*)

$$u_B(t,\cdot)=\sum_{k=1}^{|B|}\mathrm{e}^{t\lambda_k}v_k(0)v_k(\cdot) \tag{2.22}$$ SR

with respect to the eigenvalues λ_k of $\mathcal{H}_B$ and an orthonormal basis consisting of corresponding eigenfunctions v_k, both depending also on B. At least formally, (2.22) can be understood by the representation

$$u_B(t,z)=\langle\delta_z,\mathrm{e}^{t\mathcal{H}_B}\delta_0\rangle=\big(\mathrm{e}^{t\mathcal{H}_B}\delta_0\big)(z), \tag{2.23}$$ FKBpraes

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\ell^2(\mathbb{Z}^d)$. We always pick the eigenvalues $\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{|B|}$ in decreasing order and the principal eigenfunction v_1 positive.

EigenvExpPAM

2.3.2. *Relation between eigenvalue expansion and the PAM.* The eigenvalue expansion in (2.22) yields an instructive explanation of the large-t asymptotics from a spectral point of view and serves as a starting point for powerful proofs, see also Remark 2.3.3. Let us illustrate some the of the benefits for the study of the PAM that (2.22) offers. Generally, the nice thing about (2.22) is that the time-dependence sits exclusively in the exponent as a prefactor of the eigenvalues.

Rayleigh-Ritz formula. One certainly guesses that the large-t asymptotics of the function $u_B(t,\cdot)$ should be mainly governed by the principal eigenvalue, $\lambda_1 = \lambda_1(B)$, and this is true for many considerations. Therefore, the *Rayleigh-Ritz formula* is of high interest:

$$\begin{aligned}\lambda_1(B) &= \sup_{v\in\ell^2(\mathbb{Z}^d)\colon \operatorname{supp}(v)\subset B, \|v\|_2=1} \langle \mathcal{H}_B v, v\rangle \\ &= -\inf_{v\in\ell^2(\mathbb{Z}^d)\colon \operatorname{supp}(v)\subset B, \|v\|_2=1} \sum_{x,y\in\mathbb{Z}^d\colon x\sim y} \big(v_x - v_y\big)^2,\end{aligned} \tag{2.24}$$

RRformula

where we remark that the last sum is actually over x and y in B and its outer boundary.

Upper estimates for u_B. There is a standard way to estimate the total mass of u_B in terms of the principal eigenvalue with the help of the Cauchy-Schwarz inequality and Parseval's identity[4] as follows:

$$\begin{aligned}U_B(t) = \sum_{z\in B} u_B(t,z) &= \sum_{k=1}^{|B|} \mathrm{e}^{t\lambda_k} \langle v_k, \delta_0\rangle\langle v_k, 1\!\!1\rangle \\ &\leq \Big(\sum_{k=1}^{|B|} \mathrm{e}^{t\lambda_k} \langle v_k, \delta_0\rangle^2\Big)^{1/2} \Big(\sum_{k=1}^{|B|} \mathrm{e}^{t\lambda_k} \langle v_k, 1\!\!1\rangle^2\Big)^{1/2} \\ &\leq \mathrm{e}^{t\lambda_1} \Big(\sum_{k=1}^{|B|} \langle v_k, \delta_0\rangle^2\Big)^{1/2} \Big(\sum_{k=1}^{|B|} \langle v_k, 1\!\!1\rangle^2\Big)^{1/2} \\ &\leq \mathrm{e}^{t\lambda_1} \|\delta_0\|_2 \|1\!\!1\|_2 = \mathrm{e}^{t\lambda_1}\sqrt{|B|}.\end{aligned} \tag{2.25}$$

UBestieigenv

See Remark 4.1 for more about typical sets B to which the above is applied and about the approximation of U with U_B.

Lower estimates for u_B. In some proofs, it turned out to be very useful to reverse the estimate in (2.25), i.e., to estimate the eigenvalue λ_1 in terms of the solution u_B, with the help of the expansion in (2.22). This seems difficult on the first sight, since all eigenfunctions v_k, with the exception of v_1, assume positive and negative signs. However, if one plays with the initial condition, this problem is removed. So let us denote by $u_B^{(y)}$ denote the solution to (1.1) with initial condition $u_B^{(y)}(0,\cdot) = \delta_y(\cdot)$ rather than $\delta_0(\cdot)$, then we can estimate, using that every v_k is ℓ^2-normalised,

$$\mathrm{e}^{t\lambda_1} \leq \sum_{k=1}^{|B|} \mathrm{e}^{t\lambda_k} = \sum_{k=1}^{|B|} \mathrm{e}^{t\lambda_k} \sum_{x\in B} v_k(x)^2 = \sum_{x\in B}\sum_{k=1}^{|B|} \mathrm{e}^{t\lambda_k} \langle v_k, \delta_x\rangle^2 = \sum_{x\in B} u_B^{(x)}(t,x). \tag{2.26}$$

UBestieigenv

Applying the Feynman-Kac formula in (2.5), adapted to the initial condition δ_y, we arrive at expressions that can be handled further with the same means as $U_B(t) = \sum_{x\in B} u_B(t,x)$, as one will easily see later.

[4] *Parseval's identity* states that $\sum_{k=1}^{|B|} \langle v_k, f\rangle^2 = \|f\|_2^2$ for any $f \in \ell^2(B)$.

em-AnderLoc

2.3.3. *Anderson localisation.* One of the great sources of interest in the random Schrödinger operator $\Delta^{\rm d}+\xi$ is the fact that its spectral properties help describing electrical conductance properties of alloys of metals or optical properties of glasses with random impurities. Therefore, in this connection, one is naturally mainly interested in *bounded* potentials, as the potential generically models the concentration ratio of the two metals, in the conductance application. Note that the entire spectrum is concerned, when turning attention to this application and interpretation, which is in contrast to the interest of the large-t asymptotics of the PAM, where only the top of the spectrum is involved.

But applications to electrical or optical properties are not the only driving force in mathematical physics, but predominantly the exciting prediction of P.W. Anderson [And58] that the spectrum of $\Delta^{\rm d}+\xi$ should have a peculiar behaviour, which in a way interpolates between the smoothing effect of the Laplace operator and the highly concentrating effect of the multiplication operator ξ, which has only delta functions as eigenfunctions. He predicted that, at least in its spectrum close to the spectral ends (we are thinking of a bounded random potential ξ, for which also the spectrum of $\Delta^{\rm d}+\xi$ is bounded), all eigenfunctions of $\Delta^{\rm d}+\xi$ should be exponentially localised. More precisely, for all eigenvalues close to any of the two boundaries of the spectrum, the corresponding eigenfunction should decay exponentially fast away from its individual localisation centre. This predicted phenomenon is nowadays called *Anderson localisation.* It was the motivation of an intense research activity in the last decades, and its validity has meanwhile been confirmed in a great number of cases, after the invention of deep mathematical tools. See [Kir10] for an extensive survey on Anderson localisation and further reading.

ttAnderLoc

2.3.4. *Intermittency and Anderson localisation.* Let us explain how Anderson localisation is related with intermittency in the PAM. The starting point is the spectral representation in (2.22) with a large box B (depending on t) such that u_B is a very good approximation for u (see Remark 4.1). In the limit $t\to\infty$, we can neglect all the summands in (2.22) with large k, because the exponential term ${\rm e}^{t\lambda_k}$ makes them negligible. According to the Anderson localisation prediction, at least for small k, the eigenfunctions v_k should be exponentially localised in centres x_k. (Here we anticipate that the localisation property, which is predicted by Anderson localisation theory only in the entire space $\mathbb{Z}^d$, persists to large boxes.) Moreover, as extreme value statistics predicts (see Section 7.3 below) these centres are far away from each other, since they form a Poisson point process, after rescaling (see Remark 7.5 below). Hence, v_k should be small outside a finite neighbourhood of x_k and even extremely small in neighbourhoods of the other x_i's and in the origin. Hence, $u_B(t,x_k+\cdot)$ is well-approximated in a neighbourhood of zero by ${\rm e}^{t\lambda_k}v_k(0)v_k(\cdot)$. As a consequence, the field $u_B(t,\cdot)$ has high peaks in small islands (the neighbourhoods of the localisation centres of the leading eigenvalues), which are far away from each other, and is much smaller outside these islands. This is a clear picture of intermittency. Additionally, we also see that the solution $u(t,\cdot)$ should be shaped like the eigenfunctions in these islands.

Rem-IDS

2.3.5. *Integrated density of states.* We saw in Remark 2.3.4 that large-t asymptotics of the PAM have much to do with the top of the spectrum (eigenvalues and the corresponding eigenfunctions) of the Anderson Hamiltonian $\mathcal{H}=\Delta^{\rm d}+\xi$ with zero boundary condition in large boxes. Another explicit manifestation of this relation is in terms of *Lifshitz tails*, which describe the upper tails of the *integrated density of states (IDS).*

One definition of the integrated density of states is as follows, see [CarLac90, Kir10]. In order to be consistent with the literature, we consider the operator $-\Delta^{\rm d}-\xi$. By $(-\Delta^{\rm d}-\xi)_{B_R}$ we denote its restriction to the box $B_R=[-R,R]\cap\mathbb{Z}^d$ with zero boundary condition. Denote by $E_1<E_2\leq E_2\leq\cdots\leq E_{|B_R|}$ its eigenvalues, counted with multiplicity (and of course depending on R). Let

$\nu_R = \frac{1}{|B_R|}\sum_k \delta_{E_k}$ denote its spectral measure. For an energy $E \in \mathbb{R}$, let

$$\mu_R(E) = \nu_R((-\infty, E]) \tag{2.27}$$

denote the number of eigenvalues $\leq E$ of $(-\Delta^{\mathrm{d}} - \xi)_{B_R}$. Then, by the subadditive ergodic theorem, the limit

$$\mu(E) = \lim_{R\to\infty} \frac{1}{|B_R|}\mu_R(E) \tag{2.28}$$ IDSdef

exists and is almost surely constant. The function μ is called the IDS. The interpretation of $\mu(E)$ is the number of energy levels of $-\Delta^{\mathrm{d}} - \xi$ below E per unit volume. Note that $\mu(E) \in [0, 1]$, since the $B_R \times B_R$-matrix $(-\Delta^{\mathrm{d}} - \xi)_{B_R}$ cannot have more eigenvalues than the cardinality of B_R. After shifting and rescaling, μ is a distribution function, i.e., it is increasing and right-continuous with left limits and boundary values 1 as $E \to \sup \sigma(\mathcal{H})$ and 0 as $E \to -\inf \sigma(\mathcal{H})$, where $\sigma(\mathcal{H})$ denotes the spectrum of $\mathcal{H}$.

The IDS is related to the PAM as follows. Let

$$\mathcal{L}(\nu_R, t) = \int_{\mathbb{R}} \mathrm{e}^{-\lambda t}\,\nu_R(\,\mathrm{d}\lambda) = \frac{1}{|B_R|}\sum_k \mathrm{e}^{-tE_k}$$

be the Laplace transform of ν_R evaluated at $t > 0$. Using the eigenvalue expansion in (2.22), we have the representation

$$\mathcal{L}(\nu_R, t) = \frac{1}{|B_R|}\sum_{z\in B_R} \mathbb{E}_z\Big[\mathrm{e}^{\int_0^t \xi(X_s)\,\mathrm{d}s}\mathbb{1}\{X_{[0,t]} \subset B_R\}\mathbb{1}\{X_t = z\}\Big], \tag{2.29}$$ LreprFK

i.e., the mixture over $z \in B_R$ of solutions to the PAM with initial condition δ_z, evaluated at z. The existence of the limit in (2.28) is proved by showing that ν_R has an almost sure limit ν, and this in turn is proved by showing that $\mathcal{L}(\nu_R, t)$ has a non-trivial limit. Using the ergodic theorem in (2.29), it is not difficult to prove that, almost surely,

$$\lim_{R\to\infty} \mathcal{L}(\nu_R, t) = \Big\langle \mathbb{E}_0\Big[\mathrm{e}^{\int_0^t \xi(X_s)\,\mathrm{d}s}\mathbb{1}\{X_t = 0\}\Big]\Big\rangle = \langle u(t, 0)\rangle. \tag{2.30}$$ Lapformlimit

Hence, ν_R has a limit ν as $R \to \infty$, whose Laplace transform $\mathcal{L}(\nu, t)$ is given by the right-hand side of (2.30), and this is equal to the expectation of the solution to the PAM as in (1.1) evaluated at zero.

There is also a useful connection between the IDS and the principal eigenvalue in a fixed box [CarLac90, VI.15, p: 311]. Indeed, for any $R \in \mathbb{N}$,

$$\mu(E) \geq \frac{1}{|B_R|}\langle \mu_R(E)\rangle \geq \frac{1}{|B_R|}\sum_k \mathrm{Prob}(E_k \leq E) \geq \frac{1}{|B_R|}\mathrm{Prob}(E_1 \leq E). \tag{2.31}$$

This connection was uitlized in [Fuk09b] for deriving relations between asymptotics of $\mu(E)$ for $E \downarrow \sigma \inf \sigma(-\Delta^{\mathrm{d}} - \xi)$ (see Section 2.3.6) and the almost sure asymptotics of the principal eigenvalue in large boxes.

Rem-Lifshitz

2.3.6. *Lifshitz tails.* Roughly speaking, the logarithmic asymptotics of the IDS $\mu(E)$ (as defined in (2.28) above) for $E \downarrow \inf \sigma(-\Delta^{\mathrm{d}} - \xi)$ are called the *Lifshitz tails* of the operator $-\Delta^{\mathrm{d}} - \xi$, see [CarLac90, Kir10]. They are of high interest for the description of the spectrum of $-\Delta^{\mathrm{d}} - \xi$ close to its bottom. Moreover, there is a general result that proves that Anderson localisation holds close to the bottom of the spectrum if the tails of $\mu(E)$ for $E \downarrow \inf \sigma(-\Delta^{\mathrm{d}} - \xi)$ are not too thin REFERENZ?.

Hence, we have a look at the asymptotics of $\mu(E)$ for $E \downarrow \inf \sigma(-\Delta^{\mathrm{d}} - \xi)$. We are going to identify them in terms of the IDS as follows. It is not difficult to see that

$$\mathcal{L}(\nu, t) \approx \langle U(t)\rangle \qquad \text{as } t \to \infty. \tag{2.32}$$ IDSFK

Indeed, '$\leq$' is obvious from the Feynman-Kac formula in (2.1) (just drop the indicator on $\{X(t) = 0\}$), and one obtains a lower bound for $\mathcal{L}(\nu, t)$ by inserting, on the right-hand side of (2.30), the indicator on $\{X_{[0,t]} \subset B\}$ for any set B, and a good choice is a t-dependent large centred box; see Remark 4.1. Expanding this in an eigenvalue series, we obtain

$$\mathcal{L}(\nu, t) \geq \Big\langle \sum_k \mathrm{e}^{t\lambda_k(B)} v_k(0)^2 \Big\rangle \geq \big\langle \mathrm{e}^{t\lambda_1(B)} v_1(0)^2 \big\rangle.$$

(Note that it was the coincidence of the initial and terminal conditions that enabled us to drop all other summands.) Now a bit of technical work is required to deduce that the term $v_1(0)^2$ is negligible, and the fact that $\langle \mathrm{e}^{t\lambda_1(B)} \rangle \approx \langle U(t) \rangle$ is anyway one of the main mottos in the treatment of the PAM asymptotics, if the box B is properly chosen and the sense of '$\approx$' properly specified.

sec-heur

3. First heuristic observations

Based on the probabilistic and the functional analytic considerations in Sections 2.1 and 2.3, let us give now some heuristics about what to expect in the description of the solution of the PAM after a long time.

U(t)expmom

3.1 $U(t)$ as exponential moments

The first observation is that, via the Feynman-Kac formula in (2.2), $U(t)$ is equal to the t-th positive exponential moment of the quantity

$$Y_t = \frac{1}{t} \int_0^t \xi(X_s) \, \mathrm{d}s,$$

the average of the potential values along the random walk path. (The quantity tY_t is sometimes called *random walk in a random scenery*, see Remark 2.1.) It is a well-known fact from standard probability theory that, for any random variable Y, we have $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[\mathrm{e}^{tY}] = \operatorname{esssup} Y \in (-\infty, \infty]$. Hence, the limiting exponential growth rate of $U(t)$ as $t \to \infty$ will have much to do with the maximisation of Y_t over the probability space.

Actually, this maximisation has to be put into the right balance with the limiting behaviour of Y_t as $t \to \infty$, i.e., with the prefactor t in the exponent. One has to find the proper balance between the two random objects, the path and the potential. Certainly, an optimisation of Y_t is achieved by confining the random walk path $X_{[0,t]}$ to an area in which the potential ξ is extremely large, and in which it does not cost the path too much probabilistically to stay a long time. For example, just one site in which the potential is extemely huge is not necessarily optimal, in comparison to a larger area in which potential is very large, but not extremely large.

Certainly, one will has also to keep the spatial distance of the locations of the high peaks to the origin in the consideration, but it will turn out that they do not severely enter the optimisation, as long as they are located with certain bounds and as long as we are not after finer assertions than logarithmic asymptotics. As a rule of thumb, for potentials that have all positive exponential moments finite (i.e., for which $H(t) < \infty$ for any $t > 0$), the relevant optimisation takes place in a centred box with radius t, up to logarithmic corrections.

It is clear from the above that the upper tails of ξ (i.e., the asymptotics of $\mathrm{Prob}(\xi(0) > r)$ for $r \uparrow \operatorname{esssup} \xi(0))$) will be one of the most important criteria, since they quantify the probabilistic cost of making the potential large, and they give information about the size of the highest peaks of the potential. The second relevant criterion is the probabilistic cost to confine the motion to the optimal area. The balance between the two strategies is subtle and will be described in detail in Section 4.2.

Rem-MomAs

3.2 Asymptotics of $U(t)$ and of its moments

Let us now explain the difference in the thinking about the quenched and the annealed setting. The asymptotics of the moments of $U(t)$ and its almost sure asymptotics are based on quite different (but related) arguments. The phenomenological difference between the two is the following. From the

Feynman-Kac formula in (2.1) we see that the moments of $U(t)$ are the joint expectations over the path and over the potential. Hence, both random objects can 'work together' according to a joint strategy that is a compromise between the two; both give a contribution that is exponentially costly: the potential assumes high values in a suitable area, and the path does not leave that area during the time interval $[0,t]$. In particular, it will be convenient to choose this area centered at the origin, the starting point of the walk, and to pick it equal to one big ball, instead of many small ones that are widely spread (as in the heuristics on the Anderson localisation). Hence, the main contribution to the moments of $U(t)$ should come from a self-attractive behaviour of the random walk and an extreme behaviour of the potential. For details, see Section 4.

Remark 3.1. (Estimating of probabilistic costs.) Here is a simple rule of thumb for estimating the probabilistic cost for the random walk to stay in a ball of radius $r \ll \sqrt{t}$ until time t. Namely, it is of order $\mathrm{e}^{-O(t/r^2)}$, i.e.,

$$-\log \mathbb{P}_0(X_{[0,t]} \subset [-r,r]) \asymp \frac{t}{r^2}, \qquad t \to \infty, \tag{3.1}$$

probsqueeze

which can be seen, with the help of the central limit theorem as follows. Chop the random walk path into t/r^2 pieces of length r^2 to see that staying t time units in a ball with radius r with $1 \leq r^d \ll t$ is equivalent that each of these t/r^2 pieces stays in that ball. For each piece, the probability for doing this is converges towards some fixed number in $(0,1)$, according to the central limit theorem, and hence the total probability for all the pieces together, should, according to the Markov property, be of size $\exp\{-O(t/r^2)\}$. $\diamond$

ema-ProbCost

In contrast, in the almost sure setting, the potential makes no effort to make a particular contribution, and the random path has to cope with that and must 'make the best' out of it. Hence, the identification of the almost sure asymptotics depends on a closer analysis of the potential landscape, almost surely for every sufficiently large t. First, one uses the Borel-Cantelli lemma to derive assertions about the existence of regions that are optimal in this sense, and then one uses this in order to derive lower bounds for the Feynman-Kac formula for $U(t)$ by requiring that the path spend almost the total time there. The quantity that measures the quality of a local region for the random walk to spend much time in it is the local Dirichlet eigenvalue of the region (more precisely, of $\Delta^{\mathrm{d}} + \xi$ in that region). See Section 6 for details.

It turns out there that a standard spatial extreme value analysis argument explains the existence, size and potential heights in the optimal regions which contribute most to $U(t)$. For this argument to work, one needs a good control on the upper tails of the eigenvalue of $\Delta + \xi$ in local regions, i.e., on the probability that they are extremely large. This is achieved by the control on the exponential moments of this eigenvalues with large prefactor, which one has to gain in the course of the proof of the asymptotics of the moments, since $\langle U(t)\rangle \approx \langle \mathrm{e}^{t\lambda(\xi)}\rangle$. In this way, the analysis of the moments gives the necessary control on the upper tails of the eigenvalue and serves as an important input in the proof of the almost sure asymptotics.

However, the above explains only lower bounds (but very good ones). Most of the proofs in the literature for the corresponding upper bound do not reflect any details about the potential landscape and are quite abstract.

rem-heurConc

3.3 Mass concentration

Certainly, we want to go much deeper into the description of the PAM and want to understand also where the main part of the mass that it describes is located. For this, one has to go back to the eigenvalue expansion in (2.22):

$$u(t,\cdot) \approx u_{B^{(t)}}(t,\cdot) = \sum_{k=1}^{|B^{(t)}|} \mathrm{e}^{t\lambda_k} v_k(0) v_k(\cdot), \tag{3.2}$$

EigenReprtime

where $B^{(t)}$ is a centred box that is so large that the first approximation is good enough. (For this, its radius should be roughly of order t with logarithmic corrections, if all the positive exponential moments of ξ are finite, and it should be of order of an appropriate power of t for heavy-tailed potentials.) We did not stress it in our explanations in Section 3.2, but the main point there was that the local region that gives the main contribution to the eigenvalue in $B^{(t)}$ is precisely the region where the corresponding eigenfunction v_k has its main mass. More precisely, as Anderson localisation theory predicts, all the leading eigenfunctions v_k (with k not depending on t) are highly concentrated in a region of small size somewhere in $B^{(t)}$ and are extremely close to zero everywhere outside. Each of these regions gives rise to a lower bound of the kind that we explained in Section 3.2. For the above logarithmic asymptotics of $U(t)$ we just considered, as an approximation, the first term in the above sum, and did not care about the term $v_k(0)$. However, if we also want to understand the upper bound in the almost sure asymptotics, we must take into consideration a large number of summands and the union of their localisation regions.

It is not easy to determine how many of the summands must be considered in order to quantify the upper bound in this way; in [Szn98] and [GärKönMol07], their number is of order $t^{o(1)}$, in [KönLacMörSid09, LacMör12, SidTwa14] (for very heavy-tailed potentials and in [BisKön13] (for the double-exponential distribution) it is one respectively two, depending on whether one works in almost-sure sense or in distribution. The criterion which summand to chose is simple: just the k that optimises the factor $\mathrm{e}^{t\lambda_k} v_k(0)$. For details, see Section 7.

em-heurFlow

3.4 Time-evolution of the mass flow

All heuristics so far considered only the situation of the mass flow at a given fixed, large time, i.e., a snapshot. However, it must be the goal to describe the evolution of the mass flow, i.e., the function $u(t,\cdot)$ as a function of the time. This is not easy and not very explicit, as long as one has no more information about the situation at fixed large times that the main mass sits in some of the relevant regions that are characterised as the concentration centres of the eigenfunctions v_k in the eigenvalue expansion in (3.2). However, for some important distributions (heavy-tailed ones [KönLacMörSid09, LacMör12, SidTwa14] and the double-exponential distribution [BisKön13]), we currently can make the much more precise assertion that, at most of the time instances, just one of the islands in question carries the overwhelming part of the global mass, and there is a description of this island in terms of its location and the size and shape of the potential in the island. Hence, the description of the time evolution of the mass flow can now convincingly be replaced by the description of the time-evolution of the site at which the main mass sits.

This process is, after appropriate time-space scaling, a jump process, which can be intuitively described as follows (see also [Mör11]). The box $b^{(t)}$ in (3.2) is the space horizon of the main mass at time t, i.e., the space in which it can travel in the time interval $[0,t]$. If t increases, from time to time it happens that this increasing horizon suddenlly includes a new, much better local island than all islands that it all included before. Here 'better' refers to the relation between size of the local eigenvalue and the distance to the origin, as is expressed by the term $\mathrm{e}^{t\lambda_k} v_k(0)$. During a small time interval (which vanishes in the time-scaling that we are looking at) this island becomes relevant and replaces the island that was optimal before. As a result, the rescaled process that describes the location of the relevant island jumps to the new island, and the Feynman-Kac formula is mainly concentrated on paths that go in short time to this new island and spend there most of the time.

sec-Summary

3.5 Summary

Let us give a small survey on the remainder of this text. We will give a detailed explanation of the asymptotics of the moments of $U(t)$ in Section 4, where we also explain the sense in which these are universal. The most important proof techniques that turned out useful in the study of these moments are listed and explained in Section 5. The almost sure logarithmic asymptotics of $U(t)$ are explained in Section 6, based on the achievements of Section 4. In Section 7 we explain the phenomenon of mass concentration in islands, respectively in just one island. In Section 8 we explore further and deeper questions about other aspects of the model, like refined asymptotics of the moments of $U(t)$, correlated potentials and weak potentials. Related models whose study influenced the study of the PAM (or *vice versa*) are presented in Section 9. Finally, the PAM with time-dependent random potential is presented in Section 10.

sec-MomAsy

4. Logarithmic asymptotics for the moments of the total mass

In this section, we explain on a heuristic level what the asymptotics of the logarithm of the moments of $U(t)$ are determined by, and how they can be described. We do this under the basic assumption that all positive exponential moments of $\xi(0)$ are finite, in which case all the moments of $U(t)$ are finite. We recall the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$ of $\xi(0)$ defined in (2.11), which will play an important role here, since its behaviour as $t \to \infty$ describes the potential close to its essential supremum.

For the sake of simplicity, we restrict ourselves to the first moment. Firstly, we give a heuristic derivation based on the eigenvalue expansion (2.22) in Section 4.2, followed by a second derivation in terms of a large-deviation statement for the local times of the random walk in Section 4.3. We formulate the outcome of these heuristics in Section 4.4. It turns out there that we need to distinguish four different regimes only, and we will provide explicit formulas for the logarithmic asymptotics in these regimes.

ec-RoughHeur

4.1 Rough moment asymptotics in terms of the cumulant generating function

Without much efforts, we obtain the two bounds

$$\mathrm{e}^{H(t)-2dt} \leq \langle U(t)\rangle \leq \mathrm{e}^{H(t)}, \qquad t \in (0,\infty). \tag{4.1}$$
eq-MomRoughB

This easily follows from the Feynman-Kac formula $U(t) = \mathbb{E}_0[\mathrm{e}^{\int_0^t \xi(X_s)\,\mathrm{d}s}]$ from (2.2). Indeed, we obtain a first lower estimate for $\langle U(t)\rangle$ by restricting the expectation w.r.t. the random walk to the event that it does not leave the origin up to time t. This event has probability e^{-2dt} as the time of the first jump, $\tau = \inf\{t > 0 \colon X(t) \neq X(0)\}$, is exponentially distributed with parameter $2d$. Furthermore, on this event, we have that $\int_0^t \xi(X(s))\,\mathrm{d}s = t\xi(0)$. Hence,

$$\langle U(t)\rangle \geq \langle \mathbb{E}_0 \mathrm{e}^{t\xi(0)} 1\!\!1_{\{\tau > t\}}\rangle = \big\langle \mathrm{e}^{t\xi(0)}\big\rangle \mathrm{e}^{-2dt},$$

which shows the left inequality in (4.1). On the other hand, an upper estimate arises by applying Jensen's inequality in the exponential term in the Feynman-Kac representation to the probability measure on $[0,t]$ with Lebesgue density $1/t$ as follows:

$$\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\} \leq \int_0^t \frac{1}{t}\exp\Big\{t\xi(X_s)\Big\}\,\mathrm{d}s.$$

Now taking the expectation with respect to ξ and interchanging it with the integral over $\mathrm{d}s$ and the random walk expectation, we arrive at

$$\langle U(t)\rangle \leq \Big\langle \int_0^t \frac{1}{t}\mathbb{E}_0\Big[\exp\Big\{t\xi(X_s)\Big\}\Big]\,\mathrm{d}s\Big\rangle = \int_0^t \frac{1}{t}\mathbb{E}_0\big\langle \mathrm{e}^{t\xi(X_s)}\big\rangle\,\mathrm{d}s = \big\langle \mathrm{e}^{t\xi(0)}\big\rangle = \mathrm{e}^{H(t)}, \tag{4.2}$$
Umomentsfini

which shows the right inequality in (4.1).

