

Ordered random walks

Wolfgang König

Universität Leipzig

joint work with Peter Eichelsbacher (Bochum)

The Goal

Consider k i.i.d. random walks $X_i = (X_i(n))_{n \in \mathbb{N}_0}$ ($i = 1, \dots, k$) on \mathbb{R} . We ask the following questions:

- What is the conditional version given that the walkers stay in strict order for ever?
- What is the asymptotic probability that they stay in strict order until a late time?
- What is the large-time behaviour of the k walkers given that they stay in strict order until a late time or for ever?

Denote $X = (X_1, \dots, X_k)$, starting from $x \in \mathbb{R}^k$ under \mathbb{P}_x , and

$$W = \{x \in \mathbb{R}^k : x_1 < x_2 < \dots < x_k\} \quad \text{Weyl chamber}$$

$$\tau = \inf\{n \in \mathbb{N} : X(n) \notin W\} \quad \text{exit time from } W,$$

then our questions may be reformulated as follows.

- What is the conditional distribution of X given $\{\tau = \infty\}$?
- What are the asymptotics of $\mathbb{P}_x(\tau > n)$ as $n \rightarrow \infty$?
- Does the distribution of $X(n)/\sqrt{n}$ converge under $\mathbb{P}_x(\cdot \mid \tau > n)$ or $\mathbb{P}_x(\cdot \mid \tau = \infty)$?

Motivation A: Non-colliding random walks

Also called **non-intersecting paths** or **vicious walkers**.

- Appeared in JOHANSSON'S beautiful analysis of the corner-growth model (2002).
- Have remarkable connections to tandem queues (survey article [O'CONNELL '03]).
- Discrete version of **Dyson's Brownian motion** (see below).

Only rather special cases handled yet: nearest-neighbor random walks on \mathbb{Z}^k that satisfy the

continuity property: $\mathbb{P}_x(X(\tau) \in \partial W) = 1.$

Here, 'ordered' is equivalent to 'non-colliding'.

Examples: simple random walk [KATORI/TANEMURA '04], binomial walk, multinomial walk, Poisson walk [K./O'CONNELL/ROCH '02], Yule process [DOUMERC '05].

General random walks not considered before.

Motivation B: Dyson's Brownian motion

Also called **non-colliding Brownian motions**: the continuous version of our question [Dyson 1962].

$H(t) = (H_{i,j}(t))_{i,j=1,\dots,k}$ **Hermitian Brownian motion** (GUE at time $t = 1$)

$\lambda_1(t) \leq \lambda_2(t) \leq \dots \leq \lambda_k(t)$ eigenvalues of $H(t)$

$\lambda = (\lambda_1(t), \dots, \lambda_k(t))_{t \in [0, \infty)}$ eigenvalue process in \overline{W}

Theorem. [DYSON 1962] λ satisfies, for $\beta = 2$, the SDE

$$d\lambda_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt, \quad i = 1, \dots, k. \quad (1)$$

Furthermore, λ is a Brownian motion in \mathbb{R}^k , conditioned on being non-colliding for ever.

Hence, if $T = \inf\{t > 0: B(t) \notin W\}$ is the **exit time** of a BM B in \mathbb{R}^k from the Weyl chamber W , then, formally,

$$\mathcal{L}(\lambda) = \mathcal{L}(B \mid T = \infty).$$

(More about that later)

Motivation C: Fluctuation Theory

The special case $k = 2$ is equivalent to conditioning a random walk S on \mathbb{R} to stay positive at all times. Fluctuation theory studies conditioning on being nonnegative. The answer is given in terms of a **Doob h -transform**. If the walker's steps have finite mean, then

$$V(x) = \frac{x - \mathbb{E}_x[S_\sigma]}{-\mathbb{E}_0[S_\sigma]}, \quad \text{where } \sigma = \inf\{n \in \mathbb{N} : S_n < 0\},$$

turns out to be a positive harmonic function for the restriction to $[0, \infty)$. The transform

$$\widehat{\mathbb{P}}_x((S_0, \dots, S_n) \in A) = \mathbb{P}_x((S_0, \dots, S_n) \in A, \sigma > n) \frac{V(S_n)}{V(S_0)}, \quad A \subset [0, \infty)^{n+1},$$

is equal to the limiting process S , given that $\{\sigma > n\}$ as $n \rightarrow \infty$. Furthermore,

$$\lim_{n \rightarrow \infty} \sqrt{n} \mathbb{P}_x(\sigma > n) = V(x), \quad x \in [0, \infty).$$

Main tools: **duality** and **Sparre-Andersen identity** (see [FELLER '71], e.g.).

