Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Gemeinschaft

Eigenvalue order statistics and mass concentration in the parabolic Anderson model

Joint work in progress with Marek Biskup (Ceske Budejowice and Los Angeles)

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The Parabolic Anderson Model

Cauchy problem for the heat equation with random coefficients and localised initial datum:

$$\frac{\partial}{\partial t} u(t,z) = \Delta u(t,z) + \xi(z) u(t,z), \qquad \text{for } (t,z) \in (0,\infty) \times \mathbb{Z}^d, \tag{1}$$

$$u(0,z) = \mathbb{1}_0(z), \qquad \text{for } z \in \mathbb{Z}^d. \tag{2}$$

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- $\xi = (\xi(z)\colon z\in\mathbb{Z}^d)$ i.i.d. random potential, $[-\infty,\infty)$-valued.
- $\Delta f(z) = \sum_{y\sim z}\big[f(y)-f(z)\big]$ discrete Laplacian
- $\Delta+\xi$ Anderson Hamiltonian

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Interpretations / Motivations:

- Random mass transport through a random field of sinks and sources.
- Expected particle number in a branching random walk model in a field of random branching and killing rates.
- Anderson Hamiltonian $\Delta+\xi$ describes conductance properties of alloys of metals, or optical properties of glasses with impurities. Many open questions about delocalised versus extended states.

Main tools

Feynman-Kac formula

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X(t)=z\}\Big], \qquad z\in\mathbb{Z}^d, t>0,$$

where $(X(s))_{s\in[0,\infty)}$ is a simple random walk on $\mathbb{Z}^d$ with generator Δ, starting from z under $\mathbb{P}_z$.

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Eigenvalue expansion

$$u(t,z) \approx \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X(t)=z\}\mathbb{1}\{X_{[0,t]}\subset B^{(2)}(t)\}\Big]$$
$$= \sum_k \mathrm{e}^{t\lambda_k(\xi,B^{(2)}(t))}\,\varphi_k(0)\varphi_k(z),$$

where $(\lambda_k(\xi,B^{(2)}(t)),\varphi_k)_k$ is a sequence of eigenvalues $\lambda_1>\lambda_2\geq\lambda_3\geq\dots$ and L^2-orthonormal eigenfunctions $\varphi_1,\varphi_2,\varphi_3\dots$ of $\Delta+\xi$ in some box $B^{(2)}(t) = t\log^2 t\times[-1,1]^d$ with zero boundary condition.

Main objectives

- Mass concentration: The total mass $U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z)$ comes in probability from just one island (strong form of intermittency).
- Eigenvalue order statistics: The top eigenvalues and the concentration centres of the corresponding eigenfunctions (after rescaling and shifting) form a Poisson point process.

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- Mass concentration: The total mass $U(t) = \sum_{z \in \mathbb{Z}^d} u(t,z)$ comes in probability from just one island (strong form of intermittency).
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Explanation

- The top eigenvalues satisfy an order statistics in some box $B^{(1)}(t) \subset B^{(2)}(t)$. In particular, we have control on their differences, i.e., the spectral gaps.
- The corresponding eigenfunctions are exponentially localised in islands $B_{r_t}(z_k)$ whose locations z_k form a Poisson point process.
- The main contribution to $U(t)$ inside $B_t^{(1)}$ comes from precisely that summand k which maximises $\mathrm{e}^{t\lambda_k(\xi, B_t^{(1)})} \varphi_k(0) \langle \varphi_k, \mathbb{1} \rangle$.
- The contribution to $U(t)$ from the outside of the *a priori* box $B^{(2)}(t)$ is negligible.
- The contribution to $U(t)$ from $B^{(2)}(t) \setminus B^{(1)}(t)$ is negligible since the values of $\mathrm{e}^{t\lambda_k(\xi, B^{(2)}(t))} \varphi_k(0) \langle \varphi_k, \mathbb{1} \rangle$ are substantially worse.

Earlier results

- [Sznitman 98] (Brownian motion among Poisson obstacles) and [Gärtner/K./Molchanov 07] (double-exponential distribution): mass concentration a.s. in $t^{o(1)}$ islands.
- [K./Lacoin/Mörters/Sidorova 09] (Pareto distribution): mass concentration in one site in probability, and in two sites a.s.
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We are working here for ξ double-exponentially distributed, i.e., for some $\rho \in (0,\infty)$,

$$\mathrm{Prob}(\xi(0) > r) = \exp\left\{-\mathrm{e}^{r/\rho}\right\}, \qquad r \in \mathbb{R}.$$

Earlier papers:

[GÄRTNER/MOLCHANOV 98], [GÄRTNER/DEN HOLLANDER 99], [GÄRTNER/K./MOLCHANOV 07].

The potential is unbounded to $+\infty$. The islands are of bounded size. The potential and of the solution approach (after shifting and normalization) certain shapes, which are given by a characteristic variational formula.

First result: Eigenvalue order statistics

Abbreviate $B_L = L \times [-\frac{1}{2}, \frac{1}{2}]^d$.