As a consequence of (4.1), it appears appropriate to consider the term $\mathrm{e}^{-H(t)}\langle U(t)\rangle$ and to try to derive logarithmic asymptotics on the scale t. This means that the moment asymptotics are described by (at least) two terms, the first of which is the cumulant generating function. This term yields a rough information about the way in which the potential attains large values, but no information about the structure of the potential in the high peaks. Therefore, we will have to work harder on the second term. Actually, it will turn out that it is more appropriate to replace $\mathrm{e}^{-H(t)}$ by some modification.

sec-MomHeur 4.2 Heuristics via eigenvalues

We give a heuristic derivation of a lower bound for $\langle U(t)\rangle$, which will later turn out to be also equal to the upper bound, up to the precision given by logarithmic asymptotics that we will consider. However, the explanation of the lower bound here is intuitive and gives quite some insight in the behaviour of the PAM, while the proof of the upper bound does not. The main result of the heuristics of this section is (4.22).

The first observation is that,

$$U(t) \sim U_{B^{(t)}}(t), \tag{4.3}$$

appreigenv

if the centred box $B^{(t)}$ is large enough, where we recall from Remark 1.3 that U_B denotes the total mass of the solution of the PAM in the set B. More details about the introduction of $B^{(t)}$ are given in Remark 4.1; for the remainder of this section it will be enough to know that the diameter of $B^{(t)}$ is large, but not larger than a power of t.

Remark 4.1. (Approximating with a large box.) In order to approximate U with $U_{B^{(t)}}$ to obtain (4.3), one uses the Feynman-Kac formula (2.1) to see that

$$U(t) - U_B(t) = \mathbb{E}_0\Big[\mathrm{e}^{\int_0^t \xi(X_s)\,\mathrm{d}s}\mathbb{1}\{X_{[0,t]} \not\subset B\}\Big]. \tag{4.4}$$

U-UBesti

Hence, to make this error term small, the box B should be taken so large that the probability that the path travels so far up to time t that it reaches the outside of B is small. A qualitative upper bound is (see [GärMol98, Lemma 2.5(a)])

$$\mathbb{P}_0(X_{[0,t]} \not\subset [-R,R]) \leq 2^{d+1}\exp\Big\{-R\log\frac{R}{dt} + R\Big\}, \qquad R, t > 0. \tag{4.5}$$

exitesti

On a case-by-case basis, depending on the assumptions on the potential ξ, this can be used to estimate the right-hand side of (4.4), after separating the exponential from the indicator by means of Hölder's inequality, if necessary, and by choosing the diameter of B large enough. In case that ξ is unbounded from infinity, in order to use just some moment assumptions on the potential, this technique will work only when taking the expectation with respect to ξ. There is no problem to take the diameter $R = R_t$ of $B = B^{(t)}$ (depending on t) so large that $\langle U(t)\rangle = \langle U_{B^{(t)}}(t)\rangle(1 + o(1))$ as $t \to \infty$. For potentials ξ with finite positive exponential moments (i.e. with $H(t) < \infty$ for all $t > 0$), it suffices to take R_t of the order $t(\log t)^{1+\eta}$ with some $\eta > 0$. For heavy tails, for example the Pareto distribution, $B^{(t)}$ needs to be of much larger diameter, see Section 7.2. $\diamond$

n-BoxRemove

The second main step is the approximation

$$U_{B^{(t)}}(t) \approx \mathrm{e}^{t\lambda(B^{(t)},\xi)} \tag{4.6}$$

appreigenv1

(in the sense of logarithmic equivalence), where $\lambda(B,\varphi)$ denotes the principal (i.e., largest) eigenvalue of the Anderson operator $\mathcal{H} = \Delta + \varphi$ in a finite set $B \subset \mathbb{Z}^d$ with zero boundary condition. This approximation is a consequence of the spectral representation (2.22) and can be easily justified with the help of the methods that we described in Section 2.3.2. Hence, we have to understand the logarithmic asymptotics of high exponential moments of the principal eigenvalue of $\mathcal{H}_{B^{(t)}}$ in a large, time-dependent box $B^{(t)}$.

Remark 4.2. (p-th moments.) It is already heuristically clear from (4.3) (and true in all known cases) that the p-th moments of $U(t)$ should have the same asymptotics as the first moments of $U(pt)$, at least as it concerns the leading terms. $\diamond$

m-pthmoments

Remark 4.3. (Rough bounds on $\lambda(B^{(t)},\xi)$.) Observe that the leading eigenvalue $\lambda(B^{(t)},\xi)$ is of the same order as the highest peak of ξ in $B^{(t)}$. Indeed, we easily check by the Rayleigh-Ritz-formula (2.24) that

$$\max_B \xi - 4d \le \lambda(B,\xi) \le \max_B \xi, \qquad B \subset \mathbb{Z}^d \text{ finite.} \tag{4.7}$$

$\diamond$

m-peaklambda

As we indicated in Remark 2.3.3, the main contribution to $\langle \mathrm{e}^{t\lambda(B^{(t)},\xi)}\rangle$ comes from realizations of the potential ξ having high peaks of some order $L(t)$ on mutually distant islands in $B^{(t)}$, whose radii are of some order $\alpha(t)$, which is much smaller than t. As we now consider the expectation over the potential ξ, it will be much easier (i.e., much less costly on the probabilistic side) to form just one such island and to center it at the origin. The appropriate orders of $L(t)$ and $\alpha(t)$ will be identified in (4.18) and (4.17), respectively.

For definiteness, we are considering in this heuristics only the case where the diameter of this island diverges like a function of t as $t\to\infty$, i.e., that we need to rescale the discrete set $\mathbb{Z}^d$ to α_t times continuous subsets of $\mathbb{R}^d$.

Therefore, we make the ansatz that for some continuous shape function φ on $Q_R=[-R,R]$,

$$\begin{aligned}\langle \mathrm{e}^{t\lambda(B^{(t)},\xi)}\rangle &\approx \Big\langle \mathrm{e}^{t\lambda(B^{(t)},\xi)}\mathbb{1}\{\overline{\xi}_t\approx\varphi \text{ in } Q_R\}\Big\rangle\\ &\approx \Big\langle \mathrm{e}^{t\lambda(B_{R\alpha(t)},\xi)}\mathbb{1}\{\overline{\xi}_t\approx\varphi \text{ in } Q_R\}\Big\rangle,\end{aligned} \tag{4.8}$$

Ansatz1

where

$$\overline{\xi}_t(\cdot)=\gamma(t)\Big[\xi\big(\lfloor\cdot\,\alpha(t)\rfloor\big)-L(t)\Big], \tag{4.9}$$

xitdef

with an appropriate scaling factor $\gamma(t)$, and we use the notation $B_R=[-R,R]\cap\mathbb{Z}^d$ for the discrete box with diameter $\approx 2R$. Hence, in (4.8), the potential undertakes particular efforts in the 'microbox' $B_{R\alpha(t)}$, and these give the main contribution to the eigenvalue in the macrobox $B^{(t)}$. This effort consists of achieving an extraordinarily height of order $L(t)$ in a relatively small box of diameter of order $\alpha(t)$, with a deviation from $L(t)$ of order $1/\gamma(t)$, and, moreover, to assume a particular shape φ inside this box, after proper rescaling.

The scale functions $\alpha(t)$, $L(t)$ and $\gamma(t)$ are deterministic and have to be adapted to the potential distribution (more precisely to the large-t asymptotics of $H(t)$), and afterwards we have to optimize over the box diameter R and over the potential shape φ.

Let us find out what proper choices for $\alpha(t)$, $L(t)$ and $\gamma(t)$ are. For this purpose, we calculate the contribution from the event $\{\overline{\xi}_t\approx\varphi \text{ in } Q_R\}$. We have obviously

$$\overline{\xi}_t\approx\varphi \quad\text{in } Q_R \qquad\Longleftrightarrow\qquad \xi(\cdot)\approx L(t)+\tfrac{1}{\gamma(t)}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big)\quad \text{in } B_{R\alpha(t)}, \tag{4.10}$$

where we recall that $B_R=[-R,R]\cap\mathbb{Z}^d$. It is clear that a shift of the potential by a constant shifts the eigenvalue by the same constant (and leaves the eigenfunction unchanged). Furthermore, it turns out that the only reasonable choice of $\gamma(t)$ is $\alpha(t)^2$, since the asymptotic scaling properties of the discrete Laplacian, Δ^{d}, imply that

$$\lambda\big(B_{R\alpha(t)},\tfrac{1}{\alpha(t)^2}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big)\big)\approx\frac{1}{\alpha(t)^2}\lambda^{(\mathrm{c})}(Q_R,\varphi), \tag{4.11}$$

eigenvaluere

where $\lambda^{(c)}(Q,\varphi)$ denotes the principal eigenvalue of $\Delta+\varphi$ in a bounded set $Q\subset\mathbb{R}^d$ having a 'nice' boundary with zero boundary condition, and Δ is the usual 'continuous' Laplacian. The relation (4.11) reflects the convergence of the discrete Laplace operator towards the continuous one after a spatial rescaling that is in the spirit of the central limit theorem. It can be rigorously proved by use of Gamma convergence methods.

Because of (4.11), we need to choose $\gamma(t)=\alpha(t)^2$ and obtain

$$\mathrm{e}^{t\lambda(B_{R\alpha(t)},\xi)}\mathbb{1}\{\overline{\xi}_t\approx\varphi \text{ in } Q_R\}\approx \mathrm{e}^{tL(t)}\exp\Big\{\frac{t}{\alpha(t)^2}\lambda^{(c)}(Q_R,\varphi)\Big\}. \tag{4.12}$$
eigenvaluecon

Now we need to choose $\alpha(t)$ and $L(t)$ such that the probability of the event $\{\overline{\xi}_t\approx\varphi \text{ in } Q_R\}$ is also on the exponential scale $t\alpha(t)^{-2}$. The idea here is that, for choices of $\alpha(t)$ that do not give a balance of the contribution to t times the eigenvalue with the logarithm of the probability will not give the optimal conpromise, i.e., will lead to one of the two terms being negligible with respect to the other, and the joint contribution being muss less than the one of the balancing order. The probability of this event depends on the tail of the potential distribution, and in order to approximate it properly, we have to make an assumption on the tails of the potential distribution.

Assumption (J). *There is an auxiliary scale function η and a non-trivial shape function J such that*

$$\lim_{s\to\infty}\frac{1}{\eta(s)}\log \mathrm{Prob}\Big(\xi(0)>\frac{H(s)}{s}+\frac{\eta(s)}{s}x\Big)=-J(x),\qquad x\in\mathbb{R}. \tag{4.13}$$
AssJ

Under this assumption, we calculate

$$\mathrm{Prob}\Big(\xi(\cdot)\approx \tfrac{H(s)}{s}+\tfrac{\eta(s)}{s}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big)\text{ in } B_{R\alpha(t)}\Big)\approx\prod_{z\in B_{R\alpha(t)}}\mathrm{e}^{-\eta(s)J\left(\varphi(\frac{z}{\alpha(t)})\right)}\approx\exp\Big\{-\eta(s)\alpha(t)^d I_R(\varphi)\Big\}, \tag{4.14}$$
LDPpot

where

$$I_R(\varphi)=\int_{Q_R}J(\varphi(y))\,\mathrm{d}y, \tag{4.15}$$
IRdef

after using that that potential ξ is i.i.d., and after turning the Riemann sum of the $J(\varphi(\cdot))$-values into an integral. In order that this potential shape scaling is of the form that we consider and that the scale of this probability is of the same order than the contribution on the right of (4.12), we need to introduce a new scale function $s(t)$ such that

$$\frac{\eta(s(t))}{s(t)}=\frac{1}{\alpha(t)^2}\qquad\text{and}\qquad\frac{t}{\alpha(t)^2}=\eta(s(t))\alpha(t)^d. \tag{4.16}$$

Clearly, this requries that $s(t)=t\alpha(t)^{-d}$. Hence, $\alpha(t)$ is determined by the requirement

$$\frac{\eta\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}}=\frac{1}{\alpha(t)^2}, \tag{4.17}$$
alphadefannea

which in turn implies that

$$L(t)=\frac{H(t\alpha(t)^{-d})}{t\alpha(t)^{-d}}. \tag{4.18}$$
L(t)ident

Hence, we identified the right scales and see that

$$\overline{\xi}_t(\cdot)=\frac{1}{\alpha(t)^2}\Big(\xi(\lfloor\cdot\,\alpha_t\rfloor)-\frac{H(t\alpha(t)^{-d})}{t\alpha(t)^{-d}}\Big) \tag{4.19}$$
xitident

formally satisfies a *large-deviation principle (LDP)* with speed $t/\alpha(t)^2$ and rate function I_R defined in (4.15) in the box Q_R, i.e., in simple terms,

$$\mathrm{Prob}(\overline{\xi}_t \approx \varphi \text{ in } Q_R) \approx \exp\Big\{ - \frac{t}{\alpha(t)^2} I_R(\varphi)\Big\}. \tag{4.20}$$

LDPxit

The theory of large deviations is instrumental to the study of the large-t asymptotics of the moments of $U(t)$, as we see here. See [DemZei98] for an account on the theory; in Section 5.1 we summarise the most important facts.

So we arrive at

$$\begin{aligned} \big\langle U(t)\big\rangle &\geq \Big\langle \mathrm{e}^{t\lambda(B_{R\alpha(t)},\xi)}\, \mathbb{1}\{\overline{\xi}(\cdot) \approx \varphi(\cdot) \text{ in } Q_R\}\Big\rangle \\ &\approx \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)} \exp\Big\{\frac{1}{\alpha(t)^2}\lambda^{(\mathrm{c})}(Q_R,\varphi)\Big\}\mathrm{Prob}(\overline{\xi}_t \approx \varphi \text{ in } Q_R) \\ &\approx \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)} \exp\Big\{\frac{t}{\alpha(t)^2}\Big(- I_R(\varphi) + \lambda^{(\mathrm{c})}(Q_R,\varphi)\Big)\Big\}. \end{aligned} \tag{4.21}$$

appr4

Optimising on φ and R, we obtain

$$\big\langle U(t)\big\rangle \geq \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)} \exp\Big\{ - \frac{t}{\alpha(t)^2}(\chi + o(1))\Big\}, \qquad t\to\infty, \tag{4.22}$$

appr5

where the constant χ is given in terms of the characteristic variational problem

$$\begin{aligned} \chi &= \lim_{R\to\infty} \inf_{\varphi\in\mathcal{C}(Q_R)} \big[I_R(\varphi) - \lambda^{(\mathrm{c})}(Q_R,\varphi)\big] \\ &= \inf_{\varphi\in\mathcal{C}(\mathbb{R}^d)} \Big[\int_{\mathbb{R}^d} J\circ\varphi - \sup_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1} \Big(\int_{\mathbb{R}^d}\varphi g^2 - \|\nabla g\|_2^2\Big)\Big], \end{aligned} \tag{4.23}$$

chideffirst

where we used the Rayleigh-Ritz principle for the principal eigenvalue, and $\mathcal{C}(Q)$ is the set of all continuous functions $Q\to\mathbb{R}$.

The last step of the heuristics is also known as the *Laplace-Varadhan method* or *Varadhan's lemma* from the theory of large deviations; for a precise formulation see Section 5.1 below, and for more on the theory see [DemZei98]. We arrived at the lower bound for the main assertion on the moment asymptotics. Recall that we did this heuristics under the assumption that $\alpha_t\to\infty$; in the other cases the formula (4.23) does not apply (it must be replaced by a discrete version of it); see Section 4.4. In some cases, a rigorous proof of (4.22) is derived by filling the gaps left open in the above heuristics. The upper bound '$\leq$' also holds, but its proof is much more technical and difficult and does not give insight in the behaviour of the model. See Section 5 for some techniques for proving the upper bound and Section 4.4 for the main examples, which are explicit.

Remark 4.4. (Interpretation of χ.) The first term on the right of (4.21) is determined by the absolute height of the typical realizations of the potential and the second contains information about the shape of the potential close to its maximum in spectral terms of the Anderson Hamiltonian $\mathcal{H}$ in this region. More precisely, those realizations of ξ with $\overline{\xi}_t \approx \varphi_*$ in Q_R for large R and φ_* a minimizer in the variational formula in (4.23) contribute most to $\langle U(t)\rangle$. In particular, the geometry of the relevant potential peaks is hidden via χ in the second asymptotic term of $\langle U(t)\rangle$. $\diamond$

chiInterpret

Remark 4.5. (Potential confinement properties.) The above heuristics suggest, in the spirit of large-deviation theory, that the main contribution to the moments should come from those realisations of the potential ξ such that the rescaled shifted version $\overline{\xi}_t$ resembles the members of the set $\mathcal{M}$ of minimizer(s) of the variational formula in (4.23). In other words, there should be a law of large numbers for $\overline{\xi}_t$ in the sense that the event $\{\xi\colon \overline{\xi}_t \notin U(\mathcal{M})\}$, where $U(\mathcal{M})$ is some neighbourhood of

$\mathcal{M}$ in a suitable topology, is of asymptotically small probability w.r.t. the transformed measures

$$\widehat{Q}_t(\,\mathrm{d}\varphi) = \frac{1}{\langle U(t)\rangle}\mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\Big]\mathrm{Prob}(\xi\in\,\mathrm{d}\varphi), \qquad t\geq 0,\ \varphi\colon\mathbb{Z}^d\to\mathbb{R}.$$

There is no doubt that such a law of large numbers should be valid in great generality, but there are only few proofs for this in the literature. A technical problem may be that the problem in (4.23), and hence also $\mathcal{M}$, is spatially shift-invariant, i.e., one has to cope with the event that $\overline{\xi}_t$ does not resemble any shift of the minimiser(s). Such a statement has been proved in the case of an almost bounded potential ξ (in the notation introduced in Section 4.4 below) in [GrüKön09]. $\diamond$

Rem-ConfPot

4.3 Heuristics via local times

Sec-Momproof

In this section, we present another route along which the (lower bound of the) asymptotics of the moments of $U(t)$ can be identified. This route is in a sense 'dual' to the route that we described in Section 4.2: Instead of carrying out the expectation with respect to the random walk in the Feynman-Kac formula in (2.1) first, and then analysing the ξ-expectation of the resulting expression in the eigenvalue expansion, we start by carrying out the ξ-expectation and then analyse the resulting expectation over the random walk. The main result of this section is in (4.30).

To that end, we recall from (2.9) one of the main objects in the probabilistic treatment of the PAM, the *local times* $\ell_t(z) = \int_0^t \delta_z(X_s)\,\mathrm{d}s$ of the random walk $(X_s)_{s\in[0,\infty)}$. Again, the cumulant generating function $H(t) = \log\langle \mathrm{e}^{t\xi(0)}\rangle$ plays a major role; it is again assumed to be finite for any $t>0$. From (2.10) we already know that

$$\langle U(t)\rangle = \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\}\Big]. \tag{4.24}$$

Ucalc1

We are going to work under the following supposition on the asymptotics of H.

Assumption (H): *There are a function* $\widehat{H}\colon(0,\infty)\to\mathbb{R}$ *and a continuous auxiliary function* $\eta\colon(0,\infty)\to(0,\infty)$ *such that*

$$\lim_{t\uparrow\infty}\frac{H(ty)-yH(t)}{\eta(t)} = \widehat{H}(y)\neq 0 \qquad \text{for } y\neq 1, \tag{4.25}$$

Hasyass

and the limit $\eta_* = \lim_{t\to\infty}\eta(t)/t\in[0,\infty]$ *exists.*

Assumption (H) is crucial and will be discussed at length in Section 4.4 below. Let us already remark that the function η coincides with the one of Assumption (J) above. We define the scale function $\alpha(t)$ as in Section 4 by

$$\frac{\eta\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} = \frac{1}{\alpha(t)^2}\,.$$

Let us again assume that $\alpha(t)\to\infty$ for these heuristics. Then we need to consider the spatially rescaled version of the local times,

$$L_t(y) = \frac{\alpha(t)^d}{t}\ell_t(\lfloor\alpha_t y\rfloor), \qquad y\in Q_R, \tag{4.26}$$

Ltdef

which is a random, L^1-normalised step function. We continue (4.24) with

$$\langle U(t)\rangle = \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)}\mathbb{E}_0\Big[\exp\Big\{\eta(t/\alpha(t)^d)\sum_{z\in\mathbb{Z}^d}\frac{H(L_t(z/\alpha(t))t/\alpha(t)^d) - L_t(z/\alpha(t))H(t/\alpha(t)^d)}{\eta(t/\alpha(t)^d)}\Big\}\Big].$$

According to Assumption (H), we may asymptotically replace the quotient after the sum on z by $\widehat{H}(L_t(z/\alpha(t)))$, and by definition of α, replace the prefactor in the exponent by $t/\alpha(t)^{d+2}$. Reducing the sum on $z \in \mathbb{Z}^d$ to a sum on $z \in B_{R\alpha(t)}$ and turning this sum into an integral using the substitution $z = \lfloor y\alpha(t) \rfloor$, we arrive at

$$\langle U(t)\rangle \approx \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)}\mathbb{E}_0\Big[\exp\Big\{\frac{t}{\alpha(t)^2}\int_{Q_R}\widehat{H}(L_t(y))\,\mathrm{d}y\Big\}\Big]. \qquad (4.27)\ \boxed{\texttt{Ucalc2}}$$

A crucial fact is that $(L_t)_{t\in(0,\infty)}$ satisfies a large-deviation principle with speed $t/\alpha(t)^2$ and rate function $g^2 \mapsto \|\nabla g\|_2^2$, that is,

$$\mathbb{P}_0(L_t(\cdot) \approx g^2(\cdot) \text{ in } Q_R) \approx \exp\Big\{-\frac{t}{\alpha(t)^2}\|\nabla g\|_2^2\Big\}, \qquad (4.28)\ \boxed{\texttt{LDPLt}}$$

for any L^2-normalised function $g \in H^1(\mathbb{R}^d)$ with support in Q_R (see Section 5.1 for details). Now we formally apply Varadhan's lemma (see e.g. [DemZei98]) to the expectation on the right-hand side of (4.27), ignoring that the functional $g^2 \mapsto \int_{Q_R}\widehat{H}(g^2(y))\,\mathrm{d}y$ might be not bounded or not continuous (in fact, it usually fails to be). This gives that

$$\begin{aligned}\mathbb{E}_0\Big[\exp\Big\{\frac{t}{\alpha(t)^2}&\int_{Q_R}\widehat{H}(L_t(y))\,\mathrm{d}y\Big\}\Big]\\ &\approx \exp\Big\{\frac{t}{\alpha(t)^2}\sup\Big\{\int_{Q_R}\widehat{H}\circ g^2 - \|\nabla g\|_2^2\colon g\in H_0^1(Q_R), \|g\|_2 = 1\Big\}\Big\}.\end{aligned} \qquad (4.29)$$

We obtain, after letting $R \to \infty$ (which optimises over R),

$$\langle U(t)\rangle \geq \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)}\exp\Big\{-\frac{t}{\alpha(t)^2}\chi_\circ\Big\}, \qquad (4.30)\ \boxed{\texttt{Ucalc3}}$$

where $\chi_\circ$ is given as

$$\chi_\circ = \inf\Big\{\|\nabla g\|_2^2 - \int_{\mathbb{R}^d}\widehat{H}\circ g^2\colon g \in H^1(\mathbb{R}^d), \|g\|_2 = 1\Big\}. \qquad (4.31)\ \boxed{\texttt{chipath}}$$

This is the main result on the moment asymptotics for $U(t)$; it is 'dual' to (4.22) in the sense that the two variational formulas χ and $\widetilde{\chi}$ are dual to each other, see Remark 4.6. The remarks that we made below (4.22) apply here as well.

Remark 4.6. ($\chi_\circ = \chi$.) In view of the fact that in both (4.21) and (4.30) also the complementary inequality $\leq$ holds, the two variational formulas in (4.23) and (4.31) must be identical. This can also be seen in an analytical way by interchanging the infimum and the supremum in (4.23):

$$\begin{aligned}\chi &= \inf_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1}\ \inf_{\varphi\in\mathcal{C}(\mathbb{R}^d)}\Big[\int_{\mathbb{R}^d}J\circ\varphi - \Big(\int_{\mathbb{R}^d}\varphi g^2 - \|\nabla g\|_2^2\Big)\Big]\\ &= \inf_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1}\Big[\|\nabla g\|_2^2 - \sup_{\varphi\in\mathcal{C}(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}\varphi g^2 - \int_{\mathbb{R}^d}J\circ\varphi\Big)\Big]\\ &= \chi_\circ.\end{aligned} \qquad (4.32)\ \boxed{\texttt{chichitildei}}$$

Furthermore, it is not too difficult to identify the function J introduced in Assumption (J) as the Legendre transform of the function $-\widehat{H}$ introduced in Assumption (H), that is,

$$J(x) = \sup_{y>0}\big(xy + \widehat{H}(y)\big), \qquad x > 0. \qquad (4.33)\ \boxed{\texttt{LegTrafo}}$$

This is done as follows. According to Assumption (J), the random variable $X_t = (\xi(0) - H(t)/t)t/\eta(t)$ has upper tails given by $\mathrm{Prob}(X_t > x) \approx \mathrm{e}^{-\eta(t)J(x)}$ as $t \to \infty$, for any $x > 0$. Hence, according to the Laplace principle, for any $y > 0$, as $t \to \infty$,

$$\mathrm{e}^{\eta(t)\max_{x>0}(xy - J(x))} \approx \big\langle \mathrm{e}^{\eta(t)yX_t}\big\rangle \approx \langle \mathrm{e}^{ty\xi(0)}\rangle \mathrm{e}^{-yH(t)} = \mathrm{e}^{H(ty)-yH(t)} \approx \mathrm{e}^{\eta(t)\widehat{H}(y)},$$

where the last step used Assumption (H). This shows that $\widehat{H}$ is the Legendre transformation of J and implies (4.33) via the duality principle.

From (4.33), it is only a technical step to see that the two functionals $g^2 \mapsto \int_{\mathbb{R}^d} \widehat{H} \circ g^2$ and $\varphi \mapsto \int_{\mathbb{R}^d} J \circ \varphi$ are Legendre transforms of each other, i.e.,

$$\int_{\mathbb{R}^d} \widehat{H} \circ g^2 = \sup_{\varphi \in \mathcal{C}(\mathbb{R}^d)} \Big(\int_{\mathbb{R}^d} \varphi g^2 - \int_{\mathbb{R}^d} J \circ \varphi \Big).$$

Using this in (4.32), we see that $\chi_\circ = \chi$. ◇

em-Chiident

Hence, the heuristics of this and the preceding section lead us to the formulation of the following result. We also include the case where $\alpha(t)$ does not diverge as $t \to \infty$, which we excluded from the above heuristics.

n-MomentAsy **Theorem 4.7** (Moment asymptotics)**.** *Suppose that Assumption (H) holds, and define α_t by* (4.17)*.*

(1) *Assume that $\lim_{t\to\infty} \eta(t)/t = 0$. Then $\alpha(t) \to \infty$, and*

$$\langle U(t) \rangle = \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)} \exp\Big\{ -\frac{t}{\alpha(t)^2} \chi_\circ \Big\}, \tag{4.34}$$

Umomentsident

with $\chi_\circ = \chi$ given by (4.23) *and* (4.31)*.*

(2) *Assume that $\lim_{t\to\infty} \eta(t)/t = 1$. Then $\alpha(t) \to 1$, and $\widehat{H}(y) = \rho y \log y$ for some $\rho \in (0,\infty)$, and*

$$\langle U(t) \rangle = \mathrm{e}^{H(t)} \exp\Big\{ -t\chi_\rho \Big\}, \tag{4.35}$$

Umomentsident

with χ_ρ given by (4.39) *and* (4.40)*, the spatially discrete version of* (4.23) *and* (4.31)*, respectively (see Remark 4.11).*

(3) *Assume that $\lim_{t\to\infty} \eta(t)/t = \infty$. Then $\alpha(t) \to 0$ and* (4.35) *holds with $\rho = \infty$, where $\chi_\infty = 2d$ is also given by* (4.39) *and* (4.40) *(see Remark 4.10).*

Indeed, Theorem 4.7 has been proved in the literature practically verbatim, but the proof for various choices of the distribution of the potential ξ is distributed over a number of papers. In case (1) the diameter of the intermittent islands diverges, in case (2) they stay bounded and positive, and in case (3) the islands shrink to single sites. See Section 4.4 for references and explicit formulas.