More on non-colliding BMs

Proper definition in terms of **Doob h -transform** with $h = \Delta$, where

$$\Delta(x) = \prod_{1 \leq i < j \leq k} (x_j - x_i) = \det \left[(x_i^{j-1})_{i,j=1,\dots,k} \right], \quad \text{Vandermonde determinant}$$

Main properties: Δ is harmonic for $\frac{1}{2} \sum_{i=1}^k \partial_i^2$, and $\Delta > 0$ in W .

transition probability density of the h -transform:

$$\hat{p}_t(x, y) dy = \mathbb{P}_x(B(t) \in dy; T > t) \frac{\Delta(y)}{\Delta(x)}, \quad x, y \in W.$$

Lemma. [KARLIN/MCGREGOR 1958]

$$\mathbb{P}_x(B(t) \in dy; T > t) = \det \left[(p_t(x_i, y_j))_{i,j=1,\dots,k} \right] dy.$$

Main tools of the proof: reflection principle and a clever enumeration.

Some properties

Corollary.

● $\hat{p}_t(0, y) = Ct^{-\frac{k}{4}}(k-1)(2\pi t)^{-k/2}e^{-|y|^2/(2t)}\Delta(y)^2$ **Hermite ensemble**

● $\mathbb{P}_x(T > t) \sim Ct^{-\frac{k}{4}}(k-1)\Delta(x)$ as $t \rightarrow \infty$ **non-colliding probability**

● $\lim_{t \rightarrow \infty} \mathbb{P}_x(t^{-1/2}B(t) \in dy \mid T > t) = Ce^{-|y|^2/2}\Delta(x)\Delta(y)$

● $\mathbb{P}_0(B(t) \in dy \mid T > t) = Ce^{-|y|^2/(2t)}\Delta(y)$

Sketch of proof of the first observation: Observe that

$$\hat{p}_t(x, y) = C(2\pi t)^{-k/2}e^{-|x|^2/(2t)}e^{-|y|^2/(2t)} \det \left[(e^{x_i y_j / t})_{i,j} \right] \frac{\Delta(y)}{\Delta(x)}.$$

As $x \rightarrow 0$,

$$\begin{aligned} \det \left[(e^{x_i y_j / t})_{i,j} \right] &\sim \det \left[\left(\sum_{l=1}^k \frac{x_i^{l-1}}{(l-1)!t^{l-1}} y_j^{l-1} \right)_{i,j} \right] \\ &= \det \left[\left(\frac{x_i^{l-1}}{(l-1)!t^{l-1}} \right)_{i,l} \right] \det \left[(y_j^{l-1})_{l,j} \right] = Ct^{-\frac{k}{4}}(k-1)\Delta(x)\Delta(y). \end{aligned}$$

Back to the question

Hence, we are looking for a positive regular function $V: W \rightarrow (0, \infty)$ for the restriction of the kernel of the walk X on \mathbb{R}^k . (Recall: W is the Weyl chamber, and τ its exit time.)

Under the above-mentioned continuity property, the Vandermonde determinant Δ is a positive regular function for the restriction to W , and the solution is similar to the Brownian case.

What is a suitable positive regular function in the general case?

Here it is:

$$V(x) = \Delta(x) - \mathbb{E}_x [\Delta(X(\tau))], \quad x \in W.$$

Easily seen to be regular: Since $(\Delta(X(n)))_{n \in \mathbb{N}}$ is a martingale [K./O'CONNELL '01], we have

$$\begin{aligned} \mathbb{E}_x [\mathbb{1}_{\{\tau > 1\}} V(X(1))] &= \mathbb{E}_x [\mathbb{1}_{\{\tau > 1\}} \Delta(X(1))] - \mathbb{E}_x [\mathbb{1}_{\{\tau > 1\}} \mathbb{E}_{X(1)} [\Delta(X(\tau))]] \\ &= \mathbb{E}_x [\mathbb{1}_{\{\tau > 1\}} \Delta(X(1))] - \mathbb{E}_x [\mathbb{1}_{\{\tau > 1\}} \Delta(X(\tau))] \\ &= \mathbb{E}_x [\mathbb{1}_{\{\tau > 1\}} \Delta(X(1))] - \mathbb{E}_x [\Delta(X(\tau))] + \mathbb{E}_x [\Delta(X(\tau)) \mathbb{1}_{\{\tau \leq 1\}}] \\ &= \mathbb{E}_x [\Delta(X(1))] - \mathbb{E}_x [\Delta(X(\tau))] + \mathbb{E}_x [\Delta(X(\tau)) \mathbb{1}_{\{\tau \leq 1\}}] - \Delta(X(1)) \mathbb{1}_{\{\tau \leq 1\}} \\ &= V(x). \end{aligned}$$

Difficulties

Not easy to see: $V \geq 0$ on W .

more difficult to see: $V > 0$ on W .

very difficult to see: V is well-defined, i.e., $\Delta(X(\tau))$ is integrable!

Why so delicate:

Consider

$$\mathbb{E}_x [|\Delta(X(\tau))|] = \sum_{n \in \mathbb{N}} \mathbb{E}_x \left[\prod_{i < j} |X_j(n) - X_i(n)| \mathbb{1}_{\{\tau=n\}} \right].$$

All the factors $|X_j(n) - X_i(n)|$ are $O(\sqrt{n})$ with the exception of one of them.

We expect (and later prove) that $\mathbb{P}_x(\tau > n) \approx n^{-\frac{1}{4}k(k-1)}$.