Theorem 1

There is a number $\chi = \chi_\rho \in (0, 2d)$ and a sequence $(a_L)_{L \in \mathbb{N}}$ with $a_L = \rho \log\log |B_L| - \chi + o(1)$ as $L \to \infty$ and, for any $L \in \mathbb{N}$, a sequence $(X_k^{(L)})_k$ in B_L such that, in probability,

$$\lim_{L \to \infty} \sum_{z \colon |z - X_k^{(L)}| \le \log L} \varphi_k(z)^2 = 1, \qquad k \in \mathbb{N},$$

and the law of

$$\Big(\frac{X_k^{(L)}}{L}, \big(\lambda_k(\xi, B_L) - a_L\big) \log L \Big)_{k \in \mathbb{N}}$$

converges weakly to the ranking, by the value of the last coordinate, of a Poisson process on $B_1 \times \mathbb{R}$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$.

Hence, the top eigenvalues in B_L are of order $\log\log L$, leave gaps of order $1/\log L$, are in the max-domain of attraction of the Gumbel distribution, and the localisation centres are separated by a distance of order L and are uniformly distributed.

Second result: Mass concentration

Theorem 2

Put $r_L = L \log L \log\log\log L$ and

$$\Psi_{L,t}(z,\lambda) = \frac{t}{r_L}(\lambda - a_L)\log L - \frac{|z|}{L},$$

and pick k such that $\Psi_{L,t}(X_k^{(L)}, \lambda_k(\xi, B_L))$ is maximal, and put $Z_{L,t} = X_k^{(L)}/L$. Then, with L_t defined by $r_{L_t} = t$,

$$\lim_{t\to\infty} \frac{1}{U(t)} \sum_{z\colon\, |z - Z_{L_t,t}| \le R_t} u(t,z) = 1 \qquad \text{in probability,}$$

for any $R_t \gg \log t$. Furthermore, as $L \to \infty$, the process $(Z_{L,tr_L})_{t\in[0,\infty)}$ converges in distribution to the process of maximizers of $z \mapsto t\lambda - |z|$ over the points (z,λ) of a Poisson process on $[-\frac{1}{2}, \frac{1}{2}]^d \times \mathbb{R}$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$.

Hence, the total mass comes from a $\gg \log t$-island in the centred box $B^{(1)}(t)$ with radius $\approx t/(\log t \log\log\log t)$

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- We have in particular Anderson localisation at the edge of the spectrum.
- The control from Poisson process convergence holds only in the box $B^{(1)}(t)$ of radius $\asymp t/\log t \log\log\log t$, and the contribution from the outside of the box $B^{(2)}(t)$ of radius $t\log^2 t$ is easily seen to be negligible. The treatment of the region $B^{(2)}(t)\setminus B^{(1)}(t)$ is delicate and requires a comparison of the two eigenvalue expansions.

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- The two terms in the optimized functional $\Psi_{L,t}$ come from the eigenvalue and the probabilistic cost for the random walk in the Feynman-Kac formula to reach the island. The latter term can also be seen as coming from the decay of the eigenfunction term $\varphi_k(0)$.

Some elements of the proof of Theorem 1 (I)

The top eigenvalues in $B = B_L$ remain the top eigenvalues after discarding potential values significantly less than the eigenvalues. Put $\varepsilon_R = 2d\left(1+\frac{A}{2d}\right)^{1-2R}$.

Proposition 1

Fix $A > 0$ and $R \in \mathbb{N}$ and put $U = \bigcup_{z\in B:\ \xi(z)\geq\lambda_1(\xi,B)} B_R(z)$. Then

$$\lambda_k(\xi,B) \geq \lambda_1(\xi,B) - A/2 \qquad \Longrightarrow \qquad |\lambda_k(\xi,B) - \lambda_k(\xi,U)| \leq \varepsilon_R.$$

- Any ℓ^2-normalized eigenvector $v = v_{k,\xi}$ with eigenvalue $\lambda = \lambda_k(\xi,B) \geq \lambda_1 - A/2$ decays rapidly away from U.
- Proof uses the martingale $(v(Y_n)\prod_{k=0}^{n-1}\frac{2d}{2d+\lambda-\xi(Y_k)})_{n\in\mathbb{N}}$ (with $(Y_n)_n$ an SRW).

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- Proof uses the martingale $(v(Y_n)\prod_{k=0}^{n-1} \frac{2d}{2d+\lambda-\xi(Y_k)})_{n\in\mathbb{N}}$ (with $(Y_n)_n$ an SRW).
- Furthermore, we use that $\partial_{\xi(z)}\lambda_k(\xi,B) = v(z)^2$.
- Introduce $\xi_s = \xi - s\mathbb{1}_{B\setminus U}$ for $s \in [0,\infty]$. Then

$$|\partial_s\lambda_k(\xi_s,B)| = \sum_{z\in B\setminus U} v_{k,\xi_s}(z)^2,$$

which is very small. Integrating over $s \in [0,\infty]$ gives the estimate.