Remark 4.8. (Path confinement properties.) Analogously to Remark 4.5, it is also tempting to guess that the rescaled local times should satisfy a law of large numbers, i.e., should converge to the minimiser(s) in (4.31) in probability with respect to the transformed path measure given in (2.13). This property has been called the *Brownian confinement property* and was indeed proved in some of the most interesting cases, see Section 8.1. ◇

em-ConfPath

sec-Regimes 4.4 Four asymptotic regimes

In this section we explain that, under the crucial Assumption (H), which is a mild regularity assumption on the tails of $\xi(0)$ at its essential supremum, there are only four regimes (called universality classes in [HofKönMör06]) of asymptotic behaviours of the PAM. These regimes differ from each other in the order of the size of the relevant islands, in the explicit form of the rate function for the potential and in properties of the minimizers (e.g. compactness/unboundedness of support). The theory of regular functions straightly implies from the main assumption that the main asymptotic quantities in terms of which the asymptotics of the moments are given take only two different and explicit forms. Depending on unboundedly growing or vanishing diameters of the relevant islands, the asymptotic shape is continous (after spatial rescaling) or discrete or even trivially concentrated in just one site. We follow [HofKönMör06].

Recall Assumption (H) from Section 4.2, see (4.25). The function $\widehat{H}$ extracts the asymptotic scaling properties of the cumulant generating function H. In the language of the theory of regular functions, the assumption is that the logarithmic moment generating function H is in the de Haan class, which does not leave many possibilities for $\widehat{H}$:

variation **Proposition 4.9.** *Suppose that Assumption (H) holds.*

(i) *There is a $\gamma \geq 0$ such that $\lim_{t\uparrow\infty} \eta(yt)/\eta(t) = y^\gamma$ for any $y > 0$, i.e., η is regularly varying of index γ. In particular, $\eta(t) = t^{\gamma+o(1)}$ as $t \to \infty$.*

(ii) *There exists a parameter $\rho > 0$ such that, for every $y > 0$,*

$$\widehat{H}(y) = \rho \begin{cases} \frac{y-y^\gamma}{1-\gamma} & \text{if } \gamma \neq 1, \\ y \log y & \text{if } \gamma = 1. \end{cases}$$

(iii) *If $\gamma \leq 1$ and $\eta_* < \infty$, then there exists a unique solution $\alpha\colon (0,\infty) \to (0,\infty)$ to*

$$\frac{\eta\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} = \frac{1}{\alpha(t)^2}\,. \tag{4.36}$$ alphadefanne

and it satisfies $\lim_{t\to\infty} t\alpha(t)^{-d} = \infty$. Moreover,

(a) *If $\gamma = 1$ and $0 < \eta_* < \infty$, then $\lim_{t\to\infty} \alpha(t) = 1/\sqrt{\eta_*} \in (0,\infty)$.*

(b) *If $\gamma < 1$ and $\eta_* = 0$, then $\alpha(t) = t^{\nu+o(1)}$, where $\nu = (1-\gamma)/(d+2-d\gamma) \in (0, \frac{1}{d+2}]$.*

The additional assumption on the convergence of $\eta(t)/t$ is mild and is necessary only in the case $\gamma = 1$ (which will turn out to be the critical case). The function $\alpha(t)$ is the annealed scale function for the radius of the relevant islands in the parabolic Anderson model, this is the function that appeared in the heuristics described in Sections 4.2 and 4.3.

Now, under Assumption (H), we can formulate a complete distinction of the PAM into four cases:

(SP) $\eta_* = \infty$ (in particular, $\gamma \geq 1$), the *single-peak case*.
This is the boundary case $\varrho = \infty$ of the double-exponential case. It comprises all heavy-tailed distributions with finite positive exponential moments, see Example 2.8. We have $\alpha(t) \to 0$ as $t \to \infty$, as is seen from (4.36), i.e., the relevant islands consist of single lattice sites. As we will see in Section 7.2, this class phenemonologically also contains a number of potentials that have no finite positive exponential moments.

(DE) $\eta_* \in (0,\infty)$ (in particular, $\gamma = 1$), the *double exponential case*.
This is the case of the double-exponential distribution, see Example 2.6. By rescaling, one can achieve that $\eta_* = 1$. The parameter ϱ of Proposition 4.9(ii)(b) is identical to the one in (4.38) below. This case is studied in [GärMol98], [GärHol99], [GärKön00], [GärKönMol00], [GärKönMol07], [BisKön13] and more papers.

(AB) $\eta_* = 0$ and $\gamma = 1$, *the almost bounded case*.
This is the case of islands of slowly growing size, i.e., $\alpha(t) \to \infty$ as $t \to \infty$ slower than any power of t. This case comprises unbounded and bounded from above potentials, see Example 2.7. This class was introduced in [HofKönMör06] and further studied in [GrüKön09]. It lies in the union of the boundary cases $\varrho \downarrow 0$ of (DE) and $\gamma \uparrow 1$ of (B).

(B) $\gamma < 1$ (in particular, $\eta_* = 0$), *the bounded case*.
This is the case of islands of rapidly growing size, i.e., $\alpha(t) \to \infty$ as $t \to \infty$ at least as fast as some power of t. Here the potential ξ is necessarily bounded from above. This case was treated for a special subcase of $\gamma = 0$ (Bernoulli-traps, see Example 2.4) in [Ant95] and in generality in [BisKön01] (see Example 2.5).

In all these cases, the asymptotics of the moments of $U(t)$ are, for any $p \in (0,\infty)$ (see Theorem 4.7 for $p = 1$), given by

$$\frac{1}{pt} \log\langle U(t)^p\rangle = \frac{H\big(pt\,\alpha(pt)^{-d}\big)}{pt\,\alpha(pt)^{-d}} - \frac{1}{\alpha(pt)^2}\big(\chi + o(1)\big), \qquad \text{as } t\uparrow\infty, \tag{4.37}$$ generalform

where χ is alternately given in terms of two characteristic variational principles, which describe the shape of the potential and the local times that give the main contribution to $\langle U(t)^p\rangle$, respectively. Let us give some more insight in the four cases.

Remark 4.10. (The case (SP).) This case is included in [GärMol98] as the upper boundary case $\varrho = \infty$ in their notation; it comprises all heavy-tailed potentials with finite positive exponential moments, see Example 2.8. Here $\chi = 2d$, which is identical to the value of the right-hand sides of (4.39) and (4.40) for $\varrho = \infty$; the corresponding minimisers are $g = \delta_0$ and $\varphi = -\infty 1\!\!1_{\mathbb{Z}^d\setminus\{0\}}$ (with the understanding that $(-\infty)\cdot 0 = 0$). The scale function $\alpha(t) \to 0$ vanishes, and the first term on the right hand side in (4.37) dominates the sum, which diverges to infinity. $\diamond$

Rem-SP

Remark 4.11. (The case (DE).) The study of this case was initiated in [GärMol98]. The particular interest of this class comes from the fact that the intermittent islands have a discrete and non-trivial structure, since $\alpha(t)$ stays bounded (and may be put equal to one). The main representative of this class is the *double-exponential distribution* (which is indeed a reflected Gumbel distribution) given by

$$\mathrm{Prob}(\xi(0) > r) = \exp\{-\mathrm{e}^{r/\varrho}\}, \qquad r\in\mathbb{R}, \tag{4.38}$$ doubleexp4

where $\varrho \in (0,\infty)$ is a parameter. The characteristic variational problem is given as

$$\chi = \inf_{g\in\ell^2(\mathbb{Z}^d)\colon \|g\|_2=1} \left[\langle g, -\Delta^{\mathrm{d}} g\rangle + \varrho I(g^2)\right], \qquad \text{where } I(g^2) = -\sum_{z\in\mathbb{Z}^d} g^2(z)\log g^2(z). \tag{4.39}$$ chiDE

It is known [GärMol98, GärHol99] that this formula possesses minimizers g^2, which are unique (up to spatial shifts) for sufficiently large ϱ. These minimizers are not explicitly known, but they are known to decompose into a d-fold tensor product of minimisers of the formula for $d = 1$ and to approach delta-like functions for $\varrho \uparrow \infty$ and Gaussian functions (after rescaling) for $\varrho \downarrow 0$. This both is consistent with the understanding that (SP) is the boundary case of (DE) for $\varrho\uparrow\infty$, and (AB) is the boundary case of (DE) for $\varrho \downarrow 0$. The dual formula for χ is

$$\chi = \inf_{\varphi\colon \mathbb{Z}^d\to\mathbb{R}\colon \lim_{z\to\infty}\varphi(z)=-\infty} \Big(\frac{\varrho}{\mathrm{e}}\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{\varphi(z)/\varrho} - \lambda(\varphi)\Big), \tag{4.40}$$ chiDEpath

where $\lambda(\varphi) = \sup_{g\in\ell^2(\mathbb{Z}^d)\colon \|g\|_2=1}\langle g, (\Delta^{\mathrm{d}}+\varphi)g\rangle$ denotes the top of the spectrum of $\Delta^{\mathrm{d}}+\varphi$ in $\mathbb{Z}^d$; note that, due to the condition $\lim_{z\to\infty}\varphi(z) = -\infty$, it is also its principal eigenvalue with (up to shift and normalisation) precisely one eigenfunction. The first term in (4.40) is easily seen from (4.38) to be the infinite-space version of the large-deviation rate function of the potential; obviously the condition $\lim_{z\to\infty}\varphi(z) = -\infty$ is necessary for it to be finite.

In contrast with (4.39) and (4.40), the spatially continuous versions of these formulas do admit explicit minimisers; they appear in the case (AB), see Remark 4.12. With some efforts using Gamma-convergence techniques, one can show that the small-ϱ asymptotics of (4.39) and (4.40) are given by their continuous versions, including the convergence of the rescaled minimisers. This shows that the transition between the cases (DE) and (AB) is very smooth. $\diamond$

Rem-DE

Remark 4.12. (The case (AB).) This class was brought to the surface in [HofKönMör06]; it is a kind of interpolation between the cases (DE) for $\varrho \approx 0$ and (B) for $\gamma \approx 1$. One obtains examples of potentials (unbounded from above) by replacing ϱ in (4.38) by a sufficiently regular function $\varrho(r)$ that tends to 0 as $r\to\infty$, and other examples (bounded from above) by replacing γ in (4.44) by a sufficiently regular function $\gamma(x)$ tending to 1 as $x\downarrow 0$. We find that $\widehat{H}(y) = \mathrm{const}\, y\log y$, and the rate function in (4.14) turns out to be

$$I_R(\varphi) = \mathrm{const}\int_{Q_R} \mathrm{e}^{\varphi(x)/\varrho}\,\mathrm{d}x. \tag{4.41}$$

The characteristic variational problem is given by

$$\chi = \inf_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1} \Big[\|\nabla g\|_2^2 + \varrho \int_{\mathbb{R}^d} g^2 \log g^2\Big]. \tag{4.42}$$
chiAB

This is easily seen to be (up to spatial shifts, uniquely) minimised by the Gaussian density $g^2(x) = \mathrm{const}\,\mathrm{e}^{-\varrho\|x\|_2^2}$, which is the principal eigenfunction of $\Delta+\varphi$ for the parabolic function $\varphi(x) = \mathrm{const} - \varrho\|x\|_2^2$. The parabola in turn is the (up to spatial shifts, unique) minimiser of the alternate representation of χ:

$$\chi = \inf_{\varphi\in\mathcal{C}(\mathbb{R}^d)\colon \lim_{x\to\infty}\varphi(x)=-\infty} \Big(\frac{\varrho}{\mathrm{e}}\int_{\mathbb{R}^d} \mathrm{e}^{\varphi(x)/\varrho}\,\mathrm{d}x - \lambda(\varphi)\Big), \tag{4.43}$$
chiABpath

where $\lambda(\varphi) = \sup_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1}\langle g, (\Delta+\varphi)g\rangle$ is the principal eigenvalue of the operator $\Delta+\varphi$ in $\ell^2(\mathbb{Z}^d)$. Hence, in spite of a relatively odd definition of the potential distribution, the appropriately rescaled and shifted shape of the local times and of the potential that give the main contribution to the moments of the total mass are unique, explicit and elementary functions. $\diamond$

Rem-AB

Remark 4.13. (The case (B).) This class contains only distributions that are bounded from above, so without loss of generality we assume that their essential supremum is equal to zero. The upper tails at zero of the main representatives are given by

$$\log \mathrm{Prob}(\xi(0) > -x) \sim -Dx^{-\frac{\gamma}{1-\gamma}}, \qquad x\downarrow 0, \tag{4.44}$$
dist4

where $D\in(0,\infty)$ and $\gamma\in[0,1)$ are parameters. The special case $\gamma=0$ contains the Bernoulli distribution where only the values 0 and -1 are attained, and e^{-D} is the probability of the value 0. We find that $H(t) = -\mathrm{const}\,t^{\gamma+o(1)}$ and, for $\gamma\in(0,1)$, that $\widetilde H(y) = \frac{\rho}{\gamma-1}(y^\gamma - y)$. Then we have

$$\chi = \begin{cases} \inf_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1}\Big\{\|\nabla g\|_2^2 - \rho\int_{\mathbb{R}^d}\frac{g^{2\gamma}-g^2}{\gamma-1}\Big\}, & \text{if } \gamma\in(0,1),\\ \inf_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1}\Big\{\|\nabla g\|_2^2 - D|\mathrm{supp}\,(g)|\Big\}, & \text{if } \gamma=0.\end{cases} \tag{4.45}$$
Bikoe

These two formulas are well-known and well-understood. In particular, a minimiser exists, is unique up to spatial shifts, and has compact support, which is actually a ball. This was recently worked out in [Sch11] for $\gamma\in(0,1)$. For $\gamma=0$, the ball-shape of the support follows from a classic isoperimetric inequality called the *Faber-Krahn inequality*.

REFERENZ?

As a consequence, the minimiser can be explicitly written in terms of the principal eigenfunction of the Laplace operator in a ball, and the radius of that ball can easily be found using elementary analysis, using that $\lambda(B_r) = r^{-2}\lambda(B_1)$ for any $r>0$, where we wrote now $\lambda(A)$ for the principal Dirichlet eigenvalue of the Laplace operator in the set A. Now we see easily that, for $\gamma=0$,

$$\chi = \min_{A\subset\mathbb{R}^d}(D|A| + \lambda(A)) = \min_{r\in(0,\infty)}\big(D|B_r| + \lambda(B_r)\big) = \min_{r\in(0,\infty)}\big(Dr^d\omega_d + r^{-2}\lambda(B_1)\big) = c(D),$$

where ω_d is the volume of the unit ball and

$$c(D) = \Big(D\omega_d\big(\tfrac{2\lambda(B_1)}{d}\big)^d\Big)^{\frac{2}{d+2}}\big(1+\tfrac{d}{2\lambda(B_1)}\big),$$

and the optimal r is $r^* = (\frac{2}{\lambda(B_1)}D\omega_d d)^{1/(d+2)}$.

The boundary regime $\gamma\uparrow 1$ connects up smoothly with the case (AB). Indeed, using Gamma-convergence techniques, one can show that the formula in (4.45) and its minimiser(s) (properly rescaled) converge towards the formula (4.42). (Obviously, the term $\frac{g^{2\gamma}-g^2}{\gamma-1}$ converges toward the derivative of $\gamma\mapsto(g^2)^\gamma$ at $\gamma=1$, which is equal to $g^2\log g^2$.) $\diamond$

Rem-B

4.5 The spatially continuous case

In the spatially discrete case, we formulated the main assumption on the distribution of the random potential in terms of just one single random variable $\xi(0)$, since we relied on the assumption that the potential is i.i.d. In this way, one naturally covers all i.i.d. potentials. However, in the continuous case, one cannot do this so easily without determining the spatial correlations (if one does not want to mimic the i.i.d. case by putting the potential constant in the unit boxes $z+(-\frac12,\frac12]^d$ and i.i.d. over $z\in\mathbb{Z}^d$). Nevertheless, a number of potentials studied in the literature obviously belong to one of the above classes in a phenemonological sense. E.g., the case of a Poisson field of obstacles and many variants belong to the case (B) (studied in [DonVar75], many papers by Sznitman, resulting in the monograph [Szn98], [Ant94] and [Ant95] and more), the case of a Poisson field with positive cloud and many Gaussian fields (see [GärKön00] and [GärKönMol00], e.g.) belong to (SP).

4.5.1. *Brownian motion among Poisson obstacles.* Let us here discuss the moment asymptotics in the important case of a Brownian motion among Poisson obstacles, see Example 2.9. We assume that the potential is given as $V(x)=-\sum_{i\in\mathbb{N}}W(x-x_i)$ with a standard Poisson point process $\omega=(x_i)_{i\in\mathbb{N}}$ on $\mathbb{R}^d$ with intensity ν and a nonnegative nontrivial cloud $W\colon\mathbb{R}^d\to[0,\infty)$, which we want to assume here for simplicity as bounded, measurable and compactly supported. The main example is $W=\mathbb{1}_{B_a}$, the indicator on a centred ball with radius $a>0$. We want to understand the large-t asymptotics of the moments of

$$U(t)=\mathbb{E}_0\Big[\exp\Big\{-\int_0^t\sum_{i\in\mathbb{N}}W(Z_s-x_i)\,\mathrm{d}s\Big\}\Big],\tag{4.46}$$

where $Z=(Z_s)_{s\in[0,\infty)}$ is the Brownian motion with generator Δ (i.e., the time-change of the standard Brownian motion with a factor of 2).

Let us first take the expectation with respect to the Poisson process. We recall from Example 2.9 that, in the special case $W=\mathbb{1}_{B_a}$,

$$\langle U(t)\rangle=\mathbb{E}_0[\mathrm{e}^{-\nu|S_a(t)|}],$$

where $S_a(t)$ is the Wiener sausage up to time t with radius a. For finding the large-t asymptotics of this (sometimes called the *Wiener sausage problem*), a *large-deviation principle (LDP)* for the normalised occupation times of the motion, $\mu_t=\frac1t\int_0^t\delta_{Z_s}\,\mathrm{d}s$ is the most natural method. This LDP roughly says that

$$\mathbb{P}_0\Big(\mu_t(\mathrm{d}x)\approx\phi^2(x)\,\mathrm{d}x\Big)\approx\exp\Big\{-t\|\nabla\phi\|_2^2\Big)\Big\},\qquad t\to\infty,\tag{4.47}$$

i.e., the probability that μ_t resembles the probability measure with density ϕ^2 decays exponentially fast with rate given by the energy of ϕ. For a precise statement, one has to restrict to finite boxes and has to consider topologies. See Section 5.1 below for precise statements and [DemZei98] for more about the theory. The LDP in (4.47) describes the behaviour of the motion when staying in a compact part of $\mathbb{R}^d$ by time t. However, the typical spatial scale of the motion in the expectation of $\mathrm{e}^{-\nu|S_a(t)|}$ is not the finite one, but some scale α_t such that the optimal compromise between the probability to stay in a box of radius α_t and the term $\mathrm{e}^{-\nu|S_a(t)|}$ is realised. Looking only at the exponential rates, the former is $O(t\alpha_t^{-2})$[5], and the latter is $O(\alpha_t^d)$, both with the negative sign. Minimising their sum shows that the optimal scale is $\alpha_t=t^{1/(d+2)}$. Due to the Brownian scaling property for this spatial scale, we immediately obtain from (4.47) a LDP for the normalised occupation measure $\mu_t^{(t)}$ of the rescaling $Z_s^{(t)}=\alpha_t^{-1}Z_{s\alpha_t^2}$; indeed, it is the same with scale $t\alpha_t^{-2}=t^{d/(d+2)}$ instead of t. For large t, we may neglect the radius a of the Wiener sausage and can approximate

$$|S_a(t)|\approx\alpha_t^d|\mathrm{supp}\,(\mu_t^{(t)})|=t^{d/(d+2)}|\mathrm{supp}\,(\mu_t^{(t)})|.\tag{4.48}$$

[5]This can easily be seen from the argument that leads to (3.1).

Hence, using Varadhan's lemma (see Section 5.1 for a precise formulation), we now understand that

$$\langle U(t)\rangle = \mathbb{E}_0\big[\mathrm{e}^{-\nu|S_a(t)|}\big] \approx \mathbb{E}_0\Big[\mathrm{e}^{-\nu t^{d/(d+2)}|\mathrm{supp}\,(\mu_t^{(t)}|}\Big] \approx \mathrm{e}^{-t^{d/(d+2)}\chi}, \tag{4.49}$$

where

$$\chi = \inf\{\nu|\mathrm{supp}\,(\phi^2)| + \|\nabla\phi\|^2 \colon \phi\in H^1(\mathbb{R}^d)\}. \tag{4.50}$$

(The '$\approx$' in (4.49) means that the error is $\mathrm{e}^{o(t^{d/(d+2)})}$.) Note that χ is identical with the χ from the second line of (4.45) with $D=\nu$. This ends our heuristic explanation of the asymptotics of the moments in the case of Brownian motion among Poisson obstacles, using large deviations for the motion.

Another way to understand the asymptotics can be heuristically described using a joint strategy of the Poisson process and the motion as follows. We start from (4.46) and observe that the large-t asymptotics of $\langle U(t)\rangle$ are mainly concentrated on maximisation of the term in the exponent, i.e., on minimisation of $\int_0^t\sum_{i\in\mathbb{N}}W(Z_s-x_i)$. One good joint strategy of Z and ω is that ω leaves a certain bounded area $A\subset\mathbb{R}^d$ empty of sites x_i (even with a certain distance, such that A does not intersect any support of the functions $W(\cdot-x_i)$, but this extra amount asymptotically vanishes), and the path $Z_{[0,t]}$ does not leave A. In this case, the exponent is even equal to zero, which is certainly optimal. The probability cost for ω of following this strategy is $\mathrm{e}^{-\nu|A|}$, the Poisson probability to have no particle in a set with Lebesgue measure $|A|$. The cost for the motion can be found from the LDP of (4.47) as follows:

$$\mathbb{P}_0(Z_{[0,t]}\subset A) = \mathbb{P}_0(\mathrm{supp}\,(\mu_t)\subset A) \approx \exp\Big\{-t\inf_{\phi\in H_0^1(A)\colon\|\phi\|_2=1}\frac12\|\nabla\phi\|_2^2)\Big\}.$$

Note that the right-hand side is nothing but $\mathrm{e}^{-t\lambda(A)}$, where $\lambda(A)$ is the principal eigenvalue of Δ in A with zero boundary condition. Hence, the joint strategy has the probabilistic cost $\mathrm{e}^{-(\nu|A|+t\lambda(A))}$, and this is equal to the expectation of $U(t)$. Now we have to identify the optimal set A. i.e., to identify the minimiser A in the formula

$$\chi_t = \inf_{A\subset\mathbb{R}^d}(\nu|A|+t\lambda(A)).$$

But it is easily seen that this χ_t is, up to some elementary scaling, identical to the second line of (4.45), with the result that

$$\chi_t = t^{d/(d+2)}\Big(\nu\omega_d\big(\tfrac{2\lambda(B_1)}{d}\big)^d\Big)^{\frac{2}{d+2}}\big(1+\tfrac{d}{2\lambda(B_1)}\big),$$

where we recall that ω_d is the volume of the unit ball B_1.

These asymptotics do not depend on the shape of the cloud function W, at least as long as it has a compact support and does not depend on t. Actually, they are stable with respect to dependence of the support and the size of W on t; see Sections **??** and **??** for what happens if these critical scales are reached or traversed.

4.5.2. *Gaussian potentials.* Also the case of a Gaussian potential is interesting. Let $V=(V(x))_{x\in\mathbb{R}^d}$ be a Hölder continuous stationary centred Gaussian field with covariance function $B(x)=\langle V(0)V(x)\rangle$. We assume that B is twice continuously differentiable in a neighbourhood of zero with $B(0)=\sigma^2\in(0,\infty)$ and such that $-B''(0)=\Sigma^2$ is a positive definite matrix, i.e., the maximum of B at zero is strict and B parabola-shaped. Then $H(t)=\log\langle\mathrm{e}^{tV(0)}\rangle=\frac12t^2\sigma^2$, and with $\alpha_t=t^{-1/4}$, it is proved in [GärKön00] that

$$\langle U(t)\rangle = \mathrm{e}^{\frac12t^2\sigma^2}\exp\Big\{-\frac{t}{\alpha_t^2}(2^{-1/2}\mathrm{tr}(\sigma)+o(1))\Big\},\qquad t\to\infty. \tag{4.51}$$

The scale function $(\alpha_t)_{t>0}$ has the same interpretation as in the above heuristics as the order of the radius of the relevant islands. Interestingly, this is an example for *vanishing* α_t, i.e., the Gaussian field attains the relevant maxima on very small islands. Furthermore, the absolute height of the maxima is described by the variance and the (parabolic) shape of the potential in the peaks is described by the covariance matrix. (Such an interesting peak behaviour on vanishing islands can be observed only in the spatially continuous case.) The first term in the above asymptotics was already derived in [CarMol95].

Gaussian potentials with much less regularity and the singularity $B(0)=\infty$ and $B(x)\sim|x|^{-\gamma}$ as $x\to 0$ for some $\gamma\in(0,2)$ are considered in [Che12b], see Section 6.2.2.

-PoissonMom

4.5.3. *Poisson potential with high peaks.* As in Section 4.5.1 and Example 2.10, let $(x_i)_{i\in\mathbb{N}}$ be a standard Poisson point process on $\mathbb{R}^d$ with intensity $\nu\in(0,\infty)$ and $\varphi\colon\mathbb{R}^d\to[0,\infty)$ be a nonnegative, compactly supported cloud, and we consider the potential $V(x)=\sum_{i\in\mathbb{N}}\varphi(x-x_i)$. Like for the covariance function in Section 4.5.2, we assume that φ is stricly maximal in 0 with a strictly positive definite Hessian matrix $\Sigma^2=-\varphi''(0)$. Clearly, $H(t)=\nu\int(\mathrm{e}^{t\varphi(x)}-1)\,\mathrm{d}x$. Then in [GärKön00] it turns out that the order of the diameter of the relevant islands is given as $\alpha_t=t^{d/8}\mathrm{e}^{-t\varphi(0)/4}$, and that

$$\langle U(t)\rangle=\mathrm{e}^{H(t)}\exp\Big\{-\frac{t}{\alpha_t^2}\Big(\Big(\nu\frac{(2\pi)^{d/2}}{2\det(\Sigma)}\Big)^{1/2}\mathrm{tr}(\Sigma)+o(1)\Big)\Big\},\qquad t\to\infty. \tag{4.52}$$

MomasyPoiss

Note the extremely strong decay of α_t and the extremely fast asymptotics of the moments of the total mass. The first term, $H(t)$, seems to depend on all the values of the cloud φ in neighbourhood or zero, but an application of the Laplace method shows that its asymptotics depend only on $\varphi(0)$ and the Hessian matrix of φ at zero. However, the second term depends only on the Hessian matrix at zero.

Interestingly, both (4.51) and (4.52) may be summarized as

$$\frac{1}{t}\log\langle U(t)\rangle=\frac{H(t)}{t}-(\chi+o(1))\sqrt{H'(t)},\qquad t\to\infty, \tag{4.53}$$

MomasyGaussPo

with $\chi=(2\sigma^2)^{-1/2}\mathrm{tr}(\Sigma)$ where $\sigma^2=\varphi(0)$ in the Poisson case.

Gibbsfields

4.5.4. *Gibbs point fields.* MISSING

5. Some proof techniques

-techniques

In Section 4.2, we described the heuristics for the large-t exponential rate of the moments of the total mass $U(t)$, neglecting all technical issues. Nevertheless, in particular for the proofs of the upper bounds in the respective cases, there are a number of technical problems to solve, which required quite some efforts and developments. In this section, we present some of the proof techniques that have been successfully used to proof the main assertions made by these heuristics. We concentrate on explaining the methods in their simplest form, their nature, benefits and drawbacks, but we give no complete proof here and do not aim at generality.

One of the crucial assertions that need to be proved are of the type that a precise upper bound needs to be derived for the logarithmic large-t asymptotics of

$$\mathbb{E}_0\Big[\exp\Big\{\gamma_t\int_{\mathbb{R}^d}\widehat{H}(L_t(x))\,\mathrm{d}x\Big\}\Big], \tag{5.1}$$

expression

where $\gamma_t=t\alpha_t^{-2}\to\infty$ is a scale function, $\widehat{H}$ is the function introduced in Assumption (H), see (4.25), and L_t is the rescaled and normalized local times introduced in (4.26). (We stick here to the case $\alpha_t\to\infty$, i.e., the case where a spatial scaling is necessary, but the following applies also to the situation, where $\alpha_t\equiv 1$.) The main goal is to prove that the negative exponential rate on the scale γ_t is equal to $\chi_\circ$ defined in (4.31), i.e., to the supremum of the LDP rate function for the local times minus the functional $g^2\mapsto\int H\circ g^2$, taken over all L^2-normalized functions $g\in H^1(\mathbb{R}^d)$.

Another fundamental task is to find a tight logarithmic upper bound for the expression

$$\Big\langle\mathrm{e}^{\gamma_t\lambda(B,\xi)}\Big\rangle, \tag{5.2}$$

expression2

where γ_t is as above, $\lambda(B,\xi)$ is the principal Dirichlet eigenvalue of the operator $\Delta^{\rm d}+\xi$ in the box $B\subset\mathbb{Z}^d$, whose radius may depend on t and may be rather large.

