Matrix-Valued Diffusions and Non-Colliding Random Processes

Wolfgang König

Universität Leipzig

Goal

- In the ten ensembles, replace the normal variables by Brownian motions
- Derive the SDE for the eigenvalue processes
- Identify these processes in terms of Doob h-transforms (if possible)
- Interpret them as systems of non-colliding random processes

The main matrix diffusions:

$S = (S(t))_{t\in[0,\infty)} \in \mathbb{R}^{N\times N}_{\mathrm{sym}}$ symmetric Brownian motion (GOE at time 1)

$A = (A(t))_{t\in[0,\infty)} \in \mathbb{R}^{N\times N}_{\mathrm{antisym}}$ antisymmetric Brownian motion

$S + \mathrm{i}\, A$ Hermitian Brownian motion (GUE at time 1)

Dyson's Brownian motions

$\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_N(t)$ eigenvalues of $(S + \mathrm{i}\, A)(t)$

$\lambda = (\lambda_1(t), \dots, \lambda_N(t))_{t\in[0,\infty)}$ eigenvalue process in $\overline{W}$

$W = \{x \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}$ Weyl chamber (of type A)

Theorem. [DYSON 1962] λ SATISFIES, FOR $\beta = 2$, THE SDE

$$\mathrm{d}\lambda_i(t) = \mathrm{d}B_i(t) + \frac{\beta}{2} \sum_{j\neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)}\, \mathrm{d}t, \qquad i = 1, \dots, N. \tag{1}$$

FURTHERMORE, λ IS A BM IN $\mathbb{R}^N$, CONDITIONED ON BEING NON-COLLIDING FOR EVER.

Hence, if $T = \inf\{t > 0\colon B(t) \notin W\}$ is the exit time of a BM B in $\mathbb{R}^N$ from W, then, formally,

$$\mathcal{L}(\lambda) = \mathcal{L}(B \mid T = \infty).$$

Non-colliding BM

Proper definition in terms of Doob h-transform with $h = \Delta$, where

$$\Delta(x) = \prod_{1 \leq i < j \leq N} (x_j - x_i) = \det\left[\left(x_i^{j-1}\right)_{i,j=1,\dots,N}\right]$$ Vandermonde determinant

Δ is harmonic for $\frac{1}{2}\sum_{i=1}^N \partial_i^2$, and $\Delta > 0$ in W.

transition probability density of the h-transform:

$$\widehat{p}_t(x,y)\,\mathrm{d}y = \mathbb{P}_x(B(t) \in \mathrm{d}y; T > t)\frac{\Delta(y)}{\Delta(x)}, \qquad x, y \in W.$$

Lemma. [Karlin/McGregor 1958]

$$\mathbb{P}_x(B(t) \in \mathrm{d}y; T > t) = \det\left[\left(p_t(x_i, y_j)\right)_{i,j=1,\dots,N}\right]\mathrm{d}y.$$

(This is true for all diffusions with i.i.d. components in place of B, if $p_t(x,y)$ denotes their transition probability function.)

Some properties

Corollary.

- $\widehat{p}_t(0,y) = Ct^{-\frac{N}{2}(N-1)} \mathrm{e}^{-|y|^2/(2t)} \Delta(y)^2$ Hermite ensemble
- $\mathbb{P}_x(T > t) \sim Ct^{-\frac{N}{4}(N-1)} \Delta(x)$ as $t \to \infty$ non-colliding probability
- $\lim_{t\to\infty} \mathbb{P}_x(t^{-1/2}B(t) \in \mathrm{d}y \mid T > t) = C\mathrm{e}^{-|y|^2/2} \Delta(x)\Delta(y)$
- $\mathbb{P}_0(B(t) \in \mathrm{d}y \mid T > t) = C\mathrm{e}^{-|y|^2/(2t)} \Delta(y)$

Sketch of proof of the first assertion for $t = 1$: Observe that

$$\widehat{p}_t(x,y) = Ct^{-\frac{N}{4}(N-1)} \mathrm{e}^{-|x|^2/(2t)} \mathrm{e}^{-|y|^2/(2t)} \det\Big[\big(\mathrm{e}^{x_i y_j/t}\big)_{i,j}\Big] \frac{\Delta(y)}{\Delta(x)}.$$