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The top eigenvalues are the principal eigenvalues in local regions, and the corresponding eigenfunctions are exponentially localised.

A bit more precisely, with the help of the variational characterisation of the asymptotics of the PAM [GÄRTNER/K./MOLCHANOV 07], one proves the following.

- U consists of connected components of bounded size, which are far away from each other.

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- For any component C, if $\lambda_1(\xi,C)$ is close to $a_L \approx \rho \log\log L$, then $\lambda_1(\xi,C)$ is bounded away from $\lambda_2(\xi,C)$.

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- If λ is an eigenvalue of $\Delta+\xi$ larger than $\lambda_1(\xi,B_L)-A/2$ and v a corresponding ℓ^2-normalised eigenfunction such that the distance of λ to the nearest eigenvalue (spectral gap) is larger than $3\varepsilon_R$, then v decays exponentially away from one of the components of U.

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- The proof uses that the path $[0,\infty]\ni s\mapsto\lambda_k(\xi_s,B_L)$ (with $\xi_s=\xi-s1\!\!1_{B_L\setminus U}$) does not cross other eigenvalues and therefore admits a continuous choice of corresponding eigenfunctions. The one for $s=\infty$ puts all its mass in one component, and the one for $s=0$ is uniformly close.

Some elements of the proof of Theorem 1 (III)

The scale a_L satisfies $\mathrm{Prob}(\lambda_1(\xi,B_R) > a_L) = 1/|B_L|$, hence we may expect finitely many sites in B_L where the local eigenvalue is $\approx a_L$. Then the random variable $\lambda_1(\xi,B_R)$ lies in the max-domain of a Gumbel random variable:

Proposition 2

As $L \to \infty$, for any $s \in \mathbb{R}$,

$$\mathrm{Prob}(\lambda_1(\xi,B_R) > a_L + s/\log L) = \mathrm{e}^{-s}\frac{1}{|B_L|}(1+o(1)).$$

- The event $\{\lambda_1(\xi,B_R) > a\}$ is more or less the same as the event that some shift of the potential $\xi(\cdot)$ is larger than $a+\chi+\psi(\cdot)$ for some well-chosen function ψ.

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- The event $\{\lambda_1(\xi, B_R) > a\}$ is more or less the same as the event that some shift of the potential $\xi(\cdot)$ is larger than $a + \chi + \psi(\cdot)$ for some well-chosen function ψ.
- Shifting ξ by an amount of $s/\log L$ yields an additional factor of e^{-s}, using properties of ψ and of the distribution of ξ and some information from the variational characterisation.

Some elements of the proof of Theorem 2 (I)

Consider boxes $B^{(1)}(t) = B_{L_t^{(1)}}$ and $B^{(2)}(t) = B_{L_t^{(2)}}$ with

$$L_t^{(1)} = \text{const.} \times \frac{t}{\log t \log\log\log t} \qquad \text{and} \qquad L_t^{(2)} = t \log^2 t.$$

- Inside $B^{(1)}(t)$, we have the Poisson process convergence.
- Outside $B^{(2)}(t)$, the contribution is negligible.

Why is the contribution from $B^{(2)}(t) \setminus B^{(1)}(t)$ negligible?

Our Strategy:

- Consider the eigenvalue expansion in $B^{(2)}(t)$. A version of Minami's estimate gives that each spectral gap close to the top is $\geq \varepsilon_R$, with high probability.

- This enables us to prove exponential localisation of the top eigenfunctions in $B^{(2)}(t)$. This makes the top eigenvalues in $B^{(2)}(t)$ essentially independent.

Some elements of the proof of Theorem 2 (II)

Our Strategy (continued):

- The top eigenvalues of $B^{(1)}(t)$ are also top eigenvalues in $B^{(2)}(t)$. But the $B^{(2)}(t)$-eigenfunctions are located much further away (if ‘const’ is large). Hence their contributions in the eigenvalue expansion are negligible w.r.t. the optimizer of $\Psi_{L_t^{(1)},t}$ in $B^{(1)}(t)$.
- For N large enough, the eigenvalues $\lambda_k(\xi, B^{(2)}(t))$ for $k > N$ are negligible w.r.t. the optimizer of $\Psi_{L_t^{(1)},t}$ in $B^{(1)}(t)$ and hence their contribution to the eigenvalue expansion.
- The remaining N eigenvalues can be ordered with gaps $\asymp 1/\log L_t^{(1)} \approx 1/\log t$ between them, and the optimizer is among them.

Future directions

- Presumably, the mass concentration property of the PAM also holds in almost sure sense, but assertion must be adapted.
- Control on the process of localisation centres, $(Z_{L_t,tr_{L_t}})_{t\in[0,\infty)}$, opens up the possibility to study the time-evolution of the PAM, e.g. in terms of ageing properties.
- Replacing double-exponential distribution by bounded distributions will lead to other max-domains of attractions for the top eigenvalues and other rescalings of the gaps.