As we have seen in the preceding sections, both tasks are strongly related, and there are techniques to estimate the two expressions in (5.1) and (5.2) in terms of each other. Indeed, see (2.25) and (2.26) for estimating the expectation of $U_B(t)$ from below and above in terms of the expression in (5.2). Furthermore, recall Section 4.3 to see that (5.1) comes from the expectation of $U(t)$, after having taken the expectation with respect to the random potential and inserting some rescaling into the local times of the random walk.

However, a number of technical obstacles arise in deriving a tight logarithmic upper bound for (5.1) and (5.2), respectively:

(1) restriction of the integral $\int_{\mathbb{R}^d}$ respectively of the box $B_{\mathcal{R}}$ to a box of appropriate (much smaller) size,
(2) overcoming the lack of boundedness of $\widehat{H}$, respectively of the map $\xi\mapsto\lambda(B,\xi)$ and
(3) overcoming the lack of continuity of $\widehat{H}$, respectively of the map $\xi\mapsto\lambda(B,\xi)$.

These are problems that are similar also to those that arise in the analysis of other exponential functionals of the local times (for example self-intersection local times, where $\widehat{H}(l)=l^p$ for some $p>1$, see the monograph [Che09] and the short survey [Kön10]).

In Section 5.1 we first explain why the desired result should be true at all, and why items (1)–(3) are indeed problems. In the subsequent Sections 5.2–5.9, we describe some of the most often used rigorous techniques that overcome these obstacles.

sec-LDP 5.1 Large deviations

One of the cornerstones of the mathematical analysis of the expectation of exponential functionals with a large prefactor is the theory of large deviations, see [DemZei98] for a comprehensive treatment. A family $(Y_t)_{t\in(0,\infty)}$ of random variables with values in some topological space $\mathcal{X}$ is said to *satisfy a large-deviation principle (LDP) with speed* γ_t and *rate function* $I\colon\mathcal{X}\to[0,\infty]$ if the level sets $\{x\in\mathcal{X}\colon I(x)\leq c\}$ are compact for any $c\in\mathbb{R}$ and if the set functions $\frac{1}{\gamma_t}\log\mathbb{P}(Y_t\in\cdot)$ converge weakly towards the set function $A\mapsto\inf_A I=\inf_{x\in A}I(x)$ in the sense that

$$\limsup_{t\to\infty}\frac{1}{\gamma_t}\log\mathbb{P}(Y_t\in\mathcal{C})\leq-\inf_{\mathcal{C}}I\qquad\text{and}\qquad\liminf_{t\to\infty}\frac{1}{\gamma_t}\log\mathbb{P}(Y_t\in\mathcal{O})\leq-\inf_{\mathcal{O}}I$$

for any closed set $\mathcal{C}$ and and open set $\mathcal{O}$ in $\mathcal{X}$. One of the most important ideas is the following strong extension of the well-known Laplace principle, which is called *Varadhan's lemma*: If $(Y_t)_{t\in(0,\infty)}$ satisfies the above LDP and $F\colon\mathcal{X}\to\mathbb{R}$ is a continuous and bounded function, then

$$\lim_{t\to\infty}\frac{1}{\gamma_t}\log\mathbb{E}\Big[{\rm e}^{\gamma_t F(Y_t)}\Big]=\sup_{\mathcal{X}}(F-I).$$

Psformotions 5.1.1. *LDPs for the occupation measures of the random motions.* One could already guess from the representation of the moments of $U(t)$ in (2.12), and it has been used in the heuristics in Section 4.3, that the analysis of the moments of $U(t)$ may be very well attacked with the help of an LDP for the normalised local times $\frac{1}{t}\ell_t$ or spatially rescaled versions of them, see (4.28). Let us cite the relevant LDPs from [Gär77] and [DonVar75-83]. By $\mathcal{M}_1(B)$ we denote the set of probability measures on $\mathbb{Z}^d$ with support in $B\subset\mathbb{Z}^d$.

lem-LDPRW **Lemma 5.1** (LDP for the normalised local times of the random walk). *For any finite box $B\subset\mathbb{Z}^d$, the normalised local times $\frac{1}{t}\ell_t=\frac{1}{t}\int_0^t\delta_{X_s}\,{\rm d}s$ satisfies an LDP both under the distribution of the random* *walk conditioned on not exiting B by time t and under the distribution of the periodised random walk, both with speed t. The rate function of the former is the quadratic form*

$$\mathcal{M}_1(B) \ni \mu \mapsto \langle \sqrt{\mu}, -\Delta^d \sqrt{\mu} \rangle - C_B = \sum_{x,y \in B \colon x \sim y} \Big(\sqrt{\mu_x} - \sqrt{\mu_y} \Big)^2 - C_B,$$

where $C_B = \inf_{\mu \in \mathcal{M}_1(B)} \langle \sqrt{\mu}, -\Delta^d \sqrt{\mu} \rangle$. The rate function of the latter is the analogous quadratic from with Δ^d replaced by the discrete Laplace operator on B with periodised boundary condition and C_B replaced by 0.

Note that the normalisation constant C_B is also equal to the principal eigenvalue of $\Delta^{\rm d}$ in B, as one sees from comparing to the Rayleigh-Ritz formula in (2.24) for this eigenvalue. When considering zero boundary condition, it is positive, and when considering periodised boundary condition it is equal to zero, as the corresponding eigenfunction is constant.

The two LDPs of Lemma 5.1 are important tools for the case of the double-exponential distribution (i.e., the cases (DE) and (SP)) in [GärMol98], where $\alpha_t = 1$, i.e, in the absence of spatial rescaling. Note that, for time-discrete random walk, there is also a LDP like the on in Lemma 5.1, but the rate function is different, as this LDP is based on an LDP for the empirical *pair* measures via the contraction principle.

However, in the cases (AB) and (B), we need to consider the spatially rescaled version L_t of ℓ_t introduced in (4.26). A proper formulation of (4.28) is as follows; see [GanKönShi07, Lemma 3.1] for the discrete-time case, but an extension to the continuous-time case is simple, see [HofKönMör06, Prop. 3.4].

LDPscaledRW **Lemma 5.2** (LDP for the rescaled local times of the random walk)**.** *For any centred cube Q, the rescaled local times $(L_t)_{t \in (0,\infty)}$, both under the distribution of the random walk conditioned on not exiting $(\alpha_t Q) \cap \mathbb{Z}^d$ by time t and under the distribution of the periodised random walk in that box, satisfies an LDP on the set of probability densities on Q with speed $t\alpha_t^{-2}$ and rate function*

$$g^2 \mapsto \begin{cases} \frac{1}{2}\|\nabla g\|_2^2 - \frac{1}{2}\lambda(Q) & \text{if } g \in H_0^1(Q), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.3}$$ ratefunctionB

in the case of zero boundary condition, and $g^2 \mapsto \frac{1}{2}\|\nabla g\|_2^2$ in the case of periodic boundary condition. The topology is the one that is induced by test integrals with respect to continuous functions $Q \to \mathbb{R}$.

Here $H_0^1(Q)$ is the usual Sobolev space of L^2-functions g that possess a gradient in the weak sense with zero boundary condition in Q; it is usually defined as the completion of the set of all infinitely smooth functions $g\mathbb{R}^d \to \mathbb{R}$ with support in Q_R with respect to the L^2-norm of g plus the one of ∇g. The normalisation $\lambda(Q)$ is equal to the principal eigenvalue of Δ in Q with zero respectively periodic boundary condition.

In (4.47), we used a similar LDP for the normalised occupation times measures of the Brownian motion with generator Δ; this LDP holds for the two cases (1) under conditioning the motion not to leave the box Q_R by time t and (2) for periodised Brownian motion in Q_R. Both LDPs also follow from [Gär77] and [DonVar75-83] with speed equal to t, and the rate function is again equal to the function in (5.3) with zero respectively with periodic boundary condition.

The fundamental papers [DonVar75] and [DonVar79] by Donsker and Varadhan on the Wiener sausage contain apparently the first substantial annealed results on the asymptotics for the PAM, based on the large-deviation theory that they developed in [DonVar75-83] and periodisation (see Section 5.2).

forpotential

5.1.2. *LDPs for the random potential.* In Section 4.2 we used an LDP for the shifted and rescaled potential $\overline{\xi}_t$ in (4.19), based on the Assumption (J) for the upper tails of a single-site potential variable, see (4.20). Deriving precise versions of such an LDP present no deep problems, in particular in the simpler case $\alpha_t = 1$, where no spatial scaling is involved. See [BenKipLan95] for such assertions and proofs. However, it is notoriously difficult to complete the main step in the proof, the argument for the application of Varadhan's lemma, see (4.21), since the the necessary rescaling properties of the discrete principal eigenvalue of $\Delta^{\mathrm{d}} + \xi$ and the necessary continuity properties of the map $\varphi \mapsto \lambda^{(c)}(Q_R, \varphi)$ are difficult to obtain or to approximate. For this reason, there are not many rigorous proofs in the literature that are based on the methodology described in Section 4.2. One important example is the celebrated method of enlargement of obstacles by Sznitman, see his monograph [Szn98].

eriodisation

5.2 Periodisation

Since the asymptotic methods described in Section 5.1 work only for random walks confined to a box having a certain size (possibly depending on t), we first need to find upper and lower bounds for the moments of $U(t)$ in terms of its versions on these boxes. As we explained in Remark 2.1.3, the lower bound is easily obtained, since $U \geq U_B$, where we recall that U_B is the total mass of the solution to the PAM in the box B with zero boundary condition. In Remark 4.1 we described how to control the difference $U - U_B$, but is successful only for very large boxes B, depending on t, and is meant as a method to introduce just *some* finite horizon to the problem, with or without taking the moments. In this section, we rather explain how to derive an upper bound for the moments of U in terms of an arbitrary box B.

eriodisation **Lemma 5.3** (Upper bound via periodisation). *For any box $B = (-R, R]^d \cap \mathbb{Z}^d$, we have the inequality*

$$\langle U(t)\rangle \leq \langle U_B^{(\mathrm{per})}(t)\rangle, \tag{5.4}$$

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Here is the proof. Denote by $\ell_t^{(R)}(z) = \int_0^t \delta_z(X_s^{(R)})\,\mathrm{d}s$ the local times of the periodised random walk $X^{(R)}$ in the box B, then it is not difficult to see that

$$\ell_t^{(R)}(z) = \sum_{x\in\mathbb{Z}^d} \ell_t(z + 2Rx), \qquad z \in B, t \in (0,\infty).$$

Now using that H is convex and that $H(0) = 0$, we see that H is also sub-additive, i.e., $H(l) + H(l') \leq H(l + l')$ for any $l, l' \in (0,\infty)$. Indeed, first we use convexity and $H(0) = 0$ to see that $H(\lambda l) = H(\lambda l + (1-\lambda)0) \leq \lambda H(l) + (1-\lambda)H(0) = \lambda H(l)$ for any $l \in [0,\infty)$ and $\lambda \in [0,1]$, and then we see that

$$H(l) + H(l') = H\big(\tfrac{l}{l+l'}(l+l')\big) + H\big(\tfrac{l'}{l+l'}(l+l')\big) \leq \tfrac{l}{l+l'}H(l+l') + \tfrac{l'}{l+l'}H(l+l') = H(l+l').$$

The sub-additivity now shows that the interaction term for the free walk on $\mathbb{Z}^d$ is upper bounded by the same term for the periodised walk:

$$\sum_{z\in\mathbb{Z}^d} H(\ell_t(z)) = \sum_{z\in B}\sum_{x\in\mathbb{Z}^d} H(\ell_t(z+2Rx)) \leq \sum_{z\in B} H\Big(\sum_{x\in\mathbb{Z}^d} \ell_t(z+2Rx)\Big) = \sum_{z\in\mathbb{Z}^d} H(\ell_t^{(R)}(z)). \tag{5.5}$$

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Using this in (2.12) (noting the analogous formula for $U_B^{(\mathrm{per})}$), we arrive at the assertion in (5.4).

This estimate is indeed one of the canonical starting points for proof of upper bounds for the moments of $U(t)$, as we explained in Remark 2.1.4. The usage of the latter is sometimes called periodisation; it is one of a couple of methods to 'compactify' the space. It seems that [DonVar75] is the first work on the PAM that uses this technique.

In order to later find the right conclusions, one has to chose the radius of the box B as $R\alpha_t$ (using the notation of the heuristics in Section 4.3.

omaindecomp
5.3 Spectral domain decomposition

Another technique of 'compactification' is demonstrated in [GärKön00, Proposition 1] in the continuous setting and was transferred to the discrete setting in [BisKön01, Prop. 4.4]. This technique works directly for expressions of the form (5.2) and allows for an upper bound in terms of 'local' eigenvalues, i.e., of the same expression with a much smaller box instead of B, which might be rather large, as we remind. The error is of the inverse order of the square of the small box. The estimate works for the eigenvalue alone, i.e., it is not restricted to taking moments.

The idea, in the discrete-space setting, is that, for any potential $V\colon \mathbb{Z}^d \to \mathbb{R}$, the principal Dirichlet eigenvalue $\lambda(B,V)$ of $\Delta^{\rm d}+V$ in some (arbitrarily large) box $B = B_{\mathcal{R}} = [-\mathcal{R},\mathcal{R}]^d \cap \mathbb{Z}^d$ is not larger than the maximal Dirichlet eigenvalue of $\Delta^{\rm d}+V$ in certain much smaller, mutually overlapping subboxes of $B_{\mathcal{R}}$, subject to a controllable error. The precise formulation is as follows, see [BisKön01, Prop. 4.4].

epdecomdisc **Lemma 5.4** (Spectral domain decomposition, discrete version). *There is a constant C that depends only on d such that, for any potential $V\colon B_{\mathcal{R}} \to \mathbb{R}$,*

$$\lambda(B_{\mathcal{R}};V) \leq \frac{C}{r^2} + \max_{z\in B_{\mathcal{R}}} \lambda(z+B_r,V), \qquad \mathcal{R} > r > 0. \tag{5.6}$$ eigenvaluecom

In the proof one sees that already the maximum over much less (carefully chosen) boxes serves as an upper bound, but the above formulation is simpler and is good enough for the application to the PAM. In the proof, one constructs a partition of the one consisting of smooth functions that approach the indicator functions on the the interiors of the subboxes, where we mean 'interior' in the sense that these parts of the subboxes build a decomposition of B; the overlapping regions are used for pressing the functions down from 1 to 0 in a sufficiently smooth way. The error term comes from the energy of these parts, i.e., from the ℓ^2-norm of their gradients. [BisKön01, Prop. 4.4] is formulated for nonpositive potentials V only, but an inspection of the proof reveals that it actually holds for all potentials.

Here is the continuous version, see [GärKön00, Proposition 1]; here, for any smooth region $Q \subset \mathbb{R}^d$ and any Hölder-continuous potential $V\colon Q \to \mathbb{R}$, $\lambda(Q,V)$ denotes the principal Dirichlet eigenvalue of $\Delta + V$ in Q, and we write Q_r for $[-r,r]^d$.

epdecomcont **Lemma 5.5** (Spectral domain decomposition, continuous version). *For any $r \geq 2$, there is a continuous function $\Phi_r\colon \mathbb{R}^d \to [0,\infty)$, whose support is contained in the one-neighbourhood of the grid $2r\mathbb{Z}^d + \partial Q_r$, such that, for any $R > r$ and any Hölder-continuous potential $V\colon \mathbb{R}^d \to \mathbb{R}$,*

$$\lambda(Q_r, V - \Phi_r) \leq \max_{z\in\mathbb{Z}^d\colon |z|\leq R/r+1 Q_R} \lambda_1(2rz + B_r, V). \tag{5.7}$$ eigenvaluecom

Moreover, Φ_r can be chosen $2r$-periodical in each coordinate and such that $\int_{Q_r} \Phi_r \leq K|Q_r|/r$ for some constant K that does not depend on r.

In the application in the discrete-space setting, one takes $\mathcal{R}$, depending on t, so large that the *a priori* bound in (4.4), in conjunction with (4.5), gives a negligible error for $\langle U(t) - U_{B_{\mathcal{R}}}(t)\rangle$. (Recall that $\mathcal{R}$ is then larger than t.) Then, choosing the diameter r of the small subboxes as $R\alpha(t)$, with R and α_t as in the heuristics in Section 4.3, reduces the problem to the appropriate size on which one can use the LDP for L_t, and the error term $C/R^2\alpha_t^2$ is on the scale of the LDP, and vanishes in the limit $t\to\infty$, followed by $R\to\infty$. The maximum over $z \in B_{\mathcal{R}}$ is turned into a sum of exponentials, using that they all have the same distribution. Summarizing, we get the following estimate:

$$\big\langle {\rm e}^{t\lambda(B_{\mathcal{R}},\xi)}\big\rangle \leq \Big\langle {\rm e}^{t\max_{z\in B_{\mathcal{R}}}\lambda(z+B_r,\xi)}\Big\rangle {\rm e}^{tC/r^2} \leq \sum_{z\in B_{\mathcal{R}}} \big\langle {\rm e}^{t\lambda(B_r,\xi)}\big\rangle {\rm e}^{tC/r^2} = \big\langle {\rm e}^{t\lambda(B_r,\xi)}\big\rangle |B_{\mathcal{R}}| {\rm e}^{tC/r^2}.$$

Hence, the big size of the box is finally tamed down to a big pre-factor, which is negligible with respect to the exponential asymptotics that we are after.

sec-Cutting

5.4 Cutting

Difficulty (2) for the expression in (5.1) is sometimes handled by some cutting technique, which requires serious work on a case-by-case basis. The problem arises if $\widehat{H}$ is not bounded. The basic idea is to replace $\widehat{H}$ by some cut-off version $\widehat{H}_M$ that is bounded and controling the remainder $\widehat{H}^{(>M)} = \widehat{H} - \widehat{H}_M$ with the help of some additional argument in the limit $M \to \infty$, which is taken after the limit $t \to \infty$ has been taken. A typical choice is $\widehat{H}_M(l) = l \wedge M$ for a large M. One separates the factors $\mathrm{e}^{\gamma_t \int \widehat{H}_M(L_t)}$ and $\mathrm{e}^{\gamma_t \int \widehat{H}^{(>M)}(L_t)}$ from each other using Hölder's inequality with parameters p, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, such that the first term appears in the p-norm with p very close to one. The exponential rate of the second term is shown to vanish in the limit $t \to \infty$, followed by $M \to \infty$, and then the rate of the first term is shown to approach the desired one in the limit $t \to \infty$, followed by $M \to \infty$ and $p \downarrow 1$.

The details of such an approach must be carefully carried out on a case-by case basis, depending on availabilities of good upper bounds for $\widehat{H}^{(>M)}$ and additional techniques for controlling the rate of the corresponding term. E.g. in [HofKönMör06], some additional elegant inequalities could be employed to arrive in a setting, where $\widehat{H}^{(>M)}$ could be replaced by a negative power of M times a monomial with a small power, and then combinatorial techniques were used to bound the exponential rate in terms of bounds for high *polynmomial* moments.

sec-Smoothing

5.5 Smoothing

Difficulty (3) for the expression in (5.1) is often taken care of by some *smoothing procedure*, i.e., by a replacement of the rescaled local times with the convolution with a smooth approximation of the Dirac measure. This procedure is isolated in technical lemmas in a number of papers, also for the Brownian case.

As an example, let us formulate a version for Brownian motion. Let $\psi\colon \mathbb{R}^d \to [0,\infty)$ be a rotationally invariant, nonnegative, smooth function with support in Q_1 and integral equal to one. For $\delta > 0$, let $\psi_\delta(x) = \delta^{-d}\psi(x/\delta)$. We denote by $*$ the convolution, that is, $u * v(x) = \int_{R^d} u(x-y)v(y)\,\mathrm{d}y$ for integrable functions $u, v\colon \mathbb{R}^d \to \mathbb{R}$. The main idea is that, for any integrable u, the function $u * \psi_\delta$ is smooth and approaches u in the limit $\delta \downarrow 0$ in L^p-sense for any $p \geq 1$ (see [LL01], e.g.). Here is a version of this fact that works in the sense of large deviations, see [AssCas03, Lemma 3.1]. By $\mu_t = \frac{1}{t}\int_0^t \delta_{Z_s}\,\mathrm{d}s$ we denote the normalised occupation measure for a Brownian motion $(Z_s)_{s\in[0,\infty)}$ in $\mathbb{R}^d$, starting from 0 under $\mathbb{P}_0$.

lem-SmoothBM **Lemma 5.6** (Smoothing the occupation measure of Brownian motion)**.** *For any $\varepsilon > 0$,*

$$\lim_{\delta\downarrow 0}\lim_{t\to\infty}\frac{1}{t}\log \sup_{u\colon \mathbb{R}^d\to[-1,1] \text{ measurable}} \mathbb{P}_0\big(|\langle \mu_t, u - u*\psi_\delta\rangle| > \varepsilon\big) = -\infty. \tag{5.8}$$

That is, the probability for the replacement error of the potential u (which is taken to be a cut-off version of the random potential) by a smoothed version $u * \psi_\delta$ begin larger than a small amount is shown to be enormously small on the exponential scale in $t \to \infty$, if δ is taken small afterwards. Note that the application of Lemma 5.6 assumes a bounded potential, which might require a cutting pre-step; see Section 5.4.

For handling Difficulty (3) for the expression in (5.1), i.e., for the rescaled local times L_t of a random walk as defined in (4.26), one needs a version of Lemma 5.6 for this setting. This is provided in [GanKönShi07, Lemma 3.5].

lem-SmoothRW **Lemma 5.7** (Smoothing the rescaled local times of the random walk)**.** *For any $\varepsilon > 0$,*

$$\lim_{\delta\downarrow 0}\lim_{t\to\infty}\frac{\alpha_t^2}{t}\log \sup_{u\colon \mathbb{R}^d\to[-1,1] \text{ measurable}} \mathbb{P}_0\big(|\langle L_t, u - u*\psi_\delta\rangle| > \varepsilon\big) = -\infty. \tag{5.9}$$

sec-MEO 5.6 Method of Enlargement of Obstacles

In the spatially continuous case (i.e., on $\mathbb{R}^d$ instead of $\mathbb{Z}^d$), there is an additional problem to the control of the expression in (5.2): the infinite combinatorial complexity of the space. In the 1990s, Sznitman contributed a lot to the understanding and the proof techniques for Brownian motion among Poisson traps, see his monograph [Szn98]. In particular, he developed proof methods that follow the physical picture, i.e., the interpretation in terms of spectral properties of the random Schrödinger operator. For this, he had to overcome also the problem of making the principal eigenvalue, seen as a function of the random Poisson field, amenable to a large-deviation analysis. For this, he developed a method, a coarse-graining scheme, that yields an upper bound by increasing and discretising the random potential in a careful way. This method was called the *method of enlargement of obstacles*, since the obstacles (the regions in the neighbourhood of the Poisson points) are enlarged by this procedure, giving an upper bound, which is the crucial point. The method is natural, but also involved and introduces three length scales.

Since the method is explained at length in Sznitman's monograph [Szn98] and is briefly surveyed in [Kom98], and since we decided to concentrate on the spatially discrete case in these notes, we abstain from trying to explain the method here. A spatially discrete variant of this method was carried out in [Ant94] and [Ant95], but has not been deeper exploited since then.

5.7 Joint density of local times

A quite sophisticated technique for overcoming the lack of boundedness and continuity for the expression in (5.1) was derived in [BryHofKön07] and makes it possible to derive the LDPs of Lemmas 5.1 and 5.2 (with a slight restriction of the validity with respect to the choices of α_t) even in the *strong* topology, i.e., in the topology that is induced by test integrals against bounded measurable functions. This method is based on an explicit upper bound for the joint density of the family of local times for random walks on a finite state space. It was applied in [HofKönMör06, Section 5] to expressions like in (5.1). A drawback of this strategy is an error term that makes it fail in too large boxes. The precise formulation of the crucial assertion is as follows (see [BryHofKön07, Th. 3.6]).

lem-BHK **Lemma 5.8.** *Let $B \subset \mathbb{Z}^d$ be finite and A_B the generator of a continuous-time Markov chain $(X_t)_{t\in[0,\infty)}$ on B with local times $\ell_t(z) = \int_0^t \mathbb{1}_{\{X_s=z\}}$. Then, for any measurable function $F\colon \mathcal{M}_1(B) \to \mathbb{R}$ and for any $t > 0$,*

$$\mathbb{E}\Big(\exp\Big\{tF\big(\tfrac{1}{t}\ell_t\big)\Big\}\Big) \leq \exp\Big\{t \sup_{\mu\in\mathcal{M}_1(B)} \Big[F(\mu) - \|\,(-A_B)^{1/2}\sqrt{\mu}\|_2^2\Big]\Big\}C_t(B), \qquad (5.10) \text{ BHKinUse}$$

where the error term is given by $C_t(B) = \exp\big\{|B|\log\big(2d\sqrt{8\mathrm{e}}\,t\big)\big\}|B|\mathrm{e}^{|B|/4t}$.

The great value of Lemma 5.8 is that the function F is neither assumed to be continuous nor bounded, and the error term is explicit. The main term on the right-hand side is precisely the variational formula that the large-deviation principle for $\frac{1}{t}\ell_t$ (see Section 5.1), in combination with Varadhan's lemma, suggests. After applying some pre-compactification (e.g. by periodisation), the estimate in (5.10) can immediately be applied to the expression in (5.1) (note that Lemma 5.8 also applies to the periodised simple random walk on a centred box B). However, we also note that the function F that needs to be picked for the rescaled version of L_t in (5.1) depends on t in a very non-trivial way, and the appropriate box B as well. As a result, the variational formula on the right-hand side of (5.10) needs to be studied further, and techniques from the theories of Gamma-convergence and finite elements need to be adapted in order to derive a precise asymptotic. In a similar situation (intersection local times, i.e., F being a p-norm), this was carried through in [BecKön12].

Lemma 5.8 stands in the tradition of the search for more explicit, deeper and more direct evidence for an interpretation of the family of local times of a continous-time random walk as a Gaussian process with covariance structure given in terms of the generator of the walk, see the literature remarks in [BryHofKön07].

sec-Dynkin 5.8 Dynkin's isomorphism

Another fruitful attempt in the search for Gaussian descriptions of the family of local times of random walks is called the *Dynkin isomorphism theorem* [Dyn88], which says that the joint law of the local times of a symmetric recurrent Markov process stopped at an independent exponential time is related to the law of the square of a Gaussian process whose covariance function is the Green kernel of the stopped Markov process. In a version derived by Eisenbaum it proved extremely useful in the mathematical treatment of upper tails (equivalently, to the high exponential moments) of the self-intersection local times of the walk, which corresponds to the choice $\widehat{H}(l) = l^p$ for some $p > 1$ in (5.1) and taking the p-th root of the functional $\int L_t(x)^p \,\mathrm{d}x$. The Dynkin isomorphism has not yet been applied to the PAM, but is very likely to give also here very good results. Its application to self-intersection local times was initiated in [Cas10] and was brought to full bloom in [CasLauMél13]. We cite here the version by Eisenbaum [Eis95].

lem-Dynkin **Lemma 5.9** (Dynkin's isomorphism)**.** *Let $X = (X_s)_{s\in[0,\infty)}$ be a random walk on a finite set B with local times ℓ_t, and let τ be an exponentially distributed random variable, independent of the walk. By $G = G_{\lambda,B}$ we denote the Green's function of the walk stopped at time τ. Let $(Z_x)_{x\in B}$ be a centred Gaussian process with covariance matrix G, independent of τ and of the walk. For $s \in \mathbb{R}\setminus\{0\}$, consider the process $S_x := \ell_\tau(x) + \frac{1}{2}(Z_x + s)^2$ with $x \in B$. Then, for any measurable and bounded function $F\colon \mathbb{R}^B \to \mathbb{R}$,*

$$\mathbb{E}\Big[F\big((S_x)_{x\in B}\big)\Big] = \mathbb{E}\Big[F\Big(\big(\tfrac{1}{2}(Z_x + s)^2\big)_{x\in B}\Big)\Big(1 + \frac{Z_0}{s}\Big)\Big].$$

Hence, essentially the family of local times, taken at an independent exponential time, are in distribution equal to $\frac{1}{2}$ times the square of a Gaussian family with covariance matrix given by the Green's function of the stopped walk. However, there are a number of changes, due to the addition of parameter s and the density $1 + Z_0/s$. The great value of Lemma 5.9 is that Gaussian processes area lot better behaved as local times and offer a lot of more techniques for their study, like concentration inequalities and explicit calculations. See [CasLauMél13] for these techniques at work and the monograph [MarRos06] for much more on relations between local times and Gaussian processes.

scretisation 5.9 Discretisation of Rayleigh-Ritz formula

Part of Difficulty (3) comes from the fact that the eigenvalue $\lambda(B, V)$ is a supremum over a quite large set of functions. Indeed, in the Brownian case, the Rayleigh-Ritz principle says that

$$\lambda(B,V) = -\inf\{\mathcal{E}_V(\phi)\colon \phi\in H_0^1(B), \|\phi\|_2 = 1\}, \qquad \text{where } \mathcal{E}_V(\phi) = \|\nabla g\|_2^2 - \langle V, \phi^2\rangle, \tag{5.11}$$ RRformulacon

for Hölder-continuous potentials in a bounded set $B \subset \mathbb{R}^d$ with regular boundary. One natural approach to derive upper bounds for $\lambda(B, V)$ is to approximate the infimum over this large set by the infimum over a much smaller, actually finite set of functions that lie so dense that the replacement error is small. One example in the literature where this has been carried out are [MerWüt01a, MerWüt02], from which we cite now.

lem-DiscRR **Lemma 5.10** (Discretisation of the Rayleigh-Ritz formula)**.** *For any $\eta > 0$, there are $M > 0$ and $R \geq 1$ and a finite set $\Phi_R \subset \{\phi \in \mathcal{C}^1(Q_{R+1})\colon \|\phi\|_2 = 1\}$ such that, for any $\mathcal{R} > R$, and for any Hölder continuous potential $V\colon Q_{\mathcal{R}} \to \mathbb{R}$,*

$$\lambda(Q_{\mathcal{R}}, V) \leq -\min_{y\in Y_{R,\mathcal{R}}}\min_{\psi\in\Psi}\mathcal{E}_{V\wedge M}\big(\phi(\cdot - y)\big) + \eta, \tag{5.12}$$

where

$$Y_{R,\mathcal{R}} = \Big\{ y \in \frac{R}{\sqrt{d}}\mathbb{Z}^d \colon y + Q_R \subset Q_{\mathcal{R}} \neq \emptyset \Big\}.$$

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sec-AsAsy

6. Almost sure logarithmic asymptotics for the total mass

In this section, we explain the basic picture that underlies the almost sure asymptotics of the total mass $U(t)$. Like for the moments, we will only argue for a lower bound, as this gives a good insight in the behaviour of the PAM, while many proofs of the corresponding upper bounds do not.

asydiscrete

6.1 The general discrete-space case

Recall from (4.3) and (4.6) that $U(t) \approx \mathrm{e}^{t\lambda(B^{(t)},\xi)}$, an approximation that is precise enough for our heuristics. Hence, it suffices to study the asymptotics of the principal eigenvalue $\lambda(B^{(t)},\xi)$ for some centred 'macrobox' $B^{(t)}$ that is chosen so large that the mass flow from the origin does not reach the outside of that box by time t with high probability. We want to consider the case of a potential distribution with all positive exponential moments finite, in which case the diameter of $B^{(t)}$ is of order t with logarithmic corrections. We suppose that the potential distribution satisfies Assumption (J) (see (4.13)) and, equivalently, Assumption (H) (see (4.25)), as in Section 4.2. For definiteness, we again assume that $\alpha(t) \to \infty$ as $t \to \infty$, but the same heuristics applies in all other cases.

