Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

The parabolic Anderson model on $\mathbb{Z}^d$ with time-dependent potential: Frank's works

Based on Frank's works 2006 – 2016
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We consider the Cauchy problem for the heat equation with time-dependent random coefficients and localised initial datum:

$$\begin{aligned} \frac{\partial}{\partial t} u(t,z) &= \kappa \Delta^{\mathrm{d}} u(t,z) + \xi(t,z) u(t,z), && \text{for } (t,z) \in (0,\infty) \times \mathbb{Z}^d, \\ u(0,z) &= \delta_0(z), && \text{for } z \in \mathbb{Z}^d. \end{aligned}$$

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- $\xi = (\xi(t,z) \colon t \in [0,\infty), z \in \mathbb{Z}^d)$ real-valued random potential.
- $\Delta^{\mathrm{d}} f(z) = \sum_{y \sim z} \big[f(y) - f(z) \big]$ discrete Laplacian
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- General standard assumption: ξ is space-time ergodic with moment conditions on $\xi(0,0)$.
- The solution $u(t,\cdot)$ is a random time-dependent field.
- Interpretation: Expected particle number in a branching random walk model in a field of random time-dependent branching and killing rates.

Feynman-Kac formula

Background literature and surveys:

[MOLCHANOV 1994], [CARMONA/MOLCHANOV 1994], [GÄRTNER/K. 2005], [GÄRTNER/DH/MAILLARD 2009b], [K. 2016, Chapter 8].

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Total mass of the solution: $$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t>0.$$

(Alternately, take $u(0,\cdot)\equiv 1$ and consider $u(t,0)$ instead.)

Feynman-Kac formula:

$$U(t) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(t-s, X(s))\,\mathrm{d}s\Big\}\Big], \qquad t>0,$$

with $(X(s))_{s\in[0,\infty)}$ a simple random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta^{\mathrm{d}}$, starting from 0.

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MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$, also depending on κ.

Intermittency:

For large t, the main contribution to $U(t)$ comes from a concentration behaviour, i.e., $u(t,\cdot)$ is concentrated on intermittent islands

The intermittent islands and the Lyapounov exponent

- $U(t)$ is the t-th exponential power of $\frac{1}{t}\int_0^t \xi(t-s, X(s))\,\mathrm{d}s$, hence its large values matter.
- The potential $\xi(t-s,\cdot)$ is large in the islands, and X spends much time there.
- The islands are randomly located, (very!) t-dependent, not too remote.
- Unlike in the *static* case (ξ not depending on time), the islands move a lot and do not stand still for long ($\Longrightarrow$ much less pronounced concentration effect).

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- Unlike in the *static* case (ξ not depending on time), the islands move a lot and do not stand still for long ($\Longrightarrow$ much less pronounced concentration effect).
- Easier: characterisation of intermittency in terms of the Lyapounov exponents

$$\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log\left[\langle U(t)^p\rangle^{1/p}\right], \qquad p \in \mathbb{N}.$$

Definition of moment intermittency

$(u(t,\cdot))_{t>0}$ is p-intermittent $\quad :\Longleftrightarrow \quad \lambda_p(\kappa) = \infty$ or $\lambda_p(\kappa) > \lambda_{p-1}(\kappa)$.

'$\geq$' holds always (JENSEN's inequality).

Heuristics and remarks

- Interpretation: Intermittency holds if X_s can follow, for many $s \in [0,t]$, regions where $\xi(t-s,\cdot)$ is large, without paying too much in probability (mathematical formulations largely unexplored).
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- GÄRTNER/MOLCHANOV (1990) explained how moment intermittency implies concentration (at least for static potentials). The islands of concentration are disjoint for different p.
- Annealed asymptotics $\Longrightarrow$ walk X and potential ξ 'work together'.
- p-intermittency may depend on p. No proof for monotonicity in p, but many examples.
- Even the case $\kappa = 0$ (walk stands still) may be non-trivial and interesting.
- Explicit formulas for $\lambda_p(\kappa)$ not available in general.
- If ξ is time-space ergodic, then JENSEN's inequality implies that $\lambda_p(\kappa) \geq \langle \xi(0,0) \rangle$.
- The validity of '$>$' means that X and ξ have a noticeable co-operation. (Settled in many examples.)
- $\kappa \mapsto \lambda_p(\kappa)$ should be decreasing, at least for large κ. Is $\lim_{\kappa\to\infty} \lambda_p(\kappa) = \langle \xi(0,0) \rangle$? (Settled in many examples.)