For $t = 1$, as $x \to 0$,

$$\begin{aligned} \det\Big[\big(\mathrm{e}^{x_i y_j}\big)_{i,j}\Big] &\sim \det\Big[\Big(\sum_{k=1}^{N} \frac{x_i^{k-1}}{(k-1)!} y_j^{k-1}\Big)_{i,j}\Big] \\ &= \det\Big[\Big(\frac{x_i^{k-1}}{(k-1)!}\Big)_{i,k}\Big] \det\Big[\big(y_j^{k-1}\big)_{j,k}\Big] = C\Delta(x)\Delta(y). \end{aligned}$$

Other values of β

- Real variant of Dyson's theorem: the eigenvalue process of S (GOE diffusion) satisfies the Dyson SDE (1) with $\beta = 1$. This is not an h-transform of BM.
- The diffusion defined by Dyson's SDE in (1) is well-defined and non-colliding for any $\beta > 0$. Identification in terms of eigenvalue processes known only for $\beta = 1, 2$ (see above) and $\beta = 4$ (see later).

Transition from GUE to GOE

> **Lemma.** [KATORI/TANEMURA 2003] FIX $t > 0$. LET $\widetilde{A} = (\widetilde{A}(s))_{s\in[0,t]} \in \mathbb{R}^{N\times N}_{\mathrm{antisym}}$ BE AN ANTISYMMETRIC BROWNIAN BRIDGE MATRIX DIFFUSION. THEN THE EIGENVALUE PROCESS OF $S + \mathrm{i}\,\widetilde{A}$ IS IN DISTRIBUTION EQUAL TO $(B(s))_{s\in[0,t]}$, CONDITIONED ON $\{T > t\}$.

Explanation: Decompose the components of S as

$$B_{i,j}(s) = \Big(B_{i,j}(s) - \frac{s}{t} B_{i,j}(t)\Big) + \frac{s}{t} B_{i,j}(t)$$

and summarize the bridge part and the remainder, then

$$S + \mathrm{i}\,\widetilde{A} = \text{ Hermitian bridge diffusion } + \Big(\frac{s}{t} S(t)\Big)_{s\in[0,t]},$$

which interpolates between GUE and GOE as s varies.

$S + \mathrm{i}\,\widetilde{A}$ is called a two-matrix model.

More general matrix diffusions

Theorem. [Bru 1991], [Katori/Tanemura 2004] Let ξ be a complex continuous Hermitian matrix-valued semimartingale, and let $U(t)$ be unitary with

$$U(t)^*\xi(t)U(t) = \mathrm{diag}[\lambda(t)], \qquad \text{with } \lambda(t) \in W.$$

Then, for $i = 1, \dots, N$,

$$\mathrm{d}\lambda_i(t) = \mathrm{d}M_i(t) + \sum_{j=1}^{N} \frac{\Gamma_{i,j}(t)\,\mathrm{d}t}{\lambda_i(t) - \lambda_j(t)} 1\!\!1_{\{\lambda_i(t)\neq\lambda_j(t)\}} + \mathrm{d}\Theta_i(t),$$

where M_i is a martingale,

$$\Gamma_{i,j}(t)\,\mathrm{d}t = \big(U(t)^*\mathrm{d}\xi(t)U(t)\big)_{i,j}\,\big(U(t)^*\mathrm{d}\xi(t)U(t)\big)_{j,i} \quad \text{and} \quad \langle M_i\rangle_t = \int_0^t \Gamma_{i,j}(s)\,\mathrm{d}s,$$

and $\mathrm{d}\Theta_i(t)$ is the finite-variation part of $\big(U(t)^*\mathrm{d}\xi(t)U(t)\big)_{i,i}$.