The idea is to estimate $\lambda(B^{(t)},\xi) \geq \lambda(\widetilde{B},\xi)$ for some carefully chosen 'microbox' $\widetilde{B}$ in $B^{(t)}$ (in other words, we estimate $\xi \geq -\infty$ outside $\widetilde{B}$ and use that the eigenvalue is monotonous in the potential). That is, we search for some local area in which the potential is extremely high and has a particularly good shape. Our ansatz is that $\widetilde{B} = z + B_{R\widetilde{\alpha}_t}$ for some $z \in B^{(t)}$ and for some scale function $\widetilde{\alpha}_t \to \infty$ and some radius R (taken large afterwards), and that ξ is extremely high and attains some rescaled shape inside $\widetilde{B}$. Therefore, we consider the shifted and rescaled potential

$$\widetilde{\xi}_t(\cdot) = \widetilde{\alpha}_t^2 \Big[\xi\big(z + \cdot\, \widetilde{\alpha}_t\big) - h_t \Big], \qquad \text{in } Q_R = (-R,R)^d, \tag{6.1}$$ shiftresc

introducing the new scale h_t for the absolute height of the potential in $\widetilde{B}$. Note that, for any continuous shape function $\varphi\colon Q_R \to \mathbb{R}$,

$$\widetilde{\xi}_t \approx \varphi \quad \text{in } Q_R \qquad \Longleftrightarrow \qquad \xi(z+\cdot) \approx h_t + \tfrac{1}{\widetilde{\alpha}_t^2}\varphi\big(\tfrac{\cdot}{\widetilde{\alpha}_t}\big) \quad \text{in } \widetilde{B} - z. \tag{6.2}$$ potapr

For a given z, we want to use the large-deviation principle in (4.14) to derive the probability for the event in (6.2). Indeed, if we could write, for some new scale function $\beta(t) \to \infty$,

$$\widetilde{\alpha}_t = \alpha(\beta(t)) \qquad \text{and} \qquad h_t = \frac{H(\beta(t)/\alpha(\beta(t))^d)}{\beta(t)/\alpha(\beta(t))^d}, \tag{6.3}$$ picktildealpha

then $\widetilde{\xi}_t$ (with $z = 0$) is identical to $\overline{\xi}_{\beta(t)}$ defined in (4.9). Hence, an application of (4.14) with $\beta(t)$ instead of t gives that

$$\mathrm{Prob}\big(\widetilde{\xi}_t \approx \varphi \quad \text{in } Q_R\big) \approx \exp\Big\{ -\frac{\beta(t)}{\alpha(\beta(t))^2} I_R(\varphi) \Big\}, \tag{6.4}$$

where we recall the rate function $I_R(\varphi) = \int_{Q_R} J(\varphi(y))\,\mathrm{d}y$ from (4.15). Hence, the probability that ξ realizes the event in (6.2) in one of the microboxes decays exponentially on the scale $\beta(t)/\alpha(\beta(t))^2$. Pretending that the radius $\alpha(\beta(t))$ of such a microbox is quite small in comparison to t, their number is roughly equal to t^d, i.e., has exponential rate equal to $d\log t$. In order that we can expect at least one microbox in which the event in (6.2) is realised, we have to choose $\beta(t)$ according to the requirement

$$\frac{\beta(t)}{\alpha(\beta(t))^2} = d\log t, \tag{6.5}$$ betachoice

that is, $t \mapsto \beta(t)$ is the inverse of the map $t \mapsto t/\alpha(t)^2$, evaluated at $d \log t$. Furthermore, we have to restrict to potential shapes φ that satisfy $I_R(\varphi) < d$. Hence, with the choice of $\beta(t)$ in (6.5) and the choice of $\widetilde{\alpha}_t$ and h_t in (6.3), we can show (using a Borel-Cantelli argument, which involves some few technicalities) that, with probability one, if t is large enough, at least one $z \in B^{(t)}$ exists such that $\widetilde{\xi}_t \approx \varphi$ in Q_R. Hence, we obtain the lower bound

$$\frac{1}{t}\log U(t) \approx \lambda(B^{(t)}, \xi) \geq \lambda\big(\widetilde{B}, h_t + \tfrac{1}{\widetilde{\alpha}_t^2}\varphi\big(\tfrac{\cdot}{\widetilde{\alpha}_t}\big)\big) \approx h_t + \frac{1}{\widetilde{\alpha}_t^2}\lambda^{(\mathrm{c})}(Q_R, \varphi), \tag{6.6}$$

lambdascal

where we recall that $\lambda^{(\mathrm{c})}(Q, \varphi)$ is the principal Dirichlet eigenvalue of the operator $\Delta + \varphi$ in Q. Now we can summarise:

thm-asasy

Theorem 6.1 (Almost sure asymptotics of the total mass)**.** *Assume that the i.i.d. potential ξ satisfies Assumption (J) or, equivalently, Assumption (H). Then the following holds almost surely.*

(i) *If $\alpha(t) \to \infty$ as $t \to \infty$ (equivalently, if $\widetilde{\alpha}_t \to \infty$), then*

$$\frac{1}{t}\log U(t) = \frac{H(\beta(t)/\alpha(\beta(t))^d)}{\beta(t)/\alpha(\beta(t))^d} - \frac{1}{\alpha(\beta(t))^2}(\widetilde{\chi} + o(1)), \qquad t \to \infty, \tag{6.7}$$

ASlowerbound

where $\beta(t)$ is given in (6.5), and

$$\widetilde{\chi} = \inf_{\varphi \in \mathcal{C}(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} J(\varphi(y))\,\mathrm{d}y < 1} \Big[-\lambda^{(\mathrm{c})}(\mathbb{R}^d, \varphi)\Big]. \tag{6.8}$$

chitilde

(ii) *if $\alpha(t) \to 1$ as $t \to \infty$ (equivalently, if $\widetilde{\alpha}_t \to 1$), then $\widehat{H}(y) = \rho y \log y$ for some $\rho \in (0, \infty)$ and*

$$\frac{1}{t}\log U(t) = \frac{H(t)}{t} - \widetilde{\chi}_\rho + o(1), \tag{6.9}$$

ASlowerbound

where

$$\widetilde{\chi}_\rho = \inf_{\varphi \colon \mathbb{Z}^d \to \mathbb{R} \colon \frac{\rho}{\mathrm{e}}\sum_{z \in \mathbb{Z}^d} \mathrm{e}^{\varphi(z)/\rho} < d} \big[-\lambda(\mathbb{Z}^d, \varphi)\big]. \tag{6.10}$$

chitildeDE

(iii) *if $\alpha(t) \to 0$ as $t \to \infty$ (equivalently, if $\widetilde{\alpha}_t \to 0$), then (6.9) holds with $\rho = \infty$, and $\widetilde{\chi}_\infty = 2d$. MUSS MAN PRUEFEN*

Like for the moments in Theorem 4.7, there are two terms, which describe the absolute height of the potential in the 'macrobox' $B^{(t)}$, and the shape of the potential in the relevant 'microbox' $\widetilde{B}$, more precisely; the spectral properties of $\Delta^{\mathrm{d}} + \xi$ in that microbox.

The above heuristics reveal that the intermittent islands that we talked heuristically about in Remark 1.6 ($\widetilde{B}$ is one of them) have a radius of order $\widetilde{\alpha}_t$, and the precise value of the radius and the shape of the potential ξ in the island are determined by the characteristic variational formula in (6.8). Note that $\widetilde{\alpha}_t$ is logarithmic in t and hence much smaller than $\alpha(t)$, the spatial scale for the moments. A crucial feature is that the intermittent islands come out of a local optimisation procedure; they are actually the places with the highest potential values and a bit regularity. Therefore we will have a much closer look at *all* these islands in Section 7 in the light of spatial extreme-value statistical theory.

Remark 6.2. (Maximum of the potential.) A slightly different route instead of picking h_t deterministically as in (6.3) is to pick it randomly as the potential maximum $\max_{z \in B^{(t)}} \xi(z)$. Under Assumption (J), it is not too difficult to identify the asymptotics as

$$\max_{z \in B^{(t)}} \xi(z) = \frac{H(\beta(t)/\alpha(\beta(t))^d)}{\beta(t)/\alpha(\beta(t))^d} + o(\alpha(\beta(t))^{-2}), \qquad t \to \infty, \text{almost surely.} \tag{6.11}$$

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MUSS MAN PRUEFEN ◇

Rem-MaxPot

Remark 6.3. (Technical remark on the lower bound.) Actually, it is a bit nasty to use the idea of (2.26) for the box $B^{(t)}$ to justify that $\frac{1}{t}\log U(t)$ is lower bounded by the principal eigenvalue of $\Delta^{\rm d} + \xi$ in $B^{(t)}$ and to proceed with a further restriction to the microbox $\widetilde{B}$. Instead, starting from the Feynman-Kac formula in (2.1), one often uses a lower estimate by inserting the indicator on the event that the random path moves quickly to the box $\widetilde{B}$ (this is done at a negligible cost) and stays afterwards all the time until t in that box. The cost for doing the latter is $\exp\{-t\widetilde{\alpha}(t)^{-2}\lambda^{(\rm c)}(Q_R,\varphi)\}$ to high precision, which is seen from an application of (2.26) for $B = \widetilde{B}$. $\diamond$

owerBoundFK

Remark 6.4. (The upper bound in (6.7).) Like for the moment asymptotics, the proof of the upper bound in Theorem 6.1 is technically more involved and more abstract, since one has to take care of *all* the paths in the Feynman-Kac formula respectively *all* the subboxes of $B^{(t)}$ or sizes of all orders, not only some optimal ones. Some of the methods outlined in Section 5 are helpful also for the proof of the upper bound in (6.7).

Indeed, Lemma 5.4 on the spectral domain decomposition works directly on the principal eigenvalues and is precisely what one needs and gives the desired upper bound for the macrobox eigenvalue in terms of the maximum of the microbox eigenvalues. A Borel-Cantelli argument derives the asymptotics of this maximum, based on the upper tails of one eigenvalue, which one has gained from the asymptotics of the exponential moments of the eigenvalue in the course of the proof of the moment asymptotics in Theorem 4.7. See [BisKön01, GärKönMol00, HofKönMör06].

Certainly, also variants of Sznitman's method of enlargement of obstacles [Szn98] (see [Ant94]) are suitable to yield a proof in the cases of Brownian motion among Poisson obstacles and survival problems for the simple random walk, respectively.

Furthermore, also the discretisation technique for the Rayleigh-Ritz principle of Lemma 5.10 can successfully employed for deriving the corresponding upper bound, which has been demonstrated in the spatially continuous case (actually, being reduces to the discrete case in the course of the proof) for a rescaled potential in [MerWüt01a, MerWüt02], see also Section 8.4.2.

In principle, also Lemma 5.8 on the joint density of the local times of the random walk appears rather suitable for deriving proofs for almost sure upper bounds for $U_{B^{(t)}}(t)$, but has not yet been used for that, to the best of my knowledge. An application of this lemma shifts the difficulties to the analysis of the variational formula on the right-hand side of (5.10), and this is not much different from the Rayleigh-Rity formula from the principal eigenvalue of $\Delta^{\rm d} + \xi$ in $B^{(t)}$; therefore the benefit from Lemma 5.8 appears rather limited. Similar remarks apply to Dynkin's isomorphism, Lemma 5.9

Note however that the periodisation method of Lemma 5.3 is not suitable, since it works only for moments. $\diamond$

Supperbound

Remark 6.5. (Relation between the variational formulas χ and $\widetilde{\chi}$.) The variational formulas in (6.8) and (4.23) are in close connection to each other. In particular, it can be shown on a case-by-case basis that the minimizers of (6.8) are rescaled versions of the minimizers of (4.23) in the cases (B) and (AB), and they are even identical in the cases (SP) and (DE). This means that, up to rescaling, the optimal potential shapes in the annealed and in the quenched setting are identical. $\diamond$

chirelation

Remark 6.6. (Screening in one dimension.) In Remark 6.3, we say that a lower bound is obtained by requiring in the Feynman-Kac formula that the particle runs at high speed to the microbox $\widetilde{B}$, and that the cost of this is negligible. In dimensions $d \geq 2$, the negligibility is indeed always true as long as the potential is either $> -\infty$ everywhere or stays $> -\infty$ with sufficiently large probability, more precisely with a probability larger than the critical site-percolation threshold (see [Gri99] for a comprehensive treatment of percolation). In this case, it is no problem to find that there exists, with probability one, for any sufficiently large t, a path from the origin to that microbox along which the particle does not lose much mass and does not walk too long a way (here one has to make sure that the so-called chemical distance is not too long in comparison with the Euclidean distance). See [BisKön01] for details in the case of a bounded potential.

However, in dimension $d = 1$, these arguments do not work anymore, as there is alway only precisely one way from the origin to that microbox, and it may happen that the potential assumes to small (i.e., close to $-\infty$) values that the particle loses so much mass on the way that the contribution from this sprint is indeed not negligible. This effect is called *screening effect* in [BisKön01a], as the deep valleys screen the mass away from the high peaks. This effect appears as soon as the essential infimum of the potential is equal to $-\infty$ and its lower tails are too thick. See [BisKön01a] for a precise statement. $\diamond$

em-Screening

Remark 6.7. (Almost sure potential confinement.) Like for the moments, the above heuristics suggests that the shape of the potential ξ in the peaks, after appropriate shifting and rescaling, resembles the minimising shapes φ^* in the variational formula for $\widetilde{\chi}$ in (6.8) (for $\alpha(t) \to \infty$) or in (6.10) (if $\alpha(t) \to 1$), almost surely for large t. Furthermore, one can also conjecture that the solution $u(t, \cdot)$ resembles the corresponding eigenfunction of the operator $\Delta^{\rm d} + \varphi$ (if $\alpha \equiv 1$), respectively of $\Delta + \varphi$ if $\alpha(t) \to \infty$. This was indeed proved for the case (DE), the double-exponential distribution, in [GärKönMol07], see also Section 7.1. $\diamond$

lconfinement

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MPoissObstas

6.2 The spatially continuous case

6.2.1. *Brownian motion among Poisson obstacles.* Let us explain the heuristics once more in the simpler (and historically important) situation of a Brownian motion among Poisson obstacles, see Remark 2.9 and Section 4.5.1. For a proof and more details, we refer to [Szn98, Th. 4.5.1]. As in Section 4.5, we consider the potential $-\sum_i W(\cdot - x_i)$ with a standard Poisson point process $(x_i)_{i\in\mathbb{N}}$ with parameter ν and a nontrivial nonnegative measurable compactly supported cloud W. In oder to ease the notation, we slightly simplify the assertion.

We proceed as in Section 4.5 and try to get an optimal lower bound. For any (random) set $A \subset \mathbb{R}^d$ that is free of obstacles (which roughly means that $\omega(A) = 0$), we get a lower bound for the Feynman-Kac formula in (4.46) by inserting the indicator on the event that the Brownian motion travels to A within some small time interval of length $o(t)$ and then stays in A the remaining time of length $t(1 - o(1))$. Now the location of A must be chosen so well that A is not too far away from the origin (such that it is not too costly to travel there in short time) and that the motion does not lose too much mass on the way by traveling through the random potential. The first one is handled by searching for A only in a certain t-depending centred box $B^{(t)}$ (it suffices to take its radius as T times some carefully chosen poer of $\log t$), and the second can be proved by some percolation argument. Hence, the contribution of the short travel to A is negligible on the first-order scale. This means that the contribution comes only from the long stay in A, and hence we obtain the lower bound

$$U(t) \geq {\rm e}^{-t(\lambda_1(A)+o(1))}, \qquad t \to \infty,$$

provided there is such set A. More precisely, for this argument it must be guaranteed that, with probability one, for any sufficiently large t, there is an obstacle-free set A in $B^{(t)}$. Here a Borel-Cantelli argument is necessary, and this works if the probability for given set A to be obstacle-free (this is ${\rm e}^{-\nu|A|}$) is roughly equal to $1/|B^{(t)}| \approx t^{-d}$. The reason is that one has approximately t^d places where to put A, i.e., independent trials, and therefore the probability that one of them is obstacle-free, is of finite order. (Since we are only working on an exponential scale, there is quite some room for imprecisions.) Hence, the largest volume of A for which this Borel-Cantelli argument works is $|A| = \frac{d}{\nu}\log t$. Again, the optimal shape of A with given volume is a ball, whose radius therefore is equal to $\widetilde{r} = (\frac{d}{\omega_d \nu}\log t)^{1/d}$.

Hence,

$$\begin{aligned}\frac{1}{t}\log U(t) &\geq -(1+o(1)) \sup_{A:\,|A|=\frac{d}{\nu}\log t} \lambda_1(A)\\ &= -(1+o(1))\lambda_1(B_{\widetilde{r}}) = -(1+o(1))\widetilde{r}^{-2}\lambda_1(B_1)\\ &= -(\log t)^{2/d}(\widetilde{c}(d,\nu)+o(1)),\end{aligned}$$

where $\widetilde{c}(d,\nu)=\lambda_1(B_1)(\omega_d\nu/d)^{2/d}$.

6.2.2. *Gaussian potential.* The almost sure asymptotics of the total mass for a Hölder continuous Gaussian potential $V=(V(x))_{x\in\mathbb{R}^d}$ are identified in [GärKönMol00] as follows. The assumptions are as in [GärKön00], se Section 4.5.2. The covariance function B of the centred stationary Gaussian potential V is twice continuously differentiable, hence V can be assumed Hölder continuous with any parameter $\in(0,1)$. Let $L(h)=\sup_{t>0}(ht-H(t))$ denote the Legendre transform of the logarithm of the moment generating function $H(t)=\log\langle \mathrm{e}^{tV(0)}\rangle=\frac{1}{2}t^2\sigma^2$ with $\sigma^2=B(0)$, and define $(h_t)_{t>0}$ as solution to the equation $L(h_t)=d\log t$. Then the main result of [GärKönMol00] is

$$\frac{1}{t}\log U(t)=h_t-(\chi+o(1))\sqrt{h_t},\qquad t\to\infty,\ \text{almost surely,}\tag{6.12}$$

where again $\chi=(2\sigma^2)^{-1/2}\mathrm{tr}(\Sigma)$, and Σ^2 is the Hessian matrix of B at zero. Note the formal similarity to the moment asymptotics in (4.53). It is not difficult to prove that $h_t=(2d\sigma^2\log t)^{1/2}\approx\max_{x\in[-t,t]^d}V(x)$; it formally coincides with h_t in the heuristics in Section 6.1. Again, the first term in the asymptotics was earlier derived in [CarMol95]. The second term reflects the heuristics that the main contribution to $U(t)$ comes from a microbox in $[-t,t]^d$ with radius of order $h_t^{-1/4}$, where the potential V approaches the non-random parabolic shape h_tp, where $p(x)=1-\frac{1}{2\sigma^2}|\Sigma x|^2$, centred at the random localisation centre. The principal eigenvalue of $\Delta+h_tp$ is easily calculated to be $h_t-\chi\sqrt{h_t}$, which is the right-hand side in (6.12).

Interestingly, the peaks in the Gaussian potential have a parabolic shape, the description of which depends only on $B(0)$ and $B''(0)$, but not on $B^{(4)}(0)$. Indeed, one easily calculates that, for any site $x_0\in\mathbb{R}^d$, the variables $V(x_0)$, $V'(x_0)$ and $v=V''(x_0)-B''(0)V(x_0)/\sigma^2$ are independent Gaussians. In particular, $V(x_0)$ and $V''(x_0)$ are highly correlated, and large values of $V(x_0)$ enforce large values of $-V''(x_0)$. More precisely, given that V has a large local maximum $V(x_0)\approx h_t$ at x_0, then $V'(x_0)=0$ and $|v|=|V''(x_0)-B''(0)V(x_0)/\sigma^2|\ll h_t$ and therefore, in a neighbourhood of x_0,

$$V(x)\approx V(x_0)+\frac{1}{2}V''(x_0)(x-x_0)^2\approx h_t+\frac{1}{2}\Big(\frac{B''(0)}{\sigma^2}V(x_0)+v\Big)(x-x_0)^2\approx h-tp(x-x_0).$$

A Gaussian potential with much less regularity was considered in [Che12b].

MEHR DARUEBER!

7. Mass concentration

One of the fundamental questions about the PAM is the analysis of the important phenomenon of intermittency in the strongest possible sense, i.e., the phenomenon that the mass of the heat flow through the random potential is concentrated in a few small islands, which are time-dependent and randomly located. We discussed intermittency already on various levels of deepness and from different angels, see Remarks 1.6 and2.3.3, e.g. A rough understanding of this effect is already seen in the proof of the lower bound for the almost sure behaviour of the total mass (see Section 6), where we showed that certain islands give already a contribution that is asymptotically equal to the total mass, at least on the exponential scale that we looked at. Now, we want to go much deeper and show that the contribution from the complement of these islands is negligible.

This question is interesting in both the annealed and the quenched setting. Here, we concentrate on the quenched setting, where one has to consider all the islands in the entire space, in contrast to the annealed setting (see Remark 3.2), where we saw in Section 4 that only one island is relevant. In this setting, the concentration property was already discussed in Remarks 4.5 and 4.8.

The goal in this section is to explain how to find relatively small and few random subsets $A_1, \dots, A_{n_t}$ of $\mathbb{Z}^d$ such that, almost surely,

$$U(t) \sim \sum_{i=1}^{n_t} \sum_{z \in A_i} u(t,z), \qquad t \to \infty, \tag{7.1}$$

concentratio

in the sense of asymptotic equivalence. This is a strong assertion about *mass concentration* for the PAM. Certainly, the sets A_i should be equal to the sets on which the leading eigenfunctions of the Anderson operator $\mathcal{H}$ are concentrated, see Remark 2.3.3, and their distance to the origin should also play an important rôle.

Results of the type in (7.1) were derived in the literature in the cases of Brownian motion among Poisson obstacles [Szn98] (see Remark 7.2), for the double-exponential distribution [GärKönMol07], and for the Pareto distribution [KönLacMörSid09]. For the latter, concentration can be proved in an extremely strong sense: it turns out that n_t can be picked as $n_t = 2$, and the sets A_1 and A_2 are singletons. We will explain the case of the double-exponential distribution in Section 7.1 in greater detail and the case of the Pareto distribution in Section 7.2; for the case of Brownian motion among Poisson obstacles, we give a short explanation in Remark 7.2 and refer to [Szn98] for details.

It is conjectured that concentration holds even in the sense that *just one island* carries the total mass in the strong sense that the contribution from everywhere else is negligible with respect to the contribution coming from that single island. This concentration property cannot hold almost surely at every large (random) time t, since the islands change from time to time, but at each fixed large t, the concentration in just one island should hold with high probability. This was proved for the Pareto distribution in [KönLacMörSid09] and for the double-exponential distribution in [BisKönSan15+]. In Section 7.3 we will explain this and elements of the proof for the double-exponential distribution.

Note that such a strong concentration property makes it possible to describe the mass flow through the medium just in terms of the location of the single relevant island, and this opens the possibility to describe this mass flow as an process evolving in time just in terms of the stochastic $\mathbb{R}^d$-valued process that describes this location. Hence, one can then study dynamic properties like ageing. This will be discussed in Section 8.2.

sec-GeoInter

7.1 Geometric characterisation of intermittency

In this section we give a more precise formulation of (7.1) for the case of the double-exponential distribution introduced in Remark 4.11. In particular, we will describe the sets A_i and the typical shape of the potential ξ and of the solution $u(t,\cdot)$ inside these sets, almost surely. We follow [GärKönMol07], but slightly simplify some facts.

For simplification, we also assume that the parameter ϱ appearing in (4.38) is so large that, up to spatial shifts, the variational problem in (4.39) possesses a unique maximizer, which has a unique maximum [GärHol99]. By V_* we denote the unique maximizer of (4.39) which attains its unique maximum at the origin. We will call V_* *optimal potential shape*. Some crucial properties of the formula (4.39) are as follows. The operator $\Delta^{\mathrm{d}} + V_*$ has a unique nonnegative eigenfunction $w_* \in \ell^2(\mathbb{Z}^d)$ with $w_*(0) = 1$ corresponding to the eigenvalue $\lambda(V_*)$. Moreover, $w_* \in \ell^1(\mathbb{Z}^d)$ is positive everywhere on $\mathbb{Z}^d$.

One crucial object is the maximum $h_t = \max_{z \in B^{(t)}} \xi(z)$ of the potential in the macrobox $B^{(t)}$. We shall see that the main contribution to the total mass $U(t)$ comes from a neighbourhood of the set of best local coincidences of $\xi - h_t$ with spatial shifts of V_*. These neighbourhoods are widely separated from each other and hence not numerous. We may restrict ourselves further to those neighbourhoods in which, in addition, $u(t,\cdot)$, properly normalized, is close to w_*.

Denote by $B_R(y) = y + B_R$ the closed box of radius R centered at $y \in \mathbb{Z}^d$ and write $B_R(A) = \bigcup_{y\in A} B_R(y)$ for the R-box neighbourhood of a set $A \subset \mathbb{Z}^d$. In particular, $B_0(A) = A$.