Potentials built out of particles

Main type of potentials: $\xi = \gamma\xi_* - \delta$, where $\gamma, \delta \in (0, \infty)$ and

$$\xi_*(t, z) = \#\{\text{Particles present at time } t \text{ at site } z\},$$

for some auxiliary process of particles, the catalysts. Then X receives the interpretation of the reactant particle. We think of branching particle systems and of simplified models of chemical reactions between moving substances.

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Three classes of such processes:

(i) Independent simple random walks (ISRW) with generator $\rho\Delta^d$, starting from a Poisson random field (that is, in every lattice point the number of catalysts at the beginning is independently Poisson distributed).

(ii) Symmetric exclusion process (SEP): At each time, every site is either occupied by one particle or empty. Particles jump from a site x to a neighbouring site y at rate $p(x, y) = p(y, x) \in (0, 1)$, if y is empty.

(iii) Symmetric voter model (SVM): At each time, every site is either occupied by one particle or empty. Site x imposes its state on a particle at y at rate $p(x, y) = p(y, x) \in (0, 1)$. Starting either in the Bernoulli measure or in an equilibrium with 1-density ρ.

ξ can attain arbitrarily large values by letting many catalysts clump together. Put $\delta = 0$.

[GÄRTNER/DH 2006]

(i) If $\lambda_p(0) < \infty$, then $\kappa \mapsto \lambda_p(\kappa)$ is finite, convex and strictly decreasing.

(ii) $\lambda_p(\kappa) < \infty \iff p < 1/G_d^{(\rho)}\gamma$, where $G_d^{(\rho)} = \int_0^\infty p_t^{(\rho)}(0,0)\,\mathrm{d}t$.

(iii) $\lambda_p(0) = \langle \xi(0,0) \rangle (1 - p\gamma G_d)^{-1}$.

(iv) $\lim_{\kappa\to\infty} 2d\kappa(\lambda_p(\kappa) - \langle \xi(0,0) \rangle) = \rho\gamma^2 G_d^{(\rho)} + 1\!\!1_{\{d=3\}}(2d)^3(\rho\gamma^2 p)^2 \mathcal{P}_3$, where $\mathcal{P}_3$ is the *polaron variational formula*.

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- (ii) $\Longrightarrow$ p-intermittency holds for any p in $d \leq 2$ and for sufficiently small p in $d \geq 3$.
- The super-exponential growth is even $\exp$-$\exp$ fast.
- (iii) and continuity $\Longrightarrow$ p-intermittency holds for any p for small κ.
- (iii) and continuity $\Longrightarrow$ p-intermittency holds in $d = 3$ for large κ.
- Other deep, independent investigations for $\delta > 0$ by [KESTEN/SIDORAVICIUS (2006)].
- Conjecture: p-intermittency holds for any p in $d = 3$, but only in κ-dependent areas in $d \geq 4$.
- Dynamics are reversible, hence spectral methods apply.

Here $\xi(t,z) \in \{0,1\}$ for any t and z.

[Gärtner/dH/Maillard 2007, 2009a]

Put $\delta = 0$ and let $p(\cdot,\cdot)$ be the (symmetric) kernel that drives the exclusion process. Then

(i) $\lambda_p(\kappa)$ exists and is non-increasing and convex in κ.

(ii) $p(\cdot,\cdot)$ recurrent $\Longrightarrow \lambda_p(\kappa) = \lambda_p^{(\xi\equiv\mathbb{1})}(\kappa) > \langle\xi(0,0)\rangle$

(iii) $p(\cdot,\cdot)$ transient $\Longrightarrow \langle\xi(0,0)\rangle < \lambda_p(\kappa) < \lambda_p^{(\xi\equiv\mathbb{1})}(\kappa)$ and $\lambda_p(\kappa) \downarrow \langle\xi(0,0)\rangle$ as $\kappa \to \infty$.

