Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

The Mean-Field Polaron Model

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based on joint work with C. Mukherjee and E. Bolthausen

The mean-field polaron model

$W=(W_t)_{t\geq 0}$ Brownian motion in $\mathbb{R}^3$

Transformed path measure $\mathrm{d}\widehat{\mathbb{P}}_t = \frac{1}{Z_t}\,\mathrm{e}^{tH_t}\,\mathrm{d}\mathbb{P}$, with partition function $Z_t=\mathbb{E}[\mathrm{e}^{tH_t}]$ and

$$\text{energy}\qquad H_t=\frac{1}{t^2}\int_0^t\int_0^t \mathrm{d}\sigma\mathrm{d}s\,\frac{1}{\big|W_\sigma-W_s\big|}.$$

This is a function of the

$$\text{normalized occupation measure}\qquad L_t=\frac{1}{t}\int_0^t \mathrm{d}s\,\delta_{W_s}.$$

Indeed, introducing

$$\text{Coulomb potential energy}\qquad H(\mu) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mu(\mathrm{d}x)\,\mu(\mathrm{d}y)}{|x-y|},$$

$$\text{Coulomb potential functional}\qquad (\Lambda\mu)(x) = \Big(\mu\star\frac{1}{|\cdot|}\Big)(x)=\int_{\mathbb{R}^3}\frac{\mu(\mathrm{d}y)}{|x-y|},$$

then

$$H_t=H(L_t)=\langle L_t,\Lambda(L_t)\rangle.$$

Background: the polaron model

What one is really interested in is the model

$$\mathrm{d}\widehat{\mathbb{P}}_{\lambda,t} = \frac{1}{Z_{\lambda,t}} \exp\left\{\lambda \int_0^t \int_0^t \mathrm{d}\sigma \mathrm{d}s \frac{\mathrm{e}^{-\lambda|\sigma-s|}}{|W_\sigma - W_s|}\right\} \mathrm{d}\mathbb{P}.$$

Physically relevant: strong coupling limit $\lambda \to 0$ (after $t \to \infty$).

Intuition: Then all times s, σ interact equally, and the mean-field model should be approached.

Both models are self-attractive, as they should enforce that $|W_\sigma - W_s| \asymp 1$ for all s, σ.
Challenges:

- Prove behaviour of partition functions ($\Longrightarrow$ [DONSKER/VARADHAN 1983])
- Prove localisation of mean-field model ($\Longrightarrow$ this talk)
- Prove convergence of the mean-field model to the Pekar process ($\Longrightarrow$ this talk)
- Understand path behaviour in the limit $t \to \infty$, followed by $\lambda \to 0$, in particular convergence to the Pekar process ($\Longrightarrow$ future; heuristics in [SPOHN (1987)])

Behaviour of partition functions [Donsker/Varadhan 1983]

$$\lim_{\lambda\to 0}\lim_{t\to\infty}\frac{1}{t}\log Z_{\lambda,t}=\lim_{t\to\infty}\frac{1}{t}\log Z_t=\sup_{\mu\in\mathcal{M}_1(\mathbb{R}^3)}\Big(H(\mu)-I(\mu)\Big)$$
$$=\sup_{\psi\in H^1(\mathbb{R}^3)\colon\,\|\psi\|_2=1}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{d}x\mathrm{d}y\,\frac{\psi^2(x)\psi^2(y)}{|x-y|}-\frac{1}{2}\|\nabla\psi\|_2^2\Big).$$

Note that H and I are shift-invariant.

[Lieb 1976]: Up to shifts, there is precisely one minimiser $\mu_0(\mathrm{d}x)=\psi_0(x)^2\,\mathrm{d}x$, i.e., the set $\mathfrak{m}$ of minimizers is equal to

$$\mathfrak{m}=\big\{\mu_0\star\delta_x=:\mu_x\colon x\in\mathbb{R}^3\big\}.$$

ψ_0 is smooth, rotationally symmetric and centered.

[Bolthausen/Schmock (1997)]: spatially discrete version of $\widehat{\mathbb{P}}_t$; full understanding of partition function, localisation, convergence to Pekar-like process.

First substantial step for the mean-field polaron model after 1983: L_t converges to $\mathfrak{m}$.

Tube property [Mukherjee/Varadhan 2014]

For any neighbourhood $U(\mathfrak{m})$ of $\mathfrak{m}$, $\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t\big\{L_t \notin U(\mathfrak{m})\big\} < 0$.

Important idea: a true compactification of the quotient space $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ of orbits $\widetilde{\mu} = \{\mu \star \delta_x : x \in \mathbb{R}^d\}$ of $\mu \in \mathcal{M}_1(\mathbb{R}^d)$.