For any $\varepsilon > 0$, let $r(\varepsilon, \varrho)$ denote the smallest $r \in \mathbb{N}_0$ such that

$$\|w_*\|_2^2 \sum_{x\in\mathbb{Z}^d\setminus B_r} w_*(x) < \varepsilon. \tag{7.2}$$

rchoice

Given $f\colon \mathbb{Z}^d \to \mathbb{R}$ and $R > 0$, let $\|f\|_R = \sup_{x\in B_R} |f(x)|$. The main result of [GärKönMol07] is the following.

main **Theorem 7.1.** *There exists a random time-dependent subset $\Gamma^* = \Gamma^*_{t\log^2 t}$ of $B_{t\log^2 t}$ such that, almost surely,*

$$(i) \qquad \liminf_{t\to\infty} \frac{1}{U(t)} \sum_{x\in B_{r(\varepsilon,\varrho)}(\Gamma^*)} u(t,x) \geq 1-\varepsilon, \qquad \varepsilon \in (0,1); \tag{7.3}$$

$$(ii) \qquad |\Gamma^*| \leq t^{o(1)} \quad \textit{and} \quad \min_{y,\widetilde{y}\in\Gamma^*\colon y\neq\widetilde{y}} |y-\widetilde{y}| \geq t^{1-o(1)} \qquad \textit{as } t\to\infty; \tag{7.4}$$

$$(iii) \qquad \lim_{t\to\infty} \max_{y\in\Gamma^*} \big\|\xi(y+\cdot) - h_t - V_*(\cdot)\big\|_R = 0, \qquad R>0; \tag{7.5}$$

$$(iv) \qquad \lim_{t\to\infty} \max_{y\in\Gamma^*} \Big\|\frac{u(t,y+\cdot)}{u(t,y)} - w_*(\cdot)\Big\|_R = 0, \qquad R>0. \tag{7.6}$$

Theorem 7.1 states that, up to an arbitrarily small relative error ε, the islands with centers in Γ^* and radius $r(\varepsilon,\varrho)$ carry the whole mass of the solution $u(t,\cdot)$. (In other words, in terms of (7.1), $n_t = |\Gamma^*| = t^{o(1)}$, and the A_i are the R-neighbourhoods of the sites in Γ^*.) Locally, in an arbitrarily fixed R-neighbourhood of each of these centers, the shapes of the potential and the normalized solution resemble $h_t + V_*$ and w_*, respectively. The number of these islands increases at most as an arbitrarily small power of t and their distance increases almost like t.

The main strategy of [GärKönMol07] is not based on the eigenvalue expansion in Remark 2.3.1, since it is difficult to handle the possible negativity of the eigenfunctions at zero. Instead, a strategy is developed that works exclusively with *principal* eigenfunctions of $\Delta^{\rm d} + \xi$ in local neighbourhoods of high exceedances of the potential, after destroying the quality of the eigenvalues in all the other islands. One crucial point is the proof of the exponential localisation of the corresponding eigenfunctions using a decomposition technique for the paths in the Feynman-Kac representation for these principal eigenfunctions (called *probabilistic cluster expansion* in [GärKönMol07]). There is no control on the differences between any two of the top eigenvalues, but it is shown that the concentration centres of these eigenfunctions have mutual distance $t^{1-o(1)}$ from each other. This in turn implies that there are not more than $t^{o(1)}$ of them, and therefore there must be, somewhere close to the top of the spectrum, some gap of minimal size $t^{-o(1)}$. This gap played a crucial rôle in the proof of the exponential localisation.

Remark 7.2. (Brownian path concentration among Poisson obstacles.) In the case of Brownian motion among Poisson obstacles (see Remark 2.4) an assertion was proved that is closely related to Theorem 7.1. In fact, this assertion is formulated in terms of the behaviour of the motion among the obstacles rather than in terms of mass concentration of its occupation probabilities. The main result here can be roughly formulated as follows. Almost surely, as $t\to\infty$, there are $n_t = t^{o(1)}$ balls $A_1,\dots,A_{n_t} \subset \mathbb{R}^d$ of radius $\mathrm{const}\,\widetilde{\alpha}(t)$ with mutual distance $t^{1-o(1)}$ such that the Brownian motion of the Feynman-Kac formula does the following with probability tending to one under the transformed path measure $Q_{\xi,t}$ defined in (2.13). The motion arrives, after some deterministic diverging time $\ll t$, at one of these balls and does not leave it anymore up to time t.

These balls are characterised in terms of the property that they optimise, within the macrobox $B_{t\log^2 t}$, the sum of the principal eigenvalue of $\Delta+V$ in that region and a certain quantity that measures the exponential cost for a Brownian motion to travel to that region through the random potential. The latter can be formally written as a *Lyapounov exponent*, but there exists no explicit formula for it; its existence is based on subadditivity. $\diamond$

Rem-pathconc

Both results in [GärKönMol07] and [Szn98] do not provide much control on the location of the concentration centres of the leading eigenfunctions, i.e., of the sets A_i in (7.1), nor on the minimal number n_t of the relevant islands that are needed to prove (7.1). One of the reasons is that both work in the almost-sure setting, and then it is difficult to find a criterion of the "quality" of a local region that is fine enough that the difference between the best and the next-to-best island is so visible that one can show that the contribution from the latter is negligible with respect to the contribution from the best. One way to overcome this problem is to turn to heavy-tailed distributions, where the difference between the two largest potential values is huge (see Section 7.2), or to work in the setting of convergence in probability, where limiting distributions can be identified with the help of extreme-value analysis (see Section 7.3).

sec-NoExpMom

7.2 Potentials without exponential moments: a 'two-cities theorem'

In Remark 4.10 we already saw that, for heavy-tailed potentials, the structure of the intermittent islands is the simplest, as they are just singletons. This concerns the class (SP) of Section 4.4, which contains the double-exponential distribution with parameter ρ. Strictly speaking, (SP) contains only distributions with finite positive exponential moments, but it can phenomenologically easily be extended to even heavier-tailed distributions like distributions with *Weibull tails*, where $\mathrm{Prob}(\xi(0) > r) \approx \exp\{-r^\gamma\}$ with $\gamma \in (0,1)$, and the *Pareto distribution*, where $\mathrm{Prob}(\xi(0) > r) = r^{-\alpha}$ for $r \in [1,\infty)$ with a parameter $\alpha > d$. All these distributions have the two characteristic properties:

- If a sum of many independent random variables with this distribution is conditioned to be large, then, with very high probability, just one summand of them is large, and all the others are of finite order only,
- The difference between the largest and the next-to largest of a great number of independent such random variables is huge.

These properties makes them a perfect object to study with respect to concentration properties of the PAM, since they suggest that their proof should be particularly easy on the technical side, and this indeed turns out to be true. We cannot talk about moment asymptotics of the total mass $U(t)$ anymore, but we can still talk about the distribution of the solution $u(t,\cdot)$ and its limiting concentration properties, either in the almost sure sense or in the sense of convergence in distribution. The distribution with the strongest pronounciation of the two above properties is the Pareto distribution. Indeed, for the potential ξ Pareto-distributed, a very strong form of the concentration property in (7.1) can be proved: (7.1) is true with $n_t = 1$ and a singleton A_1, if the limit is understood in probability, and with $n_t = 2$ and two singletons A_1 and A_2 almost surely. This is the strongest assertion possible on the long-time behaviour of the PAM, and it opens up the possibility to analyse properties of the entire heat flow that is described by the process $(u(t,\cdot))_{t\in(0,\infty)}$ just in terms of the concentration site as a stochastic process in t. In particular, ageing properties can be studied, see Section 8.2 below.

Let us briefly summarize what has been achieved for the PAM with thick-tailed potentials. Since there is a recent survey [Mör11] on this special aspect of the PAM, we abstain from an extensive formulation.

The study of the PAM with thick-tailed potentials was initiated in [HofMörSid08], where almost sure and distributional limit theorems for the total mass $U(t)$ are derived for the Weibull and the

Pareto case. We discuss the Pareto distribution. Here, it is proved that

$$\Big(\frac{t}{\log t}\Big)^{\frac{-d}{\alpha-d}}\times\frac{1}{t}\log U(t)\Longrightarrow Y,\qquad\text{where }\mathbb{P}(Y\leq y)=\exp\{-\theta y^{d-\alpha}\},\tag{7.7}$$

ParetoU(T)

and θ is some explicit constant. Note that this is an assertion about distributional convergence. Furthermore, explicit almost sure liminf and limsup results for the logarithm of $\frac{1}{t}\log U(t)$ are derived. Note that the limiting distribution in (7.7) is the Fréchet distribution, one of the three famous limiting distributions for the maximum of i.i.d. random variables. Hence, the assertion of (7.7) is very much in line with the understanding that all the leading eigenfunctions in the expansion (2.22) are delta-like functions with extremly high values, and therefore $U(t)$ is approximately equal to the maximum of a large number of i.i.d. Pareto-distributed random variables. Since it is known that the difference between the largest and the second-largest of such random variables is huge, one can hope that this huge difference can be used to show that the main contribution ot $U(t)$ comes just from *one* of these values, i.e., from one site, and this hope is indeed not disappointed.

In the follow-up paper [KönLacMörSid09], techniques from [GärKönMol07] were added to prove the above scenario. More precisely, it was proved that there is a stochastic process $(Z_t)_{t\in(0,\infty)}$ in $\mathbb{Z}^d$ such that

$$U(t)\sim u(t,Z_t)\qquad\text{as }t\to\infty\text{ in probability.}\tag{7.8}$$

Paretoconc

This is the announced strong form of (7.1). An informal description of the site Z_t is as follows. Consider the function

$$\Psi_t(z)=\xi(z)-\frac{|z|}{t}\log\frac{|z|}{2det},\qquad z\in\mathbb{Z}^d,t>0,$$

then $\mathrm{e}^{t\Psi_t(z)}$ is roughly equal to the contribution to the Feynman-Kac formula in (2.1) coming from a path that quickly runs to the site z and stays in z for the rest of the time until t. (The first term is the potential value that is attained for $\approx t$ time units, and the second is the probability to go for a distance $|z|$ in $\approx o(t)$ time units.) Then Z_t is defined as the site that maximises Ψ_t. In particular, $\Psi_t(Z_t)=\max_{z\in\mathbb{Z}^d}\Psi_t(z)\approx\frac{1}{t}\log U(t)$. This description lies at the heart of the heuristics in Remark 2.3.3 and improves the idea outlined in Section 7.1.

Remark 7.3. (Almost sure concentration.) The asymptotics in (7.8) cannot be true almost surely. In this case, t would be a random time and would also sooner or later attain a value that lies in a time interval during which the dominant potential peak wanders from one location to another one. Such phases of wandering of the overwhelming mass from one 'city' to the next one occur, since the horizon increases as t increases, and the maximisation of the field takes place over larger and larger areas. However, in [KönLacMörSid09] it is proved that the main mass is concentrated in no more than *two* sites at any large time t, almost surely. This interpretation gave this section and the paper [KönLacMörSid09] their titles. $\diamond$

Rem-ASConc

The proof of (7.8) relies on spectral theory and on techniques from the theory of order statistics for i.i.d. random variables and implicitly on the theory of Poisson point convergence, which was later detailed and further exploited in [MörOrtSid11]. We will explain this mechanism in Section 7.3 in greater generality and give here only the main results of [MörOrtSid11] and some comments.

The description of the entire process $(Z_t)_{t\in(0,\infty)}$ is identified as follows. In [MörOrtSid11] it is proved that there is a (time-inhomogeneous) Markov process $(Y_t^{(1)},Y_t^{(2)})_{t\in(0,\infty)}$ in $\mathbb{Z}^d\times\mathbb{Z}$ such that, as $T\to\infty$,

$$\Big(\Big(\frac{\log T}{T}\Big)^{\frac{\alpha}{\alpha-d}}Z_{tT},\Big(\frac{\log T}{T}\Big)^{\frac{d}{\alpha-d}}\xi(Z_{tT})\Big)_{t\in(0,\infty)}\Longrightarrow\Big(Y_t^{(1)},Y_t^{(2)}+\frac{d}{\alpha-d}|Y_t^{(1)}|\Big)_{t\in(0,\infty)}.$$

Here $Y_t^{(1)}$ and $Y_t^{(2)}$, after rescaling, are the maximizer and next-to maximiser of Ψ_t. We also see that $(\frac{\log T}{T})^{\frac{\alpha}{\alpha-d}}Z_T$ converges in distribution to $Y=Y_1^{(1)}$, hence, the relevant area (called 'relevant macrobox' in Section 6) has a diameter of order $\gg (\frac{T}{\log T})^{\frac{\alpha}{\alpha-d}}$, which is much larger than in the case of potentials with finite exponential moments.

Remark 7.4. (Exponential and Weibull distribution.) Another interesting potential distribution that turns out to phenemonologically lie in the class (SP), is the *exponential distribution*, $\mathrm{Prob}(\xi(0) > r) = \mathrm{e}^{-r}$ for $r \in (0, \infty)$. This distribution is considered in [LacMör12], and it is found that a concentration property in one single site takes place as well. First, like for the Pareto distribution in [HofMörSid08], some distributional and almost sure liminf and limsup results for $\frac{1}{t}\log U(t)$ are given in [LacMör12]. Furthermore, it is shown that the point process

$$\frac{1}{U(t)} \sum_{z\in\mathbb{Z}^d} u(t,z)\delta_{z/r_t} \qquad \text{with } r_t = \frac{t}{\log\log t},$$

converges towards δ_Y, where Y is an $\mathbb{R}^d$-valued random variable with i.i.d. coordinates with exponential distribution with uniform random sign. According to [LacMör12], the analogous assertion for the Weibull distribution can be formulated and proved in the same way. The details are not given there, but in [SidTwa14]. ◇

Rem-ExpWeib

Remark 7.5. (Eigenvalue order statistics.) For heavy-tailed potentials ξ, which we consider in this section, the largest eigenvalues in a large box are just equal to the highest potential values, up a rather small error, and the corresponding island consists just of the single site in which the extreme potential value is located within a large box with diameter of a certain asymptotic size. (Equivalently, this site maximises in $\mathbb{Z}^d$ the sum of the potential value and a t-dependent term that measures its distance to the origin, see the proofs in [KönLacMörSid09].) Hence, the mass concentration in one site can relatively easily be derived from an assertion about the asymptotic location of all the top eigenvalues in a large box and the height of the eigenvalues, in the sense of a convergence of the point process in the 'space-spectrum plane' for the rescaled eigenvalues and locations. This is implicit in [KönLacMörSid09], [MörOrtSid11] and [LacMör12] and will be detailed in Section 7.3 below. More precise information about the eigenvalues and the eigenfunctions (i.e., their convergence towards delta-like functions) is given in [Ast08, Ast12, Ast13], see also [ComGerKle10] for a Poisson process convergence of the eigenvalues in the spatially continuous setting with a Poisson potential. ◇

erStatPareto

7.3 Concentration in one island for case (DE)

ec-OneIsland

As we argued in Remark 2.3.3, the main contribution to the total mass $U(t)$ should come from just one island of eigenfunction concentration that optimises the relation between the high value of the corresponding eigenvalue and the proximity of the region to the initial site, the origin. For the class (SP) of thick-tailed potentials, this idea was turned into rigorous proofs, as we explained in Section 7.2. However, this is a quite simple case, as the islands are just singletons, the eigenfunctions are strongly delta-like, and the differences between the largest and second-largest potential value is huge. The classes (DE), (AB) and (B) are much more interesting, since the intermittent islands carry some non-trivial structure that is asymptotically given by deterministic variational formulas. In this section, we explain the concentration phenomenon in just one island in the interesting case (DE), and we outline a proof that is inspired by ideas from the theory of Anderson localisation. We follow [BisKön13].

The main idea is to achieve some control on the differences between subsequent macro eigenvalues close to the top of the spectrum of $\mathcal{H} = \Delta^{\mathrm{d}} + \xi$ in terms of an *order statistics*. Furthermore, one must show that, up to some small error, any macro eigenvalue/eigenfunction pair is equal to a principal pair in a local microbox with Dirichlet boundary condition. Then the principal micro eigenvalues are practically independent and should satisfy an order statistics, provided their distribution lies in the max-domain of attraction of one of the three famous max-distributions. For having this, one needs an assertion of the form

$$\begin{aligned} &\mathrm{Prob}(\lambda_1(B_{\log L},\xi)\geq a_L)=\frac{1}{L^d}\\ \Longrightarrow\qquad &\mathrm{Prob}(\lambda_1(B_{\log L},\xi)\geq a_L+sb_L)=\frac{\mathrm{e}^{-s}(1+o(1))}{L^d}\text{ for any }s\in\mathbb{R}, \end{aligned}\tag{7.9}$$

orderstatcore

as $L\to\infty$, for some scale functions a_L and b_L, where we recall that $B_L=[-L,L]\cap\mathbb{Z}^d$ and $\lambda_k(B,\xi)$ the k-th largest eigenvalue of $\mathcal{H}_B$ for $B\subset\mathbb{Z}^d$. (For technical reasons, we took the microbox of size $\log L$, which should be seen as a large, fixed radius.)

For the potential given precisely by (4.38), this implication is shown to be true with the values

$$a_L=\rho\log\log L^d-\chi+o(1)\qquad\text{and}\qquad b_L=\frac{d}{\rho}\frac{1}{\log L}.$$

(The main point in the proof of (7.9) is that the behaviour of the potential distribution in (4.38) under shifts $\xi\mapsto\xi+c$ is rather easily identified explicitly.) This is the core of a proof not only of an eigenvalue order statistics in the domain of attraction of the Gumbel distribution, but even of the convergence of the point process of rescaled eigenvalues, together with the rescaled localisation centres of the corresponding eigenfunctions, towards an explicit Poisson point process. A (slightly imprecise) formulation is the following. Let $v_B^{(k)}$ denote an ℓ^2-normalised eigenfunction of $\mathcal{H}$ corresponding to $\lambda_k(B,\xi)$, such that $v_B^{(1)}$ is positive and $(v_B^{(k)})_{k\in\mathbb{N}}$ is an orthonormal basis of the sequence space $\ell^2(B)$.

n-orderstat **Theorem 7.6** (Eigenvalue order-statistics). *For each $L\geq 1$ there is a sequence $X_1^{(L)},X_2^{(L)},\dots$ of random sites in B_L and a number $a_L=\rho\log\log L^d-\chi+o(1)$ such that, for any $R_L\to\infty$,*

$$\sum_{z\colon |z-X_k|\leq R_L}\big|v_{B_L}^{(k)}(z)\big|^2\underset{L\to\infty}{\longrightarrow}1\tag{7.10}$$

in probability, for each $k\in\mathbb{N}$. Moreover, the law of the point process

$$\sum_{k\in\mathbb{N}}\delta_{(X_k^{(L)}/L,(\lambda_k(B_L,\xi)-a_L)/\log L)}\tag{7.11}$$

eigenvPointPr

converges weakly towards a Poisson point process on $B_1\times\mathbb{R}$ with intensity measure $\mathrm{d}x\otimes\mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$.

In particular, any two neighbouring eigenvalues have distance of order $1/\log L$ to each other, the rescaled eigenvalue $(\lambda_1(B_L,\xi)-\rho\log\log L^d+\chi)\log L$ converges weakly towards a standard Gumbel variable G_1, and the eigenfunction localisation centres converge towards a standard Poisson process, after rescaling with the box diameter. Even more, the second rescaled eigenvalue converges towards the sum of G_1 with another independent standard Gumbel variable G_2, and so forth for all the other rescaled eigenvalues. In particular, the gaps between two subsequent eigenvalues are in distribution equal to $1/\log L$ times some explicit random variables. Note that all these convergences are in distribution.

Remark 7.7. (Which max-domain?.) Note from Section 7.2 that, in the case (SP), the local eigenvalues turned out to lie in the max-domain of a Fréchet distribution (a one-sided max-domain that appears if the distribution of maxima is close to the boundary of the support), while for the case (DE) it is the Gumbel distribution, which arises if the maxima have a certain distance to the boundary. On base of the work in [BisKön13], we conjecture that a similar picture can be proved for the distributions in the classes (B) and (AB), at least for some prominent representatives. In particular, we conjecture that in both these cases the eigenvalues lie in the max-domain of a Gumbel distribution. However, this should hold in the case (B) only for $\gamma>0$, where the potential distribution does not feel the boundary of its support immediately. The case $\gamma=0$ should lead to the Weibull distribution.

One important difficulty that one has to overcome for proving (7.9) is that, for bounded potentials, the increase from $\{\lambda_k(B_{\log L},\xi)\geq a_L\}$ to $\{\lambda_k(B_{\log L},\xi)\geq a_l+sb_L\}$ does not predominantly come (like for the double-exponential distribution) from making each single potential site greater by the amount s_L, but from making the radius of the ball larger in which the potential gives the main contribution.$\diamond$

em-maxdomain

Remark 7.8. (Some remarks on the proof.) One of the main technical points in the proof of Theorem 7.6 is the proof of eigenfunction localisation if the eigenvalue is large enough and sufficiently far from all the other large eigenvalues, i.e., has a sufficiently large spectral gap. Here an argument is employed that shows that the eigenfunction remains practically unchanged if the potential is shifted to $-\infty$ outside of a neighbourhood of the local island of sites that give extremely high potential values and carries some mass $\geq 1/2$ of the eigenfunction. For this, one first shows that

$$\frac{\partial}{\partial \xi(z)}\lambda_k(B^{(t)},\xi) = v_{B^{(t)}}^{(k)}(z)^2, \qquad z \in B^{(t)}. \tag{7.12}$$

eigenvderi

Furthermore, it turns out that the process

$$\Big(v_{B^{(t)}}^{(k)}(Y_n)\prod_{l=1}^{n}\frac{2d}{2d-\lambda_k(B^{(t)},\xi)-\xi(Y_l)}\Big)_{n\in\mathbb{N}}$$

is a martingale, where $(Y_n)_{n\in\mathbb{N}}$ is a discrete-time simple random walk. Since the quotient in the product lies in $(1,\infty)$ and is bounded away from 1 as long as Y runs outside the highest peaks, this property makes it quite easy to show that the eigenfunction decays exponentially fast away from the area of high exceedances of ξ. Using this in (7.12), we see that the eigenvalue does not change if the potential is drastically changed there. This argument in particular proves the phenomenon of Anderson localisation at the top of $\mathcal{H}_{B^{(t)}}$. $\diamond$

envorderstat

Remark 7.9. (Poisson point process convergence of eigenvalues.) In the community of random Schrödinger operators, the joint distribution of the eigenvalues and the concentration centres of the eigenfunctions is of great interest, as it gives important information about Anderson localisation. An early example is [Mol81], where the eigenvalues of the one-dimensional Anderson Hamiltonian on $\mathbb{Z}$ are shown to have a Poisson process structure. An important progress was made in [Min96], where this kind of assertion is extended to the d-dimensional setting in $\mathbb{R}^d$. The main value of [Min96] was the introduction of a flexible estimate that establishes the existence of a gap between two subsequent eigenvalues, an estimate that is now called the *Minami estimate*. The first result on the convergence of point processes of both the eigenvalues and the concentration centres of the eigenfunctions is [KilNak07]. An extension of the results of [KilNak07] to some discrete systems was given in [**?**]. The currently strongest available results are [GerKlo11a] and [GerKlo11b], where [GerKlo11a] works in the bulk of the spectrum and [GerKlo11b] close to the top.

The latter works assume that the potential distribution has a bounded density and that Anderson localisation holds in the spectral interval considered, and they make a couple of assumptions on the validity of Wegner and Minami estimates, which are known to hold for large classes of random operators. They pick a growing number of eigenvalues in the interval considered and corresponding eigenfunction centres and show that their point process, after 'unfolding', converges towards a standard Poisson process with intensity measure $\mathrm{d}x \otimes \mathrm{d}\lambda$. More precisely, they do not look at a rescaling $(\lambda_k - a_L)b_L$ for box-depending quantities a_L and b_L, but on the unfolded eigenvalues

$$\big[N(\lambda_k) - N(\lambda_0)\big]|\Lambda_L|,$$

where $N\colon \mathbb{R} \to [0,1]$ is the integrated density of states (IDS), see Remark 2.3.6, and λ_0 is a certain value in the spectrum of the global operator $\mathcal{H}$ that satisfies some additional properties. Since λ_0 is assumed to lie in the interior of the support of the IDS, also [GerKlo11b] makes assertions only for eigenvalues that are substantially away from the boundary of the spectrum (however, it contains also a restricted assertion precisely at the boundary for the one-dimensional operator $\mathcal{H} = \Delta^{\mathrm{d}} + \xi$).

All the assertions proved in [GerKlo11a] and [GerKlo11b] do not come from any kind of maximisation and therefore have nothing to do with extreme-value analysis, nor there are assertions about the shape of the potential inside the islands. A comparison to the approach described above is not immediately clear. $\diamond$

Having now control on the top eigenvalues and their eigenfunctions, we can turn to the Cauchy problem in (1.1) and couple the eigenvalue order statistics with time, see [BisKönSan15+]. It turns out that L has to be picked of order $t/\log t \log\log\log t$ in order to match the scales. Indeed, put

$$\Psi_{L,t}(\lambda, r) = \frac{t}{r_L}(\lambda - a_L)\frac{1}{b_L} - \frac{|z|}{L}, \qquad \text{where } r_L = L\log L \log\log\log L,$$

and pick $k \in \mathbb{N}$ as the maximiser of $k \mapsto \Psi_{L,t}(\lambda_k(B_L), X_k)$ and put $Z_{L,t} = X_k/L$. As we explained above for the Pareto distribution, the two terms of $\Psi_{L,t}$ describe the exponential rate for the gain of a large eigenvalue and the distance to the origin, respectively. Then, with $L = L_t = t/\log t \log\log\log t$,

$$\lim_{t\to\infty} \frac{1}{U(t)} \sum_{z\colon |z - LZ_{L_t,t}| \leq R_t} u(t,z) = 1 \qquad \text{in probability,} \tag{7.13}$$

for any $R_t \gg \log t$. Furthermore, as $L \to \infty$, the process $(Z_{L,sr_L})_{s\in[0,\infty)}$ converges towards the process of maximisers of the map $z \mapsto t\lambda - |z|$ over a Poisson process on $B_1 \times (0,\infty)$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$. (7.13) specifies the concentration property of (7.1) for the double-exponential distribution.

The proof of (7.13) needs quite some technical work, since the gap between the centred box with radius of order $t/\log t \log\log\log t$ (in which the eigenvalue order statistics holds and therefore the control on their gaps and the eigenfunction localisation is perfect) and the outside of the box with radius of order $t\log^2 t$ (where rough arguments suffice to show that this region is negligible) must be closed. Additional work and a slightly different formulation are necessary if the convergence in probability should be strengthened to almost sure convergence.

8. Further refined questions

Let us survey a number of questions around the PAM that go beyond the basic questions that we have treated so far.

8.1 Beyond logarithmic asymptotics, and confinement properties.

The moment asymptotics of the solution of the PAM in Theorem 4.7 describes just the two leading terms, more precisely, the logarithmic asymptotics. Here we discuss what can be said about the next terms. This is intimately connected with a closer analysis of the behaviours of those realisations of the path and of the potential that give the main contribution to the moments, i.e., with the confinement properties that we briefly mentioned in Remarks 4.5 and 4.8.

First we recall, on a heuristic level, the well-known (refined) Laplace method in a simple example. The large-t asymptotics of the integral $\int_0^1 \mathrm{e}^{tf(x)}\,\mathrm{d}x$ for some continuous function $f\colon [0,1] \to \mathbb{R}$ can be described just as $\mathrm{e}^{t(\max_{[0,1]} f + o(1))}$, but one can also give much more precise asymptotics of the integral, and one can closer describe the set of x that give the biggest contribution to the integral, under more restrictive assumptions on f concerning its shape close to the maximiser(s). Usually, one assumes that f possesses precisely one minimiser $x^* \in (0,1)$, and f is twice continuously differentiable in a neighbourhood of x^*. Then one can use a Taylor expansion to approximate

$$f(x) = f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^* + o(1)) \approx \max_{[0,1]} f - \frac{(x - x^*)^2}{2\sigma^2}, \qquad x \to x^*,$$

where $\sigma^2 = -1/f''(x^*) \in (0,\infty)$. Using this for x in a $O(1/\sqrt{t})$-neighbourhood of x^* gives the more precise asymptotics

$$\begin{aligned}\int_0^1 \mathrm{e}^{tf(x)}\,\mathrm{d}x &\approx \mathrm{e}^{t\max_{[0,1]}f}\int_{x^*-R/\sqrt{t}}^{x^*+R/\sqrt{t}} \mathrm{e}^{-t\frac{(x-x^*)^2}{2\sigma^2}}\,\mathrm{d}x = \mathrm{e}^{t\max_{[0,1]}f}\int_{-R}^{R}\mathrm{e}^{-\frac{y^2}{2\sigma^2}}\,\frac{\mathrm{d}y}{\sqrt{t}}\\ &\approx \mathrm{e}^{t\max_{[0,1]}f}\sqrt{\frac{2\pi\sigma^2}{t}},\end{aligned} \tag{8.1}$$

LapPrincPrec

where R is a large auxiliary parameter that is sent to ∞ at the end of the proof. Indeed, the asymptotics in (8.1) are precise up to equivalence, i.e., up to a factor of $1+o(1)$. Both asymptotics, the exponential one and the one more precise one from (8.1), are known as the *Laplace principle*. Besides that this method brings the second term of the asymptotics to the surface, it also strongly specifies the region that gives the main contribution to the integral of e^{tf}, namely an interval of size $O(1/\sqrt{t})$ around the maximiser x^*. The decisive inputs in the method are the uniqueness of the minimizer of the characteristic variational problem (here x^* for $\max f$) and a smooth behaviour of the functional (here f) in a neighbourhood of the maximiser. There are a number of abstract and high-dimensional versions of this methods in combination with large-deviation theory; we only want to mention [Bol89].