Wishart Processes

[Bru 1991], [K./O'Connell 2001], [Katori/Tanemura 2004]

Consider $\xi = D^* D$, where, with $\nu \in \mathbb{N}$ a parameter,

$$D(t) = \big(B_{i,j}(t) + \mathrm{i}\,\widetilde{B}_{i,j}(t)\big)_{\substack{i=1,\dots,N+\nu \\ j=1,\dots,N}}.$$

Every component of ξ is a BESQ$^{2\nu+2}$. ξ is called the Wishart process and has nonnegative eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_N$. One calculates

$$\mathrm{d}\Theta_i(t) = 2(N+\nu)\,\mathrm{d}t, \quad \Gamma_{i,j}(t) = 2(\lambda_i(t) + \lambda_j(t)), \qquad \langle M_i \rangle_t = 4\int_0^t \lambda_i(s)\,\mathrm{d}s.$$

Hence:

The eigenvalue process, λ, of ξ satisfies, for any $i = 1, \dots, N$, the SDE

$$\mathrm{d}\lambda_i(t) = 2\sqrt{\lambda_i(t)}\,\mathrm{d}B_i(t) + \beta\Big(N + \nu + \sum_{j \neq i} \frac{\lambda_i(t) + \lambda_j(t)}{\lambda_i(t) - \lambda_j(t)}\Big)\,\mathrm{d}t, \tag{2}$$

with $\beta = 2$.

Remarks

- The squareroot $\kappa_i = \sqrt{\lambda_i}$ satisfies, for $\beta = 2$ and $\gamma = \nu + \frac{1}{2}$, the SDE

$$\mathrm{d}\kappa_i(t) = \mathrm{d}B_i(t) + \frac{\beta}{2}\Big(\frac{\gamma}{\kappa_i(t)} + \sum_{j \neq i}\Big(\frac{1}{\kappa_i(t) - \kappa_j(t)} + \frac{1}{\kappa_i(t) + \kappa_j(t)}\Big)\Big)\,\mathrm{d}t \tag{3}$$

- The distribution of $\lambda(t)$ is the Laguerre ensemble with density

$$y \mapsto C t^{-N(N+\nu)} \Delta(y)^2 \prod_{i=1}^{N} \big(\mathrm{e}^{-y_i/(2t)} y_i^{\nu}\big).$$

  Hence, $\lambda(1)$ is the eigenvalue distribution of the chiral GUE, the class AIII.

- Real version: if the complex part of D is dropped, then λ satisfies (2) with $\beta = 1$. Then $\kappa_i = \sqrt{\lambda_i}$ satisfies (3) with $\beta = 1$ and $\gamma = \nu$, and $\kappa(t)$ has density

$$z \mapsto C t^{-\frac{N}{2}(N+\nu)} \Delta(z^2) \prod_{i=1}^{N} \big(\mathrm{e}^{-z_i^2/(2t)} z_i^{\nu}\big).$$

  In particular, $\lambda(1)$ is the eigenvalue distribution of the chiral GOE, the class BDI.

Laguerre process and non-colliding BESQ

The Wishart analogon to Dyson's theorem:

Lemma For $\beta = 2$ and $\gamma = \nu + \frac{1}{2} \in \mathbb{N} + \frac{1}{2}$, the process λ is distributed as the Doob h-transform of N independent BESQ$^{2\nu+2}$ with $h = \Delta$; i.e., λ is non-colliding BESQ$^{2\nu+2}$.

Hence, Δ is also a positive harmonic function for the generator of N independent BESQ$^{2\nu+2}$-processes,

$$Gf(x) = 2\sum_{i=1}^{N} x_i \partial_i^2 f(x) + \nu \sum_{i=1}^{N} \partial_i f(x).$$

The transition probability function of λ may be expressed as

$$\widehat{p}_t(x, y)\,\mathrm{d}y = \mathbb{P}_x(Y(t) \in \mathrm{d}y; T > t)\frac{\Delta(y)}{\Delta(x)},$$

where T is the exit time of BESQ$^{2\nu+2}(\mathbb{R}^N)$, Y, from W. The Karlin-McGregor formula applies, and analogs of the above corollary hold.