Results for simple exclusion processes

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- Interpretation: In the recurrent case, the catalysts fill a large ball completely, but in the transient case, they only increase their density there. This effect vanishes as $\kappa \to \infty$.
- Asymptotics for $\kappa\lambda_p(\kappa)$ and conjectures about intermittency as in the case of ISRWs.
- Dynamics are reversible, hence spectral methods apply.

[GÄRTNER/dH/MAILLARD 2010]

Put $\delta = 0$ and let $p(\cdot,\cdot)$ be the (symmetric) kernel that drives the voter process. Assume that it has zero mean and finite variance.

(i) $\lambda_p(\kappa)$ exists and is continuous in $\kappa \in [0,\infty)$.

(ii) $d \le 4 \Longrightarrow \lambda_p(\kappa) = \lambda_p^{(\xi\equiv\mathbb{1})}(\kappa)$.

(iii) $d > 4 \Longrightarrow \lambda_p^{(\xi\equiv\rho)}(\kappa) = \langle\xi(0,0)\rangle < \lambda_p(\kappa) < \lambda_p^{(\xi\equiv\mathbb{1})}(\kappa)$, and $\lim_{\kappa\to\infty}\lambda_p(\kappa) = \langle\xi(0,0)\rangle$.

(iv) $d > 4 \Longrightarrow\ p \mapsto \lambda_p(\kappa)$ is strictly increasing for small κ.

- (iv) $\Longrightarrow$ p-intermittency for all p for small κ in $d > 4$.
- Open: convexity of $\kappa \mapsto \lambda_p(\kappa)$ in $d > 4$.
- Conjecture: analogous variational description in $d > 4$ of the asymptotics of $\kappa\lambda_p(\kappa)$
- More difficult to analyse because of lack of reversibility (in contrast with ISRWs and SEP) and therefore absence of spectral theoretic methods
- Dynamics are not reversible (hence spectral methods do not apply), but duality with coalescing processes and the graphical representation help.

Replace the simple random walk as the reactant by the random walk among random conductances, that is, replace $\kappa\Delta^{\mathrm{d}}$ by

$$\Delta^{\mathrm{d}}_{\omega} f(z) = \sum_{x\in\mathbb{Z}^d\colon x\sim z} \omega_{x,z}(f(x) - f(z)),$$

with symmetric random conductances $\omega_{x,z} = \omega_{z,x}$, assumed to be uniformly elliptic, i.e., $\omega_e \in [\varepsilon, 1/\varepsilon]$ for some non-random $\varepsilon > 0$ and any edge e. The conductances do not have to be i.i.d., but satisfy a certain clustering property.

[Erhard, dH, Maillard 2015b]

Let ξ be either an i.i.d. field of Gaussian white noises or a field of (finitely many or infinitely many) ISRWs, or a splin-flip system. Then, ω-almost surely, the Lyapounov exponent exists and

$$\lambda_p(\omega) = \sup\{\lambda_p(\kappa)\colon \kappa \in \mathrm{supp}(\omega_e)\}, \qquad p \in \mathbb{N}.$$

That is, the main mass flows through large areas with (close-to) maximal diffusivity.

Almost-sure setting

Also the quenched Lyapounov exponent is of interest:

$$\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log U(t).$$

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- Existence (in particular, finiteness) not at all clear if ξ is unbounded to ∞. Better chances if ξ is sufficiently mixing in time and space.
- Dependence on initial condition for $u(0,\cdot)$ not at all clear (in particular bounded versus unbounded support).