It is now tempting to try to apply the same idea to the study of the moments of the total mass of the PAM, to derive more precise asymptotics and to gain more insight in the critical behaviour of the path in the Feynman-Kac formula (2.1) or in the behaviour of the potential. More precisely, as we already mentioned in Remarks 4.5 and 4.8, one conceives the moments of $U(t)$ as an exponential moment of some functional whose maximiser and behaviour close to the maximiser is smooth, and derives a high-dimensional variant of the above idea. Since, by the Feynman-Kac formula, the moments of $U(t)$ can be seen as exponential moments in a two-fold way (for the path and for the potential), there are also basically two ways to apply this idea. The path-wise version is usually carried out by a Girsanov transformation, while the potential-wise approach needs less standard means. The first approach is strongly linked with the variaitonal formula in (4.31), the second with the one in (4.23).

There is an analytic difficulty that has to be handled prior to the application of probabilistic estimates: One must know that the minimisers of (4.31) are approachable in the topology in which one wants to prove the law of large numbers for the rescaled local times. In other words, one needs a statement like 'if a sequence of admissible functions is such that its values of the target function approach the minimum, then (at least along a subsequence) this sequence converges to some shift of the minimiser (in the topology that one would like to work with probabilistically).' Proving such a statement is by far not trivial and must be done on a case-by-case basis.

This programme has been carried out for the PAM in the discrete-space setting for range problem (see Example 2.4) in $d=2$ [Bol94] and for the double-exponential distribution in any dimension [GärHol99], and in the continuous-space setting for the Brownian motion among Poisson obstacles in $d=2$ [Szn91] and, at least partly, in $d\geq 3$ [Pov99]. The works in [Szn91] and [Pov99] are sometimes called the *Brownian confinement property*, see Remark 4.8. The main motivation there and in [Bol94] was to gain a better understanding of the behaviour of those path behaviours that give the main contribution to the moments of the PAM, while in [GärHol99] the authors wanted to understand the first asymptotic term that depends on the initial condition in (1.1).

Let us here describe this transformation in the discrete-space setting with the potential ξ double-exponentially distributed and follow [GärHol99].

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sec-Ageing 8.2 Time-correlations and ageing

The main goal in the study of the PAM is of course the description of the heat flow through the random potential as a stochastic process in time, i.e., the description of the entire process $(u(t,\cdot))_{t\in(0,\infty)}$ of solutions for one realisation of the potential ξ. This is certainly a formidable task, which can be thought of on various levels of deepness. Recently, there were some first serious attempts to attack this problem. These concern (see Section 7.2) the class (SP), where concentration is known to occur in just one site Z_t in the sense of convergence in probability, and in two, if one considers the almost-sure setting, and the class (DE), see Section 7.3. In a sense, the description of the heat flow is reduced to the description of just the process $Z = (Z_t)_{t\in(0,\infty)}$ of the sites where the overwhelming mass of the systems sits. Now one can study *ageing properties* of the PAM in terms of ageing properties of this process of concentration sites.

Generally speaking, *ageing* is the phenomenon that the most prominent, drastic changes of the system occur after longer and longer time periods. Hence, the observer is in principle able to say how old the system is if he can measure the time period that passes between two changes. The system sits for a long time in some well-defined location and leaves it afterwards in order to quickly move to another one. For the PAM, the most obvious expression of ageing is of course the localisation behaviour of the main part of the total mass of the solution $u(t,\cdot)$, which we roughly explained in Remark 3.4. The concentration area is a function of time and makes large macroscopic jumps in short time since the search horizon for an optimal region increases with time, and better and better islands appear at the horizon, if it widens and widens. Describing this ageing picture and its main characteristics, like the time and length scales, is one of the main tasks in the study of the PAM.

This was carried out in [MörOrtSid11] in the case of the Pareto distribution, i.e., in the most heavy-tailed distribution of the class (SP), see also the survey [Mör11]. Indeed, it is shown there that Z ages in the sense that, on some time scale $s(t)$, the probability that Z makes a jump between time t and time $t+\theta s_t$, tends, as $t\to\infty$, to some quantity that approaches one for $x\to\infty$ and to zero for $x\to-\infty$. More precisely, it is shown that

???

In [MörOrtSid11], ageing of the PAM with Pareto-distributed potentail is also shown in a second convincing sense as follows.

...

One of the most popular definitions of ageing is in terms of correlations. A process $Y=(Y_t)_{t\in[0,\infty)}$ is said to satisfy *correlation ageing* if, for some scale functions $s_1(t)$ and $s_2(t)$, tending to infinity as $t\to\infty$,

$$\lim_{t\to\infty}\frac{\operatorname{cov}\left(Y(t),Y(t+s_i(t))\right)}{\operatorname{Var}(Y(t))\operatorname{Var}(Y(t+s_i(t))}=\begin{cases}0 & \text{if } i=1,\\ 1 & \text{if } i=2.\end{cases}$$

It appears unclear in which way this definition makes any intuitive sense for Y equal to the process of total masses, $(U(t))_{t\in(0,\infty)}$, e.g., even though a proof of correlation ageing seems to be within reach for a number of interesting potentials. However, it does make a lot of sense to study time correlations of the solution to the PAM for many choices of two time instants, and this line of research has been initiated in [GärSch11a].

8.3 Long-range correlated potential

One of the main questions in the study of the PAM is how the spatial correlations that are induced by the presence of the Laplace operator affect the correlation length of the potential ξ and how this is reflected in the solution $u(t,\cdot)$ of the PAM. Here we want to work with a non-rigorous notion of correlation length, a spatial scale that roughly indicates the least distance of independent potential values. A white-noise potential has correlation length zero, and an i.i.d. potential $\xi=(\xi(z))_{z\in\mathbb{Z}^d}$ should be conceived as a potential with a positive, fixed correlation length. Poisson potentials of the form $\sum_{x\in\xi}\varphi(\cdot-x)$ with ξ a Poisson point process in $\mathbb{R}^d$ have correlation length equal to the diameter of the support of the cloud φ, but most Gaussian fields on $\mathbb{R}^d$ have infinitely long correlations, but with a certain decay.

It turned out that the main terms in the asymptotics of the total mass $U(t)$, both for the moments and almost surely, persist without change if the length of the correlation is increased by some controlled amount. Let us illustrate this with two examples.

Example 8.1. (Correlated shift-invariant potentials on $\mathbb{Z}^d$.) In [GärMol00], the field $\xi = (\xi(z))_{z\in\mathbb{Z}^d}$ was not assumed independent (but, however, shift-invariant in distribution), but only the existence of

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{\Big\langle \exp\Big\{t \sum_{z\in\mathbb{Z}^d} \mu(z)\xi(z)\Big\}\Big\rangle}{\langle \mathrm{e}^{t\xi(0)}\rangle}$$

for any probability measure μ on $\mathbb{Z}^d$ with compact support was assumed. A similar approach was done in the $\mathbb{R}^d$-setting in [GärKön00]. In both works, the first to terms of the logarithmic asymptotics of the moments of the total mass were derived in the usual manner in terms of a variational formula that certainly depends on the above limit, but does not show any new additional effect; the i.i.d. case is contained as a special case. $\diamond$

Example 8.2. (Correlated shift-invariant potentials on $\mathbb{R}^d$.) A Hölder continuous Gaussian potential on $\mathbb{R}^d$ was considered in [GärKön00] and [GärKönMol00], see Sections 4.5.2 and 6.2.2. The covariance function $B\colon \mathbb{R}^d \to \mathbb{R}$ was assumed to be twice continuously differentiable and to take its maximum at zero with a local parabolic shape. The first two terms in the almost sure asymptotics were shown not to depend on the details of B, as long as it satisfies

$$\int_{([-R,R]^d)^{\mathrm{c}}} g^2(x)\,\mathrm{d}x = o\big((\log R)^{-2/3}\big), \qquad R\to\infty,$$

where B can be presented as $B(x) = \int_{\mathbb{R}^d} g(x-y)g(y)\,\mathrm{d}y$, and g is the Fourier transform of the square root of the spectral density of the Gaussian field. The same paper also studied the Poisson potential of the form $\sum_{x\in\xi}\varphi(\cdot - x)$ (with φ a nonnegative cloud). The cloud φ was assumed nonnegative, and again it was assumed to be smooth and to take its maximum at zero with a local parabolic shape. The first two terms of the asymptotics of the total mass were shown not to depend on the details of φ, as long as it decays fast enough, i.e.,

$$\max_{([-R,R]^d)^{\mathrm{c}}} \varphi(x) = o\big((\log R)^{-1}\big), \qquad R\to\infty.$$

This shows that the second-order asymptotics are very stable against long-term correlations in these two cases. The main goal of these investigations in [GärKönMol00] was the dependence of the second term of the asymptotics of the parabolic shape of B resp. of φ. $\diamond$

In the following example, however, a very large choice of the correlation length was proved to have a quite different effect on both terms of the asymptotics.

Example 8.3. (Brownian motion among Poisson obstacles with long range.) In the spatially continuous case, one considers the potential $V(\cdot) = -\sum_{x\in\xi} W(\cdot - x)$ with $\xi = (x_i)_{i\in\mathbb{N}}$ a standard Poisson point process with parameter $\nu\in(0,\infty)$ in $\mathbb{R}^d$ and W a non-negative potential, i.e., the trap case. Let us assume that the cloud decays like $W(x)\approx |x|^{-\alpha}$ as $|x|\to\infty$, for some $\alpha>0$. It was already shown in [DonVar75] that the first two terms in the asymptotics of the moments of the total mass are independent (and indeed the same as for $\varphi = -1\!\!1_{B_1}$, see ...), as long as the decay is strong enough in the sense that $\alpha > d+2$. In the interesting case $d<\alpha<d+2$, the moment asymptotics are different and are given, for any $p\in[0,\infty)$, as

$$\langle U(t)^p\rangle = \exp\Big\{-a_1(pt)^{d/\alpha} + (a_2+o(1))(pt)^{\frac{\alpha+d-2}{2\alpha}}\Big\}, \qquad t\to\infty, \tag{8.2}$$

momentasyhea

where

$$a_1 = \nu\omega_d \Gamma\Big(\frac{\alpha-d}{\alpha}\Big) \qquad \text{and} \qquad a_2 = \Big(\frac{\kappa\nu\alpha\sigma_d}{2}\Gamma\Big(\frac{2\alpha-d+2}{\alpha}\Big)\Big)^{1/2}, \tag{8.3}$$

and ω_d and σ_d, respectively, are the volume and the surface of the unit ball in $\mathbb{R}^d$, and we have replaced Δ by $\kappa\Delta$ with some diffusion constant $\kappa \in (0,\infty)$. The first term in (8.2) was derived in [Pas77]. It is asymptotically equivalent to $\langle \mathrm{e}^{tV(0)}\rangle$ and depends only on the potential; it contains no information about the Brownian motion. This is a common effect; in the i.i.d. case, we called this term $\mathrm{e}^{H(t)}$. Let us also mention that the critical case $\alpha = d+2$ was studied by [Oku81].

The second term in (8.2) was studied and interpreted in [Fuk11]. The number a_2 admits a representation in terms of the variational formula

$$a_2 = \inf_{\phi\in H^1(\mathbb{R}^d)\colon \|\phi\|_2 01} \Big(\kappa\|\nabla\phi\|_2^2 + \frac{a_2^2}{\kappa d}\int_{\mathbb{R}^d} |x|^2\phi(x)^2 \,\mathrm{d}x\Big).$$

The main joint strategy of the motion and the potential to contribute optimally to the Feynman-Kac formula is informally described as follows. Unlike in the standard case $\alpha < d+2$, there is no sharp interface between the area that is free of obstacles and is therefore covered by the Brownian motion local times. Instead, the obstacle density gets only gradually thinner away from the origin, and the influence of the long tails of the cloud stretches practically over all the space, but gets sufficiently weak only in a very small neighbourhood of the origin. The potential assumes its minimum at the origin with $|V(0)| \sim a_1\frac{d}{\alpha}t^{-(\alpha-d)/\alpha}$ and assumes a parabolic shape $V(x)-V(0) \sim \frac{a_2^2}{\kappa d}t^{-(\alpha-d+2)/\alpha}|x|^2$ for $|x| = o(t^{1/\alpha})$. The Brownian motion does not leave the centred ball with radius $o(t^{1/\alpha})$. This directly explains both terms of the asymptotics in (8.2), on base of the Donsker-Varadhan-Gärtner LDP for the occupation times measures of the Brownian motion.

In [Fuk11], consequences for the second-order asymptotics of the Lifshitz tails and of the almost sure asymptotics of $U(t)$ are drawn as well, but we do not formulate them here; this follows the usual patterns, given (8.2). $\diamond$

In view of the current fruitful developments in the analysis of extreme-value properties of the Gaussian free field and other log-correlated random fields, it appears interesting to study the PAM with such a potential.

8.4 Weak disorder and accelerated motion

We recall from our first considerations in Section 1.1 that the potential ξ makes the solution $u(t,\cdot)$ to the PAM in (1.1) irregular, and the Laplace operator makes it smooth. However, as we saw when discussing intermittency, the smoothening effect is not so strong that the solution would not show a strongly localised picture, even though the local areas of the high peaks show some smooth structures. Nevertheless, the localisation effect is very strong in the PAM, unlike in other models of random motions through random media, for example the much-studied model of a random walk in random environment, which has a strong tendency to homogenisation. Hence, it appears interesting to study the transition in the PAM from localised to homogenised behaviour by modifying the Anderson Hamiltonian in (1.1) by either weakening the potential or speeding-up the diffusivity. Highly interesting new phenomena arise here.

To this end, we look at the operators

$$\mathrm{e}^{t(\Delta^{\mathrm{d}}+\varepsilon_t\xi)} \qquad \text{with } \varepsilon_t\downarrow 0, \qquad \text{and } \mathrm{e}^{t(\kappa_t\Delta^{\mathrm{d}}+\xi)} \qquad \text{with } \kappa_t\to\infty. \tag{8.4}$$

Certainly, the consideration of these two operators is mathematically equivalent via the relation $\varepsilon_{t\kappa_t} = 1/\kappa_t$. The small factor ε_t tames down the influence of the potential and makes the disorder weak, and the large prefactor κ_t induces an acceleration of the random motion and makes the diffusion fast.

Generally, it is expected (and has been proved in a number of cases) that scale functions ε_t respectively κ_t that are not too fast will not change the general picture in the asymptotics that we have for the standard case $\varepsilon_t \equiv 1 = \kappa_t$, but only the scales. On the other hand, extremely fast choices will make the influence of the random potential so marginal that its influence vanishes. Naturally, the question arises whether there are further interesting regimes, in particular critical ones, between these two. There are a number of aspects under which this is interesting:

- What is a critical scale on which the random walk is so fast that he cannot spend much time in the highest peaks of the potential? What does he do instead?
- What is the critical scale on which the extremely high potential peaks do not attract the main flow of the mass? What happens instead?
- The logarithmic asymptotics in the standard case $\varepsilon_t \equiv 1 = \kappa_t$ is described by two terms, a function $H(t)$ describing the height of the relevant peaks, and a variational formula χ describing their shape. What is a critical scale of ε_t such that these two terms are merged into each other, and how does this work?

Having found the critical scale, one naturally asks also for the size and the structure of the relevant regions and for the mechanism that is behind the main contribution to the total mass of the PAM (moderate deviations? central limit theorem? what else?).

Like in the standard case $\varepsilon_t \equiv 1 = \kappa_t$, the asymptotics of the operators in (8.4) is closely connected with the upper-tail behaviour of the principal eigenvalue of the Anderson operator $\Delta^{\rm d} + \varepsilon\xi$ with small prefactor $\varepsilon \in (0, \infty)$ in front of the disorder in large ε-dependent boxes, which is an interesting object to study on its own. One can expect (and this is one of the fundamental questions here) that, for sufficiently small boxes, the corresponding (random) principal eigenfunction shows a *homogeneous behaviour*, i. e., stretches its entire homogeneously over the entire box, while for very large boxes, it shows a *localized behaviour*, i. e., concentrates its mass in some small islands, in the way that we know from the description of the almost sure behaviour of the PAM in Section 6. In the first case, the influence of the random potential ξ should come only in terms of its expected value $\langle \xi(0) \rangle$, and in the second it should come via an extreme-value analysis of the principal eigenvalue in small boxes. However, it is not clear what should happen for boxes of intermediate sizes: is the mass stretched thinner and thinner homogeneously, or does it develep a number of bumps?

Let us describe some explicit examples that have been handled in the literature.

sec-AcDecel

8.4.1. *Acceleration of motion.* In [Sch10], various interesting choices of the velocity function $(\kappa_t)_{t>0}$ for various choices of potential classes (introduced in Section 4.3) are considered, see also the summary [KönSch12]. In all the considered cases, the moment asymptotics for the total mass are identified in terms of a characteristic variational formula analogously to (4.37). An interesting competition between the growth of κ_t and the upper tails of ξ arises: the faster κ_t grows, the stronger the flattening effect of the diffusion term is. As usual, it is supposed that Assumption (H) holds (see (4.25)), i. e., regularity of the logaithmic moment generating function H at infinity.

A lower critical scale for κ_t is identified which marks the threshold between unboundedly growing intermittent islands and concentration in just one site. This scale depends on the upper tails of ξ and is equal to $\eta(t)/t$ in (4.25). Precisely at the critical scale $\eta(t)/t$, we have a discrete picture, i. e., the relevant islands have a non-trivial, discrete shape, like in the class (DE) in the standard case. Interestingly, for upper tails of ξ in the class (B), on this critical scale, the characteristic variational formula is equal to the discrete version of the formula for χ in the case (B), i.e., with $\mathbb{R}^d$ replaced by $\mathbb{Z}^d$.

Now, still assuming that $\kappa_t \gg \eta(t)/t$, we now come to the second critical scale for κ_t, the one that is asked for at the beginning of this section, i. e., the one that marks the threshold between extreme values of the local principal eigenvalues and moderately large ones. This transition is also reflected by the fact that for slower functions κ the asymptotics are described in terms of just the upper tails of $\xi(0)$ like in the standard case, and for faster ones, the entire distribution of $\xi(0)$ enters the description. This critical scale is characterised by the fact that the local times per site stays bounded, the path covers a region of radius $\asymp t^{1/d}$ (i. e., of volume $\asymp t$), the term $\int_0 \xi(X_s))\,\mathrm{d}s$ is essentially a sum over $O(t)$ i.i.d. random variables. Hence, a kind of moderate-deviation mechanism for the sum of about t potential values is combined with a large-deviation principle for the rescaled local times on a box of radius $\asymp t^{1/d}$, and both runs on the exponential scale t.

Let us formulate the main result for κ_t being on the critical scale, see [KönSch12, Theorem 3.2]. We assume that $\langle \xi(0)\rangle = 0$ and write $U^{(\kappa_t)}$ for the total mass of the PAM with additional prefactor κ_t in front of the Laplace operator. Fix $\theta \in (0,\infty)$ such that $\kappa_t t^{-2/d} \to \frac{1}{\theta}$, then

$$\big\langle U^{(\kappa_t)}(t)\big\rangle = \exp\Big\{ -\frac{t}{\theta}\big(\chi_H(\theta)+o(1)\big)\Big\}, \tag{8.5}$$ criticalaccel

where

$$\chi_H(\theta) = \inf_{g\in H^1(\mathbb{R})\colon \|g\|_2=1}\Big(\|\nabla g\|_2^2 - \theta\int H\circ g^2\Big), \tag{8.6}$$ chiHdef

where we wrote $\int H\circ g^2$ for short for $\int_{\mathbb{R}^d} H(g^2(x))\,\mathrm{d}x$. An elementary substitution of g with $g(\beta\,\cdot)\beta^{d/2}$ shows that $\frac{1}{\theta}\chi_H(\theta)$ is equal to $\theta^{-d/(d+2)}\chi_{H(\cdot\theta)}(1)$, a remark that helps understanding the result in (8.5) when following the heuristics in Section 4.3 (which we are not doing here) and helps also comparing to the main result (8.7) of Section 8.4.2. Note that all the variational formulas called χ in Section 4.4 are versions of $\chi_H(\theta)$ with special choices of H and θ.

This critical phase has not been deeper analysed, nor the (conjecturally, homogenised) phase where $\kappa_t \gg \eta(t)/t$. Analogously to the example in Section 8.4.2, some interesting phase transition(s) are to be expected in the behaviour of the minimisers of $\chi_H(\theta)$ in the parameter θ. Neither an analysis of the almost-sure behaviour of $U^{(\kappa_t)}(t)$ has yet been carried out.

nscaledpot

8.4.2. *Brownian motion in a scaled Poisson potential.* In a series of papers [MerWüt01a, MerWüt01b, MerWüt02], Merkl and Wüthrich considered Brownian motion among soft Poisson traps (see Remark 2.9 and Section 4.5) with the potential $V(x) = -\beta\varepsilon_t\sum_{i\in\mathbb{N}} W(x-x_i)$, where $(x_i)_{i\in\mathbb{N}}$ is a standard Poisson point process in $\mathbb{R}^d$, $W\colon\mathbb{R}^d\to[0,\infty)$ is a bounded measurable and compactly supported cloud, and $\beta\in(0,\infty)$ is a parameter that gives rise to interesting phase transitions, as it turned out. We consider the moments of

$$U(t) = \mathbb{E}_0\Big[\exp\Big\{-\beta\varepsilon_t\int_0^t\sum_{i\in\mathbb{N}} W(Z_s-x_i)\,\mathrm{d}s\Big\}\Big].$$

Note that we are in the case (B) with $\gamma = 0$ in the terminology of Section 4.4. The scale function $(\varepsilon_t)_{t\in(0,\infty)}$ is chosen in critical way, i.e., in such a way that a new phenomenon arises. Recall from Section 4.5 that, in the standard case $\varepsilon_t \equiv 1$, the main contribution to the moments of $U(t)$ comes from Brownian paths running on the scale $t^{1/(d+2)}$, and the potential enters the leading asymptotic term, which is on the exponential scale $t^{d/(d+2)}$, only via the density parameter of the Poisson process (which we put equal to one here). In order to achieve a significant influence from the cloud W, one has to multiply it with $\varepsilon_t = t^{-2/(d+2)}$, which makes the term in the exponential running on the scale $t\varepsilon_t = t^{d/(d+2)}$, which is the scale on which the Brownian probability of not leaving a boall of radius $t^{1/(d+2)}$ runs. This choice of ε_t is also consistent with the choice $\kappa_t \asymp t^{2/d}$ in Section 8.4.1 and the above mentioned relation $\varepsilon_{t\kappa_t} = 1/\kappa_t$.

With this choice of ε_t, [MerWüt01a, Theorem 0.2(b)] says that

$$\langle U(t)\rangle = \exp\Big\{-t^{d/(d+2)}\beta^{-2/(d+2)}\big(\chi_H(\beta)+o(1)\big)\Big\},\qquad t\to\infty, \tag{8.7}$$ MerklWuethmom

where $\chi_H(\beta)$ is as in (8.6) with $H(t) = \mathrm{e}^{-t} - 1 = \log\langle \mathrm{e}^{tV(0)}\rangle$, the logarithmic moment generating function of the Poisson process. Interestingly, for dimensions $d \geq 2$, there is a phase transition from small β to large β as to the structure of minimisers of $\chi_H(\beta)$; indeed, in [MerWüt01a, Theorem 0.3] it turns out that $\chi_H(\beta) = \beta$ for all sufficiently small positive β, but $\chi_H(\beta) = \beta$ for all other β. This can be interpreted by saying that the homogeneous phase (which is encountered in [MerWüt01a, Theorem 0.2(a)] when taking $\varepsilon_t \gg t^{-1/(d+2)}$) arises also on the critical scale, if the parameter β is small enough. Let us also remark that for $\varepsilon_t \gg t^{-1/(d+2)}$, i.e., if the damping of the potential is not too strong, [MerWüt01a, Theorem 0.2(c)] proves asymptotics that are practically the same as in the standard case $\varepsilon_t = 1$, with a suitable adaptation of the scales.

In the follow-up papers [MerWüt01b, MerWüt02], the almost sure asymptotics of $U(t)$ and large-deviation properties of the principal eigenvalue of $\frac{1}{2}\Delta + \varepsilon V$ in large, ε-dependent boxes are deduced from the result in (8.7) in the usual way that we described in Section 6.2.1. The correct choice for obtaining interesting new effects is $\varepsilon_t = (\log t)^{-2/d}$; similarly to the annealed setting, the arising variational formulas are proved to show 'homogenised' behaviour for $d \leq 3$ for small positive β and 'localised' behaviour for large β, but only 'localised' behaviour for $d \geq 4$. Note that the critical dimension is two for the annealed setting and four for the quenched one.

8.4.3. *Scaled Gaussian field.* Using a combination of some results on stretched exponential moments of (renormalised) self-intersections of random walks on $\mathbb{Z}^2$, which are also interesting on their own, one finds another explicit example.

MISSING.

sec-RWRSc 8.5 Upper deviations of random walk in random scenery

As we mentioned in Remark 2.1, the term $\int_0^t \xi(X_s)\,\mathrm{d}s$ in the exponent of the Feynman-Kac formula in (2.1) is sometimes called the *random walk in random scenery (RWRSc)*. In the study of the PAM, we are most interested in the behaviour of its exponential moments. Furthermore, in Section 8.4, we also discussed exponential moments with vanishing prefactors. All these questions are intimately connected with the question for *upper deviations* of the RWRSc on various scales, i. e., with theorems of the type

$$\log \mathrm{Prob}\otimes\mathbb{P}\Big(\int_0^t \xi(X_s)\,\mathrm{d}s > \lambda a_t\Big) \sim -b_t I(\lambda), \qquad t\to\infty,\ \text{for } \lambda\in(0,\infty), \tag{8.8}$$ upperdeviati

where a_t and b_t are scale functions that tend to ∞ as $t \to \infty$. Such a result would be more or less equivalent to the statement

$$\log\Big\langle \mathbb{E}\Big[\exp\Big\{\beta\frac{b_t}{a_t}\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\Big]\Big\rangle \sim b_t \sup_{\lambda\in(0,\infty)}\big[\lambda\beta - I(\lambda)\big], \qquad t\to\infty,\ \text{for } \beta\in(0,\infty), \tag{8.9}$$ exponMomentp

as one deduces with the help of the well-known proof of Cramér's theorem [DemZei98, Theorem ???]. Note that the supremum on the right-hand side of (8.9) is equal to the *Legendre transform* of I (with extension by the value ∞ on $(-\infty,0]$). Hence, (8.8) may prove very useful for understanding the PAM with potential $\frac{b_t}{a_t}\xi$. However, it may happen that the supremum on the right-hand side of (8.9) is trivial (i.e., constantly equal to zero or to infinity), and hence it must be checked on a case-by-case basis whether (8.8) is useful for the study of the PAM or not.

Nevertheless, the way of thinking about and the proof techniques for deriving results like in (8.8) are very close to the methods that we encountered in this survey text, and both topics benefitted from each other over the past two decades. At this place, we only mention that [AssCas07] provides a survey on results of the type (8.8). They are sometimes also called *moderate-deviations results.*

Note that, for most of the potential distributions that we considered and for most of the prefactors in front of the potential ξ, the moment asymptotics of the total mass show two quite different terms, not just one like in (8.9). Hence, the above duality can be helpful in this form only in cases described in Section 8.4, or they have to be modified accordingly. This is the reason that most of the results of the form (8.8) are not helpful for the understanding of the PAM.

8.6 Upper deviations of self-attractive functionals of local times

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Carrying out the expectation over the random potential ξ, we saw in Section 2.1.4 that the expectation of the total mass of the PAM is equal to the exponential moment of $\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))$, which is a self-attractive functional of the random walk local times ℓ_t. Analogously to the idea outlined in Section 8.5, moderate-deviation results for this functional like in (8.9) may prove helpful for the study of the moments of the PAM; however, they are interesting on their own and are studied without reference to the PAM. In particular, the function H does not have to be equal to the logarithmic moment generating function.

Analogously to what we said at the end of Section 8.5, not all moderate-deviations results for $\sum_z H(\ell_t(z))$ are interesting for the PAM, not even with additional scaling factors multiplying the potential ξ. Even more, when one would like to deduce moments of the total mass for the potential $\varepsilon_t\xi$ for some scale function ε_t, H must be taken as the logarithmic moment generating function of $\varepsilon_t\xi$, and this causes an additional complication.

The most interesting functionals H studied in the literature are

- $H(t) = \mathbb{1}_{(0,\infty)}(t)$

AUCH DIE MODERATEN ABWEICHUNGEN FUER DIE SHRINKING WIENERWURST GEHOERT PHAENEMONOLOGISCH HINEIN, DENN VERKLEINERUNG DER FALLEN IST ANALOG ZU EINEM KLEINEN VORFAKTOR BEIM POTENTIAL, ABER DER ZUSAMMENHANG IST NICHT DIREKT.