AII and Dyson's model with $\beta = 4$

Let S_1, S_2, S_3 be three independent copies of the symmetric Brownian $N \times N$-matrix diffusion S (GOE-diffusion), and consider the $(2N \times 2N)$-dimensional matrix diffusion

$$\xi = \begin{pmatrix} S_1 + \mathrm{i}\, A & S_2 + \mathrm{i}\, S_3 \\ S_2 - \mathrm{i}\, S_3 & S_1 - \mathrm{i}\, A \end{pmatrix}.$$

Then $\xi(t)$ is in the class AII and possesses the eigenvalues $\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_N(t)$, each one precisely twice. Put $\lambda = (\lambda_1, \dots, \lambda_N)$.

Lemma [Katori/Tanemura 2004] λ solves Dyson's SDE in (1) with $\beta = 4$.

Remark. The same is true for the eigenvalue process of

$$\xi = \begin{pmatrix} S - A_1 & \mathrm{i}\, A_2 - A_3 \\ \mathrm{i}\, A_2 + A_3 & S + A_1 \end{pmatrix},$$

where A_1, A_2, A_3 are three independent copies of the antisymmetric Brownian matrix diffusion A.

Class C and non-colliding BM with wall at 0

Let S_1, S_2, S_3 be three independent copies of the symmetric Brownian matrix diffusion S (GOE-diffusion), and consider the $(2N \times 2N)$-dimensional matrix diffusion

$$\xi = \begin{pmatrix} S_1 + \mathrm{i}\, A & S_2 - \mathrm{i}\, S_3 \\ S_2 + \mathrm{i}\, S_3 & -S_1 + \mathrm{i}\, A \end{pmatrix}.$$

Then $\xi(t)$ is in the class C and possesses the eigenvalues

$$-\lambda_N(t) \leq \cdots \leq -\lambda_1(t) \leq 0 \leq \lambda_1(t) \leq \cdots \leq \lambda_N(t).$$

Lemma [Katori/Tanemura 2004] λ solves the SDE in (3) with $\beta = 2$ and $\gamma = 1$ (i.e., $\nu = \frac{1}{2}$).

Hence, it is in distribution equal to N BMs, conditioned on never leaving $W^{\mathrm{C}} = \{x \in \mathbb{R}^N : 0 < x_1 < x_2 < \cdots < x_n\}$.

- W^{C} is the Weyl chamber of type C.
- λ is the Doob h-transform of a tuple of a BM in $\mathbb{R}^N$ with $h(x) = \Delta(x^2) \prod_{i=1}^N x_i$.
- $\lambda(t)$ has the density

$$y \mapsto Ct^{-\frac{N}{2}(2N+1)} \Delta(y^2)^2 \Big(\prod_{i=1}^N y_i^2 \Big) \mathrm{e}^{-|y|^2/(2t)} = Ct^{-\frac{N}{2}(2N+1)} h(y)^2 \mathrm{e}^{-|y|^2/(2t)}.$$

Class D and non-colliding reflected BMs

Let A_1, A_2, A_3 be three independent copies of the antisymmetric Brownian matrix diffusion A, and consider the $(2N \times 2N)$-dimensional matrix diffusion

$$\xi = \begin{pmatrix} S + \mathrm{i}\, A_1 & \mathrm{i}\, A_2 - A_3 \\ \mathrm{i}\, A_2 + A_3 & -S + \mathrm{i}\, A_1 \end{pmatrix}.$$

Then $\xi(t)$ is in the class D and possesses the eigenvalues

$$-\lambda_N(t) \leq \cdots \leq -\lambda_1(t) \leq 0 \leq \lambda_1(t) \leq \cdots \leq \lambda_N(t).$$

Lemma [KATORI/TANEMURA 2004] λ SOLVES THE SDE IN (3) WITH $\beta = 2$ AND $\gamma = 0$ (I.E., $\nu = -\frac{1}{2}$).

HENCE, IT IS IN DISTRIBUTION EQUAL TO N REFLECTED BMS, CONDITIONED ON NON-COLLISION.