Here is the idea.

Let a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_1(\mathbb{R}^d)$ be given.

How can we extract a converging subsequence?

Problem: Mass may escape and leak out (e.g., $\mu_n = \frac{1}{2}(\delta_{nx} + \delta_{-nx})$) or spread too flat (e.g., $\mu_n = \mathcal{N}(0, n)$) or many mixtures of these.

Here is the procedure: Along a subsequence,

$$\sup_{x\in\mathbb{R}^d} \mu_n(B_R + x) \quad \stackrel{n\to\infty}{\longrightarrow} \quad p(R)$$

L_t approaches $\mathfrak{m}$

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$$\sup_{x\in\mathbb{R}^d} \mu_n(B_R + x) \quad \stackrel{n\to\infty}{\longrightarrow} \quad p(R) \quad \stackrel{R\to\infty}{\longrightarrow} \quad p_1 \in [0,1]$$

Peel off the corresponding mass distribution. Work with the leftover. Iterate. Collect all partial masses in a sequence.

Compactification

In other words, decompose

$$\mu_n = \underbrace{\mu_n\big|_{B_R(x_n)}}_{\gamma_n} + \underbrace{\text{rest}}_{\beta_n} \quad \text{so that} \quad \gamma_n \star \delta_{-x_n} \overset{n\to\infty, R\to\infty}{\Longrightarrow} p_1\alpha_1 \quad \text{(along some subsequence)}$$

Peel off γ_n from μ_n and apply the first step to the leftover β_n in place of μ_n. Repetition yields an element in

$$\widetilde{X} = \Big\{(p_n, \widetilde{\alpha}_n)_{n\in\mathbb{N}} \in \big([0,1] \times \widetilde{\mathcal{M}}_1(\mathbb{R}^d)\big)^{\mathbb{N}} \colon \sum_{n\in\mathbb{N}} p_n \le 1\Big\}.$$

- This is the compactification of $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$.
- There is a metric on $\widetilde{X}$ that makes it a compact Polish space.
- Any $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ can be seen as $((1, \widetilde{\mu}), (0,0), \dots) \in \widetilde{X}$.

With the help of this compactification, we can lift the famous weak large-deviation principle (LDP)

$$\mathbb{P}(L_t \approx \mu) \approx \mathrm{e}^{-tI(\mu)}, \qquad t \to \infty,$$

to the compact space $\widetilde{X}$:

LDP for $\widetilde{L_t}$ [Mukherjee/Varadhan (2014)]

The family of distributions $\widetilde{L_t}$ satisfies a (strong) LDP in $\widetilde{X}$ with rate function

$$(p_n, \widetilde{\alpha}_n)_{n\in\mathbb{N}} \mapsto \sum_j p_j I(\alpha_j).$$

Important improvement of DONSKER-GÄRTNER-VARADHAN LDP theory!

Can generally be applied to shift-invariant exponential functionals of the Brownian occupation measures.

Tube property of $\Lambda(L_t)$

We proceed with the analysis of the mean-field polaron model. First a translation of the tube property from L_t to $\Lambda(L_t)$, i.e., the convergence of $\Lambda(L_t)$ towards $\Lambda(\mathfrak{m})$:

Tube property for $\Lambda(L_t)$ in the uniform metric [König/Mukherjee (2016)]

For any $\varepsilon > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \Big\{ \inf_{x\in\mathbb{R}^3} \big\|\Lambda(L_t) - \Lambda\psi_x^2\big\|_\infty > \varepsilon \Big\} < 0.$$

In particular, $H(L_t)$ converges weakly under $\widehat{\mathbb{P}}_t$ to

$$H(\psi_0^2) = \langle \psi_0^2, \Lambda(\psi_0^2)\rangle = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\psi_0^2(x)\psi_0^2(y)}{|x-y|}\, \mathrm{d}x\mathrm{d}y.$$

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- Note that $\inf_{x\in\mathbb{R}^3} \left\|\Lambda(L_t) - \Lambda\psi_x^2\right\|_\infty = \mathrm{dist}_\infty(\Lambda(L_t), \Lambda(\mathfrak{m}))$.
- Greatest obstacle: singularity of Coulomb kernel $x \mapsto |x|^{-1}$.
- Important step: for every $b > 0$,

$$\lim_{\delta\to 0} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left\{ \sup_{x_1,x_2\in\mathbb{R}^3\colon |x_1-x_2|\le\delta} \left|\Lambda_t(x_1) - \Lambda_t(x_2)\right| \ge b \right\} = -\infty.$$

- Corollary: $\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\|\Lambda(L_t)\|_\infty \ge a) < 0$ for every $a > 0$.