8.7 Self-attractive path measures

attractive

As we showed in (2.18), the moments of $U(t)$ are equal to the negative exponential moment of the volume of a Wiener sausage up to time t. If the radius a of that sausage is taken as a function of t, new effects arise. This idea has been taken as the starting point in a series of papers [BerBolHol01] about the clumping behaviour of a Brownian motion if the volume of its sausage is small on critical scales.

8.8 Transition between quenched and annealed behaviour

sec-Trans

Via the spatial ergodic theorem, the expectation of the total mass can be seen as an almost sure ergodic limit of mixtures of shifts of the solution. This is formulated as follows. Let $v\colon [0,\infty)\times\mathbb{Z}^d$ be the solution of (1.1) with the localised initial condition δ_0 replaced by the homogeneous condition $v(0,z)=1$ for every z. Then $v(t,z)$ can be represented by the Feynman-Kac formula in (2.1) for $U(t)$ with intial point 0 replaced by z, as one sees from the superposition principle for solutions with superposed initial conditions. In particular, $U(t)=v(t,0)$. Now the spatial ergodic theorem gives, for fixed $t>0$,

$$\langle U(t)\rangle = \langle v(t,0)\rangle = \lim_{B\to\mathbb{Z}^d}\frac{1}{|B|}\sum_{x\in B} v(t,x), \qquad (8.10)$$

Ergodic

where the limit is along centred boxes. Hence, an interesting question is to describe the transition in the limit $t\to\infty$ from the quenched behaviour described in Section 6 to the annealed behaviour described in Section 4, by using t-depending boxes $B_{r(t)}$ instead of B in (8.10). This has been studied in [BenAroMolRam05] and [BenAroMolRam07] as well as in [GärSch11b]

8.9 Other diffusivities

c-otherdiff

So far, we considered the PAM with the diffusivity given only by the (discrete or continous, respectively) Laplace operator, i.e., with a driving random motion that lies in the domain of attraction of the

Brownian motion. In practically all cases, this type of diffusivity pertained to the characteristic variational formula, in some cases after alluding to the typical rescaling of time and space in the spirit of Donsker's invariance principle.

However, instead of a motion taken from the Brownian universality class, it is certainly also highly interesting to study the case of random walks in the domain of attraction of Lévy processes or directly a Lévy process. So far, the first rigorous work in that direction seems to be [MolZha12], where

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8.10 PAM in a random environment

Another interesting direction is the study of the PAM with the driving Laplace operator replaced by a random version of it, or, equivalently, with the underlying random walk replaced by a *random walk in random environment (RWRE)*. This makes the PAM as a model for random motions through random potential much more realistic, as the diffusion is now itself taken random, which represents impurities in the diffusive medium, hampering or accelerating locally the conductivity. There are numerous potential applications of such a additional randomness.

8.10.1. *The local times of a random walk in random scenery in boxes.* One very natural way to introduce randomness in the diffusion is to replace Δ by the *randomised Laplace operator*,

$$\Delta^\omega f(x) = \sum_{y\in\mathbb{Z}^d\colon y\sim x} \omega_{x,y}(f(y)-f(x)), \qquad f\colon \mathbb{Z}^d\to\mathbb{R},\ x\in\mathbb{Z}^d, \tag{8.11}$$

where $\omega = (\omega_{x,y})_{x,y\in\mathbb{Z}^d, x\sim y}$ is a random i.i.d. field of positive weights on the nearest-neighbour bonds of $\mathbb{Z}^d$. In order to obtain a symmetric operator Δ^ω, one usually assumes that $\omega_{x,y} = \omega_{y,x}$ for any edge $\{x,y\}$, i.e., one gives the weights to the undirected edges. One often also speaks of the *random conductance model*, since $\omega_{x,y}$ is interpreted as the conductance of the edge $\{x,y\}$. The operator Δ^ω generates the continuous-time random walk $(X_t)_{t\in[0,\infty)}$ in $\mathbb{Z}^d$, the *random walk among random conductances (RWRC)*. When located at y, it waits an exponential random time with parameter $\sum_{z\sim y}\omega_{y,z}$ (i.e., with expectation $1/\sum_{z\sim y}\omega_{y,z}$) and then jumps to a neighbouring site z' with probability $\omega_{y,z'}/\sum_{z\sim y}\omega_{y,z}$. The RWRC is studied a lot in the last decade like many other types of RWREs, with strong emphasis on the search for laws of large numbers or (functional) central limit theorems.

However, for studying the PAM with the RWRC as the underlying random motion, other properties of the RWRC turn out to be important. The PAM with RWRC has not yet been studied, but important prerequisities have been derived: annealed large-deviation principles (LDPs) for the normalised local times of the RWRC in fixed boxes [KönSalWol12] and in time-depending, growing boxes [KönWol13]. These LDPs are interesting only because of the assumption on the conductances that they are not uniformly elliptic (i.e., not bounded away from zero and from infinity), but can attain arbitrarily small values. This is precisely what creates an interesting interaction between the random walk and the random conductances, since small conductances help the random walk to lose much time within the box, i.e., to increase the probability of not leaving it. Putting assumptions on the lower tails of the conductances of the form

$$\log P(\omega_e\le\varepsilon) \sim -D\varepsilon^{-\eta}, \qquad \varepsilon\downarrow 0,$$

with parameters $D,\eta\in(0,\infty)$, makes it possible to derive an explicit LDP rate function for the normalized local times, which is in the spirit of the famous Donsker-Varadhan-Gärtner LDP [DonVar75, Gär77]. In the case of a time-dependent growing box, interestingly there arise two cases, one of which, the case $\eta > d/2$, is just the continuous version of the fixed-box version, where one obtains a full LDP for the properly rescaled version of the local times. Here the local times spread out over the entire larg box in a more or less homogeneous way. However, in the case $\eta < d/2$, it turns out in [KönWol13] that the LDP asymptotics follow the formulas of the fixed-box version, which seems to suggest that the random walk fills only a small part of the growing box. Further studies will be necessary to understand this effect and to use the results of [KönSalWol12, KönWol13] for the study of the almost-sure setting and later for the study of the PAM with RWRC.

8.10.2. *Localisation in the Bouchaud-Anderson model.* Recently [MuiPym14], the PAM was studied in another class of random environment, more precisely, the underlying random walk was replaced by the *Bouchaud random walk*, better known under the name *Bouchaud trap model*, where the randomness of the holding times does not sit in the bonds, but in the sites, and is chosen very heavy-tailed. More explicitly, the generator is given by

$$\Delta_\sigma f(z) = \frac{1}{2d\sigma(z)} \sum_{y\in\mathbb{Z}^d\colon y\sim z} (f(y) - f(z)), \tag{8.12}$$

DeltaBouchaud

where $\sigma = (\sigma(z))_{z\in\mathbb{Z}^d}$, the *trapping landscape*, is a random i.i.d. field of positive numbers. The Bouchaud random walk, when standing in z, waits a random time that is exponentially distributed with expectation $2d\sigma(z)$ and then jumps with equal probability to any of the $2d$ neighbours. If $\log\sigma$ is Pareto-distributed (i.e., $\mathbb{P}(\sigma(0) > r) = (\log r)^{-\alpha}$ for all $r \geq 1$ for some $\alpha \in (0,\infty)$), then, almost surely with respect to the landscape σ, the Bouchaud random walk exhibits some peculiar behaviour that is caused by the existence of lattice sites with extra-ordinarily long holding time parameter; the random walk is trapped. This effect is most pronounced if the Pareto parameter α is smaller than one, and leads then to *ageing phenomena*. E.g., the trapping takes place after longer and longer time lags and in further and further remote sites, and both the time lags and the trapping sites show nice asymptotic scaling behaviours. See [BenČer06] for a summary of these properties of the Bouchaud random walk.

Combining the PAM with the Bouchaud random walk and considering concentration and ageing phenomena is tempting, as each of them show these phenomena, the PAM at least for sufficiently heavy-tailed potential distributions, as we outlined in Section 7.2. Indeed, in [MuiPym14], the potential distribution is taken as Weibull-distributed, in which case these phenomena for the PAM are known from [?] and [SidTwa14]. However, the assumptions on the Bouchaud random walk made in [MuiPym14] are less restrictive, but do contain the Pareto case with parameter in $(0, 1)$.

The main result of [MuiPym14] is the complete localisation of the solution $u(t,\cdot)$ in one random, t-dependent site $Z_t \in \mathbb{Z}^d$ in the sense that $u(t, Z_t)/U(t)$ converges to one in probability. Furthermore, depending on the details of the distribution of potential, the potential values in the neighbouring sites of the concentration site Z_t are characterised; they indeed show an interesting limiting behaviour, and it can be seen in some sense that the one-site island slowly starts to emerge an interesting shape as the potential distribution gets less heavy-distributed.

NOCH EIN WENIG MEHR UEBER DIE RESULTATE?

9. Related models

atedModels

sec-DirPol

9.1 Drifted PAM and polymers

New interesting questions arise if a drift is added to the diffusion, i.e., if the generator $\Delta^{\rm d}$ of the simple random walk is replaced by the one of a random walk with drift. The main conjecture is that, if the strength of the drift is small enough, at least the first terms of the asymptotics of the PAM will then be the same as for the drift-free case, and the scale of the size of the intermittent islands should be the same. However, there should be a critical threshold for the strength of the drift beyond which the trajectory in the Feynman-Kac formula should run on the scale of the time, i.e., should a non-zero effective drift in the same direction as the inserted one. Further questions ask whether or not there might be an intermediate regime, how to characterise the critical threshold, what the influence of the inserted drift is in the case of zero effective drift, how large the effective drift might be, and more detailed questions.

The other extreme case of drifts, the case where each step leads the trajectory by a fixed amount further in one direction, is called *directed polymer in random environment*, a name that reflects that this trajectory never hits a site more than once. Usually, one considers the time-discrete setting and picks the direction of the drift parallel to the first axis, such that the path is indeed of the form $(n, S_n)_{n\in\mathbb{N}_0}$ with a d-dimensional simple random walk (or other types of random walks). This is a $(1+d)$-dimensional polymer, which is indeed the graph of a d-dimensional walk.

Directed polymers in random environment are a subject of high importance, since they are believed to show behaviours that lie in the universality class of a number of prominent models, one of the most well-known of which is the directed last-passsage percolation and the largest eigenvalue of a random matrix drawn from the Gaussian unitary ensemble and the KPZ equation. Since also the methodology is drastically different from the treatment of the PAM, directed polymers are out of proportion to the scope of the present text, and we refer only to the survey [ComShiYos04] on directed polymers from 2004 and the survey on the KPZ equation [Cor12].

[IofVel12b]

sec-BRWRE **9.2 Branching random walks in random environment** In Section 2.1.1 we introduced the model of a branching random walk in a random environment of branching rates (BRWRE) and mentioned that the expected number of particles at time t in the site z, where the expectation is taken over the branching/killing mechanism and the migration, but not over the branching rates, is a solution of the PAM, where the potential ξ is calculated from the branching and the killing rates. This suggests to exploit the knowledge on the PAM that has been gained over the last 20 years for the study of the BRWRE, but actually this has been done to a little extent yet.

Let us phenemonologically explain how much information out of the Feynman-Kac formula (2.23) can be extracted about the BRWRE. We want to demonstrate that the path $X = (X_s)_{s\in[0,t]}$ stands for a subtree of the migrating Galton-Watson tree that describes the complete genealogy of the BRWRE, more precisely, it stands for the expectation of the total number of all branching particle trajectories that make all the steps of the random walk in full coincidence. Along the time and way, these particles branch into two or may disappear due to killing, but we consider, among all the trajectories in the entire genealogy tree, only those trajectories that make all the way precisely as the random walk of the Feynman-Kac formula. The term $\mathrm{e}^{\int_0^t \xi(X_s)\,\mathrm{d}s}$ summarizes all the expected killing and branching along the trajectory.

In the light of this interpretation, each realization of X in the Feynman-Kac formula expresses only a small part of the entire branching process. The entire branching process comes in only via taking the expectation over X. The time coincidence of all the branches of the genealogical tree cannot be seen in the Feynman-Kac formula; looking at it pathwise shows only a small excerpt of the branching tree. However, we see the entire time evolution of this excerpt over the interval $[0,t]$.

From the understanding of the behaviour of the PAM, one can now guess that the behaviour of the BRWRE should have quite some features in common with the PAM. In particular, the arising picture should have nothing to do with a homogenised behaviour, and Brownian approximations in the sprit of Donsker's invariance principle should drastically fail. Instead, the branching particles should also enjoy an intermittency effect, i.e., they should be strongly concentrated in the same intermittent islands as the solution of the PAM is. Of high interest is then the description of of the trajectories between these islands and the identification of the time scales and much more.

9.2.1. *One-dimensional models.* Some early work in that direction was carried out in a series of papers [GreHol92] for one-dimensional BRWRE in discrete time and space, with spectial attention to the influence of a drift to the expected number of particles, comparing and contrasting the annealed and quenched settings. No connection with the PAM (whose mathematical treatment was in its infancy at that time) was made, and the methods used there (Ray-Knight-type descriptions of the local time as a process in the space parameter) are strictly limited to one dimension.

9.2.2. *moment asymptotics for the population size.* Motivated and influenced by [GärKön00], the d-dimensional continuous-time version of the BRWRE was studied in [AlbBogMolYar00] for Weibull-distributed branching rates with parameter $\alpha \in (1,\infty)$, such that the corresponding PAM lies in the class of double-exponentially distributed potentials with $\rho = \infty$. The main focus was on deriving a Feynman-Kac-like formula for the expectation of the n-th power of the number of particles at time t at site z, $\eta(t,z)$, and the total particle number at time t, $\eta(t)$. This formula is formulated there in a recursive fashion, which made it difficult to analyse its asymptotics as $t \to \infty$. The result identified the first term only, and it turned out that, for $p, n \in \mathbb{N}$,

$$\langle \mathtt{E}[\eta(t)^n]^p \rangle = \mathrm{e}^{H(npt)(1+o(1))}, \qquad t \to \infty.$$

That is, the asymptotics for the p-th moment (over the branching rates) of the n-th moment (over the branching/killing and migration, denoted by $\mathtt{E}$) at time t are the same as the one of the first moment of the first moment at time tnp, at least as it concerns the first term. We know that this phenomenon can be easily interpreted for $n = 1$ (see Remark 4.2), but this is not so easily for arbitrary $n \in \mathbb{N}$.

This effect was later studied in greater detail and precision [GünKönSek13], where a direct version of that Feynman-Kac-like formula was derived, which admits deeper studies. The main tool there is the many-to-few lemma, an extension of the well-known many-to-one lemma from the theory of branching processes. For the branching rates doubly-exponentially distributed with any value of $\rho \in (0,\infty)$, also the second term in the asymptotics of $\langle \mathtt{E}[\eta(t)^n]^p \rangle$ was derived, and the above phenomenon is shown to hold true also for the second term (which is by the way, again given by the characteristic variational formula in (4.39)). A closer inspection of the proof shows that this phenomenon should come from the fact that the potential ξ can attain positive values (in which case these values will determine the asymptotics). However, for strictly negative potentials, the asymptotics of the p-th moment of the n-th moment should behave as the first moments at time tp, i.e., as if n would be one.

9.2.3. *Intermittency for the particle flow.* For the branching rates Pareto-distributed, a deep analysis of the entire particle flow over the time is carried out in [OrtRob14] in terms of limiting assertions in probability. Indeed, it is shown there that the branching particles are concentrated on the intermittent islands of the PAM (which are single sites now, see Section 7.2), but are traversed in a possibly different order than the main bulk of the mass of $u(t,\cdot)$ traverses it. This proves an appealing ageing picture of the BRWRE in great detail. The main difference between the time-evolution of the main mass of the PAM and the main particle concentration of the BRWRE is the following. If the time t exceeds the threshold beyond which new, more preferable intermittent islands appear at the horizon, then the sample trajectory in the Feynman-Kac formula is completely rearranged from scratch and immediately walks from the origin to the new optimal potential site, without paying attention to where the last intermittent peak was located. The PAM searches for new islands only from the origin. In contrast, the branching particles are already located at the last intermittent island and have to travel from that location to a new one, which is now optimal as seen from that current location and does not have to be the one that the PAM-trajectory would choose.

sec-TimeDep

10. Time-dependent potentials

Of fundamental importance is the parabolic Anderson model in (1.1) also if the random potential is allowed to be time-dependent. Here we consider the Cauchy problem

$$\frac{\partial}{\partial t}u(t,z) = \kappa\Delta^{\mathrm{d}}u(t,z)+\xi(t,z)u(t,z), \qquad \text{for } (t,z)\in(0,\infty)\times\mathbb{Z}^d, \tag{10.1}$$

$$u(0,z) = u_0(z), \qquad \text{for } z\in\mathbb{Z}^d. \tag{10.2}$$

where $\xi\colon[0,\infty]\times\mathbb{Z}^d\to\mathbb{R}$ is a space-time random field that drives the equation, and u_0 is the initial datum. This case is called the *dynamic* case, and the potential ξ is often called a *dynamic random environment* or a dynamic potential. Usual general assumptions are that the field ξ is time-space ergodic, and some integrability condition on the marginal, i.e., on the distribution of $\xi(0,0)$. Again, the main goal is the analysis of the solution $u(t,\cdot)$ in the large-t limit.

Example 10.1. (Population dynamics.) One important interpretation is in terms of population dynamics [ErhHolMai13b], which is a variant of the interpretation in terms of spatial branching that we discussed in Sections 2.1.1 and 9.2. Consider the special case where $\xi=\gamma\xi_*-\delta$, where $\xi_*=(\xi_*(t,z))_{z\in\mathbb{Z}^d,t>0}$ is an $\mathbb{N}_0$-valued random field, and $\gamma,\delta\in(0,\infty)$. Consider a system of two types of particles, namely A (catalysts) and B (reactants), subject to the rules

- A-particles evolve antonomously according to a prescribed dynamics with $\xi_*(t,z)$ denoting the number of A-particles at site z at time t,
- B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location at the same time,
- B-particles die at rate δ,
- the average number of B-particles at site x at time t is equal to $u_0(x)$.

Then $u(t,z)$ is equal to the average number of B-particles at time t at site z, conditional on the evolution of the A-particles. $\diamond$

m-Population

The problem in (10.1) is much more difficult to analyse, and the results are much less explicit than in the static case, which we considered in the preceding sections. It is not a fixed environment in which the diffusion takes place, but the potential randomly varies over time. Even more, the operator $\Delta^{\mathrm{d}}+\xi(t,\cdot)$ on the right-hand side of (10.1) depends on time, and therefore it is *a priori* not possible to make use of spectral theory here. The Feynman-Kac formula now reads

$$u(t,z)=\mathbb{E}_z\Big[\exp\Big\{\int_0^t \xi(X_s,t-s)\,\mathrm{d}s\Big\}u_0(X_t)\Big], \qquad z\in\mathbb{Z}^d, t\in(0,\infty); \tag{10.3}$$ FKtimedep

where $(X_s)_{s\in[0,\infty)}$ is the continuous-time random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta^{\mathrm{d}}$. For the comparison with (2.1), it is helpful to recall that, unlike in (10.3), a time-reversal of the path $(X_s)_{s\in[0,t]}$ was used to bring the formula into a form in which the path starts from the initial condition and terminates at the site z considered. From (10.3), one already sees that the picture is quite different from the static case. Let us try to argue heuristically in the quenched case, i.e., with probability one with respect to ξ. Indeed, again the large-t asymptotics should come from a behaviour of the path that spends as much time as possible in sites where the potential is extremely large, but when it varies in time, this is much more difficult for the path, as the locations of the optimal regions move. If the potential is mixing in time, then the fraction of time in which the path really is in the extremal regions is very small, and the large- t behaviour of the solution is not likely to come from any kind of maximisation.

The survey article [GärHolMai09b] provides an overview of annealed results obtained in four previous papers by the authors. In [GärHolMai12], the authors analyse the quenched asyptotics (i.e., conditioned on the evolution of catalyst particles). Three types of catalyst particles are considered:

(i) Independent simple symmetric random walks starting from a Poisson random field, that is, in every lattice point the number of catalysts at the beginning is independently Poisson distributed.
(ii) Symmetric exclusion process: At each time, every site is either occupied by one particle or empty. Particles jump from a site x to a neighbouring site y at rate $p(x, y) = p(y, x) \in (0, 1)$.
(iii) Symmetric voter model: As above, every site is either occupied by one particle or empty. Site x imposes its state on a neighbouring particle at rate $p(x, y) = p(y, x) \in (0, 1)$.

In each of the above models, the initial distribution of catalyst particles is such that the corresponding process is in equilibrium, i.e., the number of particles in each site is stationary. In the model (i), there are two key quantities that are of special interest as they characterise the so-called *catalytic* and *intermittent* behaviour of the model. Let us assume that the initial Poisson distribution has parameter $\nu > 0$ and the catalyst particles have jump rate $2d\rho > 0$. We define

$$\Lambda_p(t) = \frac{1}{t} \log(\mathrm{e}^{-\nu\gamma t} \langle u(0,t)^p \rangle^{1/p}).$$

This measures the effect of randomness in the model, as the term $\mathrm{e}^{\nu\gamma t}$ is exactly what the solution u would be if the random potential was replaced by its average. Then, the so-called *Lyapunov exponents* are defined as

$$\widehat{\lambda}_p = \lim_{t\to\infty} \frac{1}{t} \log \Lambda_p(t) \tag{10.4}$$

$$\lambda_p = \lim_{t\to\infty} \Lambda_p(t) \tag{10.5}$$

For $p \in \mathbb{N}$. We say that the solution u is *strongly p-catalytic* if $\widehat{\lambda}_p > 0$. This corresponds to a double-exponential effect of randomness and is caused by extreme clumping behaviour of the catalysts in certain regions. We call the solution u *strongly p-intermittent* if either $\lambda_p = \infty$ (which is, of course, always the case in the catalytic regime) or $\lambda_p > \lambda_{p-1}$.

As in many examples of the PAM with time-dependent potential, there is a strong connection with spectral properties of the operator $\Delta + r\delta_0$ with a parameter $r > 0$. For example, the quantity $\widehat{\lambda}_p$ is equal to ρ times the upper boundary of the spectrum of $\Delta + r\delta_0$ with the choice $r = p\gamma/\rho$.

Remark 10.2. (Connection with the polaron model.) In the case $d = 3$, there is an interesting connection between the Lyapunov exponent $\lambda_p = \lambda_p(\kappa)$ in the limit $\kappa \to \infty$ and the limiting behaviour of the so-called *polaron model*. This model is introduced as follows. We consider

$$\Theta(t;\alpha) = \frac{1}{\alpha^2 t} \log \mathbb{E}_0^W \Big(\exp\Big\{ \alpha \int_0^t \mathrm{d}s \int_s^t \mathrm{d}u \, \frac{\mathrm{e}^{-|u-s|}}{|W(u) - W(s)|} \Big\} \Big) \tag{10.6}$$

with a standard Brownian motion W on $\mathbb{R}^3$ and a positive parameter α. As an interesting observation, we may characterise the behaviour of $\kappa\lambda_p(\kappa)$ as $\kappa \to \infty$ by the same quantity that arises in the analysis of $\lim_{t\to\infty} \Theta(t;\alpha)$ when α tends to infinity. This quantity is obtained by a certain variational formula. We refer the reader to [GärHol06, Section 1.5] for heuristic explanations and [GärHol06, Sections 5-8] for a proof of the limiting behaviour of $\lambda_p(\kappa)$. Asymptotics for the polaron model have been analysed in [DonVar83]. $\diamond$

Remark 10.3. (Survival and extinction of branching random walks with catalysts.) The interpretation of interacting reactants and catalysts in the model i) above has also been studied in [KesSid03] with the additional assumption that reactant particles die at a certain deterministic rate $\delta > 0$. The authors make the intriguing observation that, in dimensions 1 and 2, the expected number of reactants at a site grows to infinity in time regardless of the choice of the other model parameters, including the death rate δ. Additionally, the corresponding growth rate is always faster than exponential. On the converse, conditioning on the behaviour of the catalysts, the reactants die out in all dimensions if the death parameter is chosen large enough. This interesting behaviour implies that there are immensely high peaks in the concentration of reactants along the space of different reealisations of the catalyst behaviour. ◇

Remark 10.4. (Randomly moving traps.) We obtain a model for reactant movement among randomly moving traps if we choose $\gamma < 0$. In this case, the solution $u(t,z)$ under the initial condition $u(t,\cdot) = \delta_0$ describes the survival probability of a randomly moving particle that is killed at rate $-\gamma$ times the number of traps present at the same site. In [DreGärRamSun12], the asymptotic exponential decay rate (quenched and annealed) of the survival probability is characterised in terms of the model parameters. ◇

Remark 10.5. (Finitely many particles.) Instead of looking at infinitely many catalysts or traps, one could also consider a finite number $n \in \mathbb{N}$ of particles starting from the origin. With only finitely many catalysts in the transient cases, it depends on the model parameters if there is exponential growth in the total mass at all, which is necessary in order to describe intermittency in terms of the growth rate of moments. Key to analysing the exponential growth of the first moment in the single catalyst case is spectral analysis of the operator $\kappa\Delta + \gamma\delta_0$. Exponential first moment growth corresponds to a positive upper boundary of the spectrum of that operator, whereas intermittency is present if there exists an eigenfunction corresponding to this largest spectral value.

The existence of such an eigenfunction follows as $\gamma\delta_0$ is a compact perturbation of the Laplacian. In the case of multiple catalysts and/or higher moments, the perturbing term that appears is not compact any more, which adds an additional degree of complexity to the problem. Annealed asymptotics in the case of a single particle have been addressed in [GärHey06] and [SchWol12], whereas multiple catalysts are treated in [CasGünMai12].

The case of a single randomly moving trap, i.e., $\gamma < 0$, is not accessible by the above spectral theoretic approach as the spectrum of the Laplacian is concentrated on the negative half-axis and a perturbation by $\gamma\delta_0$ does not create an isolated positive eigenvalue. Large time asymptotics in this case, including the decay of survival probability of a single reactant, have been treated in [SchWol12]. ◇

Remark 10.6. (White noise potential.) The problem of intermittency also arises in the so-called *white noise potential* case, where we consider

$$\xi(t,x) = \frac{\partial}{\partial t} W_t^x$$

with $(W^x)_{x\in\mathbb{Z}^d}$ a collection of independent Brownian motions. In this context it is common to define the annealed Lyapunov exponents by

$$\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log\langle u(t,0)^p\rangle^{1/p}, \qquad p \in \mathbb{N},$$

and the so-called *quenched Lyapunov exponent* by

$$\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log\langle u(t,0),$$

considered as functions of the diffusion constant κ on the right-hand side of (10.1). The model is said to be p-intermittent if $\lambda_{p-1}(\kappa) < \lambda_p(\kappa)$ and to be fully intermittent if it is 2-intermittent. In [CarMol94] and [GreHol07], the authors establish the following picture: Intermittency is present in the recurrent cases $d = 1,2$, whereas in higher-dimensional cases, it occurs only if the diffusion constant κ is smaller than a certain critical threshold. More precisely the first moment Lyapunov exponent $\lambda_1(\kappa)$ is equal to $1/2$ for all choices of $\kappa > 0$ in all dimensions $d > 0$, whereas the other Lyapunov exponents behave as follows.

- If $d = 1, 2$, we have $\lambda_0(\kappa) < 1/2 < \lambda_p(\kappa)$ for $p > 1$ and $\kappa > 0$.
- If $d \geq 3$, then there exist real numbers $0 < \kappa_1 \leq \kappa_2 \leq \dots$ such that

$$\lambda_0(\kappa) \begin{cases} < 1/2 & \text{for } \kappa \in [0, \kappa_1), \\ = 1/2 & \text{for } \kappa \in [\kappa_1, \infty) \end{cases}$$

and

$$\lambda_p(\kappa) \begin{cases} > 1/2 & \text{for } \kappa \in [0, \kappa_p), \\ = 1/2, & \text{for } \kappa \in [\kappa_p, \infty), \end{cases} \qquad p > 1.$$

The asymptotic behaviour of the quenched Lyapunov exponent $\lambda_0(\kappa)$ as $\kappa \to 0$ has been analysed thoroughly in [CarMol94] for the case where the initial datum v_0 has compact support. This has been extended to general initial conditions in [CarMolVie96, CarKorMol01, CraMouShi02].

The spatially continuous analogue, i.e., the solution to the so-called *stochastic heat equation*

$$\partial_t Z = \frac{1}{2}\nabla^2 Z + WZ$$

with a space-time white noise W, admits an interesting representation as the *Cole-Hopf-transformation* $Z(x,t) = \mathrm{e}^{h(x,t)}$ of the solution to the *KPZ equation*

$$\partial_t h = \frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + W,$$

which is of great interest because it appears in many models in a universal way and is believed to exhibit a highly interesting asymptotic behaviour as it concerns the order of the fluctuations. We refer the reader to [Cor12] for a detailed survey on the KPZ equation and adjacent topics. $\diamond$

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