- Equivalently, λ is a BM in $\mathbb{R}^N$, conditioned on never leaving the Weyl chamber of type D, $W^{\mathrm{D}} = \{x \in \mathbb{R}^N : |x_1| < x_2 < \cdots < x_N\}$.
- λ is the Doob h-transform of a tuple of N reflected BMs in $[0, \infty)$, with $h(x) = \Delta(x^2)$.
- $\lambda(t)$ has the density

$$y \mapsto Ct^{-\frac{N}{2}(2N+1)} \Delta(y^2)^2 \mathrm{e}^{-|y|^2/(2t)} = Ct^{-\frac{N}{2}(2N+1)} h(y)^2 \mathrm{e}^{-|y|^2/(2t)}.$$

Remarks on classes C and D

- The SDE that the eigenvalue process λ satisfies in classes C and D is the one that is satisfied by $\kappa_i = \sqrt{\lambda_i}$ in the complex Wishart case (i.e., $\beta = 2$) for $\nu \in \mathbb{N}$:

$$\mathrm{d}\kappa_i(t) = \mathrm{d}B_i(T) + \Big(\frac{2\nu+1}{2\kappa_i(t)} + \sum_{j\neq i}\Big(\frac{1}{\kappa_i(t)-\kappa_j(t)} + \frac{1}{\kappa_i(t)+\kappa_j(t)}\Big)\Big)\,\mathrm{d}t, \qquad (4)$$

  but now for $\nu = \frac{1}{2}$ (class C) and $\nu = -\frac{1}{2}$ (class D).

- The process κ in (4) is well-defined for any $\nu \in (-1, \infty)$. Its transition densities are explicit in terms of Bessel functions and determinants, using the Karlin-McGregor formula. It is a Doob h-transform only in the two special cases $\nu = \frac{1}{2}$ and $\nu = -\frac{1}{2}$.

Real versions of classes C and D

Real version of class C: If the complex part of case-C ξ is dropped, i.e.,

$$\xi = \begin{pmatrix} S_1 & S_2 \\ S_2 & -S_1 \end{pmatrix},$$

then ξ is in the class CI. Its eigenvalue process λ solves the SDE

$$\mathrm{d}\kappa_i(t) = \mathrm{d}B_i(T) + \frac{\beta}{2}\Big(\frac{\gamma}{\kappa_i(t)} + \sum_{j\neq i}\Big(\frac{1}{\kappa_i(t)-\kappa_j(t)} + \frac{1}{\kappa_i(t)+\kappa_j(t)}\Big)\Big)\,\mathrm{d}t,$$

with $\beta = 1$ and $\gamma = 1$ (i.e., $\nu = \frac{1}{2}$). However, λ is not a Doob h-transform. The density of $\lambda(t)$ is

$$y \mapsto Ct^{-\frac{N}{2}(N+1)}\Delta(y^2)\Big(\prod_{i=1}^N y_i\Big)\mathrm{e}^{-|y|^2/(2t)} = Ct^{-\frac{N}{2}(N+1)}h(y)\mathrm{e}^{-|y|^2/(2t)}.$$

Real version of class D: If the complex part of case-D ξ is dropped, then the eigenvalue process solves the above SDE with $\beta = 1$ and $\gamma = 0$ (i.e., $\nu = -\frac{1}{2}$). The density of $\lambda(t)$ is

$$y \mapsto Ct^{-\frac{N}{2}(N-1)}\Delta(y^2)\mathrm{e}^{-|y|^2/(2t)} = Ct^{-\frac{N}{2}(N-1)}h(y)\mathrm{e}^{-|y|^2/(2t)}.$$

Transitions between the classes

[Katori/Tanemura 2004]

- Analogously to the two-matrix model that interpolates from GUE to GOE in the time interval $[0, t]$ ($\Longrightarrow$ BMs non-colliding up to time t), there is a non-colliding family of generalised Brownian meanders (with parameters choosen appropriately) which interpolate between chGUE and chGOE respectively between C and CI in the interval $[0, t]$.
- **Explanation:** A Brownian meander is a one-dimensional BM conditioned to stay positive up to time t. In the limit $t \to \infty$, it converges to a three-dimensional Bessel process, which is in distribution equal to a one-dimensional BM conditioned on staying positive at all times.
- A related transition is observed for non-colliding generalised Brownian meanders if the non-colliding property collapses at time t in the way that all paths collide simultaneously or that only pairwise collisions occur. Here one has transitions from GUE to GSE, from chGUE to chGSE, and from D to DIII, respectively.