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- Dependence on initial condition for $u(0,\cdot)$ not at all clear (in particular bounded versus unbounded support).
- Generally conjectured: if ξ is sufficiently mixing, then $\kappa \mapsto \lambda_0(\kappa)$ starts from $\langle \xi(0,0)\rangle$, increases a bit and then decreases back to that number. Not yet settled in general (but see later).
- Interpretation of $\lim_{\kappa\to\infty} \lambda_0(\kappa) = \langle \xi(0,0)\rangle$: If the reactant is forced to diffuse very fast, then it cannot make any effort to exploit high-value areas of the catalyst field, but sees on an time-average only its mean value, almost surely. ($\Longrightarrow$ asymptotic space-time ergodicity of the reactant particles, or asymptotic absence of intermittency.)
- Efforts are made for finding the asymptotics of $\lambda_0(\kappa)$ for $\kappa \downarrow 0$.

Some quenched results

Assume that ξ is stationary and ergodic w.r.t. translations in $\mathbb{Z}^d$. As before, $u(0,\cdot) = \delta_0(\cdot)$.

[Gärtner, dH, Maillard 2012]

(i) If $\langle \log u(t,0) \rangle \leq ct$ for all $t > 0$ with some suitable $c > 0$, then $\lambda_0(\kappa)$ exists almost surely and in L^1-sense and is finite. Furthermore, outside a neighbourhood of 0, it is $> \langle \xi(0,0) \rangle$, and $\kappa \mapsto \lambda_0(\kappa)$ is Lipschitz continuous.

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(ii) If ξ is bounded, under some technical assumption, $\limsup_{\kappa\downarrow 0}(\lambda_0(\kappa) - \langle \xi(0,0)\rangle) \log \frac{1}{\kappa} / \log\log \frac{1}{\kappa} < \infty$.

(iii) The above applies to the three potential examples ISRWs, SEP and SVM ([Kesten, Sidoravicius 2003]). Here $\lim_{\kappa\to\infty} \lambda_0(\kappa) = \langle \xi(0,0)\rangle$. Furthermore, λ_0 is not Lipschitz-continuous at $\kappa = 0$. For ISRWs and SEP, $\liminf_{\kappa\downarrow 0}(\lambda_0(\kappa) - \langle \xi(0,0)\rangle) \log \frac{1}{\kappa} > 0$. For ISRWs, $\lambda_0(\kappa) < \lambda_1(\kappa)$.

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- See [Gärtner, dH, Maillard 2012] also for a summary of related results, in particular asymptotics for $\kappa\downarrow 0$.
- See [Drewitz, Gärtner, Ramirez, Sun 2012] for extensions of some of these results to the case that $u(0,\cdot)$ has an unbounded support (with a small modification).

Space-time ergodicity of λ_0

Assume that ξ is space-time ergodic, $\xi(0,0)$ is integrable, and $\sup_{t\in[0,T]} \xi(t,\cdot)$ is percolating from below for any $T>0$ (i.e., each of its level sets contains an unbounded connected set). Let u satisfy an unbounded nonnegative initial condition. Then [ERHARD, DH, MAILLARD 2014] shows the existence of a unique solution.

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This was only a preparation for a general negative result about 'quenched intermittency':

[ERHARD, DH, MAILLARD 2015a]

If ξ is even GÄRTNER-hypermixing, then $\lim_{\kappa\to\infty} \lambda_0(\kappa) = \langle \xi(0,0) \rangle$.

- There are GÄRTNER-hypermixing potentials with $\lambda_1(\kappa) = \infty$ for any κ, i.e., such that 1-intermittency holds.
- Proof is lengthy and technical; it involves a microscale analysis for ξ, rearrangement inequalities for the local times of the random walk, and spectral bounds for discrete Schrödinger operators.

Conclusion and outlook

What has Frank achieved (with co-authors)?

- derived methods adapted to a number of important catalyst processes ...
- ... and for general space-time mixing potentials ...
- ... also in random environment.
- derived criteria for moment intermittency ...
- ... and asymptotics for high and low diffusivity.

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- Some few cases are left open.
- Derive more geometric information about intermittent islands (number, time dependence, size, height of potential, ...).
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I wish you much more deep concentration properties in all kinds of random environments, dear Frank!

Warmest congratulations for your birthday!