Hence, we should have (and do not prove) that $\mathbb{P}_x(\tau = n) \approx n^{-\frac{1}{4}k(k-1)-1}$.

Hence, we should have

$$\mathbb{E}_x \left[\prod_{i < j} |X_j(n) - X_i(n)| \mathbb{1}_{\{\tau=n\}} \right] \approx n^{-\frac{3}{2}},$$

which is enough.

The main result

Assume that the steps have mean zero and variance one, and that the local central limit theorem holds.

Theorem. [EICHELSBACHER/K. 06] If sufficiently high step moments are finite, the following hold.

- (i) $\Delta(X(\tau))$ is integrable under \mathbb{P}_x for any $x \in W$.
- (ii) V is a positive regular function for the restriction of the transition kernel to W .
- (iii) The Doob h -transform with $h = V$ is equal to X , given $\{\tau > n\}$ as $n \rightarrow \infty$.
- (iv)

$$\lim_{n \rightarrow \infty} \mathbb{P}_x(n^{-\frac{1}{2}} X(n) \in A \mid \tau > n) = \frac{1}{Z_1} \int_A e^{-\frac{1}{2}|y|^2} \Delta(y) dy \quad \text{weakly.}$$

and, for some $K \in (0, \infty)$,

$$\lim_{n \rightarrow \infty} n^{\frac{k}{4}(k-1)} \mathbb{P}_x(\tau > n) = KV(x), \quad x \in W.$$

- (v) Uniformly on compacts, $\lim_{n \rightarrow \infty} n^{-\frac{k}{4}(k-1)} V(\sqrt{n}x) = \Delta(x)$.
- (vi) For any $x \in W$, the distribution of the process $(n^{-\frac{1}{2}} X(\lfloor nt \rfloor))_{t \in [0, \infty)}$ under $\hat{\mathbb{P}}_{\sqrt{n}x}$ converges towards Dyson's Brownian motions started at x .

Main tool: generalized Karlin-McGregor formula

The transform is given by

$$\widehat{\mathbb{P}}_x^{(V)}(X(n) \in dy) = \mathbb{P}_x(\tau > n; X(n) \in dy) \frac{V(y)}{V(x)}, \quad x, y \in W, n \in \mathbb{N}.$$

Introduce the signed measure

$$\mathcal{D}_n(x, dy) = \det \left[\left(\mathbb{P}_{x_i}(X_1(n) \in dy_j) \right)_{i,j=1,\dots,k} \right].$$

Proposition. Let $X = (X(n))_{n \in \mathbb{N}_0}$ be an arbitrary random walk on \mathbb{R}^k with i.i.d. components. Then, for any $n \in \mathbb{N}$ and any $x, y \in W$,

$$\mathbb{P}_x(\tau > n; X(n) \in dy) = \mathcal{D}_n(x, dy) - \mathbb{E}_x \left[\mathbb{1}_{\{\tau \leq n\}} \mathcal{D}_{n-\tau}(X(\tau), dy) \right].$$

- Proof via strong Markov property and clever enumeration. No reflection involved.
- Under the continuity property, some two components of $X(\tau)$ are identical, and the determinant vanishes.

The main approximation argument

How do we obtain the Vandermonde determinant from the asymptotics of $\mathcal{D}_n(x, dy)$?

Via the **local central limit theorem**. Using similar approximations as in the Brownian case, this gives that

$$\mathcal{D}_n(x, d(\sqrt{n}y)) \approx K' n^{-\frac{1}{4}k(k-1)} e^{-\frac{1}{2}|y|^2} \Delta(y) \Delta(x) dy,$$

uniformly on compacts. Using the generalized Karlin-McGregor formula, we should have

$$\begin{aligned} \frac{\mathbb{P}_x(\tau > n, n^{-\frac{1}{2}} X(n) \in dy)}{dy} &\approx K' n^{-\frac{1}{4}k(k-1)} e^{-\frac{1}{2}|y|^2} \Delta(y) [\Delta(x) - \mathbb{E}_x[\mathbb{1}_{\{\tau \leq n\}} \Delta(X(\tau))]] \\ &\approx K' n^{-\frac{1}{4}k(k-1)} e^{-\frac{1}{2}|y|^2} \Delta(y) V(x), \end{aligned}$$

if one is allowed to interchange asymptotics and expectation, and if the latter is well-defined.

Technical steps: distinguish $\{\tau \leq \mathcal{O}(\sqrt{n})\}$, $\{\mathcal{O}(\sqrt{n}) < \tau \leq n - \mathcal{O}(\sqrt{n})\}$ and $\{n - \mathcal{O}(\sqrt{n}) < \tau \leq n\}$, and use some clever recursive bounds.

Open questions

- Relation to the eigenvalue processes of some matrix-valued random walks?
- Relation to general corner-growth process?
- How to construct ordered random walks under less integrability?
- Is there a useful duality principle?
- Behaviour of the k ordered random walks if $k \rightarrow \infty$? Convergence of the empirical measure of some marginal distribution (version of WIGNER's semicircle law)? (See [BAIK/SUIDAN 06] for some partial answer.)