Important step in the proof: There are constants $\rho > 1$, $a \in (0,1)$ and $\beta \in (0,\infty)$ such that

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^3\\|x_1-x_2|\leq 1}} \sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left[\exp\left\{\beta\left(\frac{|\Lambda_1(x_1)-\Lambda_1(x_2)|}{|x_1-x_2|^a}\right)^\rho\right\}\right] < \infty.$$

On the proof of regularity of $\Lambda(L_t)$

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For proving this, we write $\Lambda_1(x_1)-\Lambda_1(x_2)=\int_0^1 V_{x_1,x_2}(W_s)\,\mathrm{d}s$, where

$$V_{x_1,x_2}(y)=\frac{1}{|y-x_1|}-\frac{1}{|y-x_2|},$$

use KHAS'MINSKI'S lemma

$$\sup_x \mathbb{E}_x\Big[\int_0^1 V(W_s)\,\mathrm{d}s\Big]\leq \eta<1 \quad\Rightarrow\quad \sup_x \mathbb{E}_x\Big[\exp\Big\{\int_0^1 V(W_s)\,\mathrm{d}s\Big\}\Big]\leq \frac{\eta}{1-\eta},$$

and the GARSIA-RODEMICH-RUMSEY estimate (a variant of the KOLOMOGOROV-CHENTSOV criterion).

Hence, L_t approaches under $\widehat{\mathbb{P}}_t$ shifts of μ_0. Which ones? – Here is our main result:

Convergence of L_t [Bolthausen/König/Mukherjee (2016)]

The distribution of L_t under $\widehat{\mathbb{P}}_t$ converges towards the random shift $\mu_X = \mu_0 \star \delta_X$ of μ_0, where X has the distribution with density $\psi_0 / \int \psi_0$.

Furthermore, the distribution of the endpoint W_t under $\widehat{\mathbb{P}}_t$ converges towards the one of $X + X'$, where X and X' are two independent copies of X.

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Explanation in terms of the Pekar process: For $1 \ll t_0 \ll t$, split

$$L_t = \frac{t_0}{t} L_{t_0} + \frac{t - t_0}{t} L_{t_0,t},$$

and accordingly the energy

$$tH(L_t) = \frac{t_0^2}{t} H(L_{t_0}) + 2\frac{t_0(t-t_0)}{t} \langle L_{t_0}, \Lambda(L_{t_0,t}) \rangle + \frac{(t-t_0)^2}{t} H(L_{t_0,t})$$

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Explanation (continued)

On the event $\{L_t \approx \mu_x\}$, we have $\{L_{t_0,t} \approx \mu_x\}$. By the tube property for $\Lambda(L_t)$, we then have $\Lambda(L_{t_0,t}) \approx \Lambda(\mu_x)$. Hence, on this event

$$tH(L_t) \approx 2t_0\langle L_{t_0}, \Lambda(\mu_x))\rangle + tH(L_{t_0,t}) = 2\int_0^{t_0} \Lambda(\psi_x^2)(W_s)\,\mathrm{d}s + tH(L_{t_0,t}).$$

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Now note that the density

$$\frac{1}{Z_t^{(\psi_x)}} \exp\Big\{\int_0^{t_0} \Lambda(\psi_x^2)(W_s)\,\mathrm{d}s\Big\}$$

defines a Girsanov transformation from Brownian motion to the Pekar process with parameter x, which is driven by the SDE

$$\mathrm{d}Y_t^{(x)} = \mathrm{d}W_t + \Big(\frac{\nabla\psi_x}{\psi_x}\Big)(W_t)\,\mathrm{d}t.$$

This process is ergodic with invariant distribution $\mu_x(\mathrm{d}y) = \psi_x(y)^2\,\mathrm{d}y$. From here, it is not far to conclude that our main result should hold.

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This process is ergodic with invariant distribution $\mu_x(\mathrm{d}y) = \psi_x(y)^2\,\mathrm{d}y$. From here, it is not far to conclude that our main result should hold. Furthermore, it is also not really difficult to extend it to

Process convergence

The distribution of the process $(W_s)_{s\in[0,t]}$ under $\widehat{\mathbb{P}}_t$ as $t\to\infty$ is the mixture $Y^{(X)}$ of the Pekar process, where X is a random vector with density $\psi_0/\int\psi_0$.

Herzlichen Glückwunsch zum Geburtstag, lieber Erwin!

Hier ist mein Geburtstagsgeschenk: Unser erster Artikel!

[BKM16] E. Bolthausen, W. König and C. Mukherjee,
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Comm. Pure Appl. Math, to appear (2016).