Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Large deviations for the local times of random walk among random conductances

Wolfgang König

joint work with M. Salvi (Berlin) and T. Wolff (Berlin);

based on [K., Salvi, Wolff 2012] in *EJP* and [K., Wolff 2013], preprint soon

Why are we interested in random motions in random media?

Important objects of interest (with references specialising to the random conductance model):

- Long-time trajectories of particles in random environment (Law of large numbers, central limit theorems, invariance principle) $\Longrightarrow$ [BISKUP/PRESCOTT 2007], [BARLOW/DEUSCHEL 2010], [ANDRES, BARLOW, DEUSCHEL, HAMBLY 2012] and many more for other models.
- Random heat kernels [BERGER et al. 2008]
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Our main questions:

- Concentration properties of random mass flow?
- Non-exit probabilities from large boxes?
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The probabilistic treatment of these questions is based on the study of the local times of the random walk.

Motivation: the parabolic Anderson model (I)

Total mass of the solution to Cauchy problem for the Laplace operator with random potential, $\Delta + \xi$:

$$U(t) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\Big], \qquad t > 0,$$

where E_0 is expectation w.r.t. a simple random walk $(X_s)_{s\in[0,\infty)}$,

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$$\ell_t(z) = \int_0^t \delta_{X_s}(z)\,\mathrm{d}s,$$

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Moments of $U(t)$:

$$\langle U(t)\rangle = \mathbb{E}_0\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d} H(\ell_t(z))\Big\}\Big],$$

with the cumulant generating function

$$H(l) = \log\langle \mathrm{e}^{l\xi(0)}\rangle, \qquad l > 0.$$

Motivation: the parabolic Anderson model (II)

Necessary inputs:

- some assumption on the asymptotics of $H(l)$ and
- a large-deviation principle (LDP) for ℓ_t.

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Example: Assume that $H(yt) - yH(t) \sim t\, y \log y$ for $t \to \infty$ ($\Longrightarrow$ double exponential distribution), see [Gärtner and Molchanov 1998]. Hence,

$$\langle U(t)\rangle = \mathrm{e}^{H(t)}\mathbb{E}_0\Big[\exp\Big\{t\sum_{z\in\mathbb{Z}^d}\frac{H(t\frac{1}{t}\ell_t(z)) - \frac{1}{t}\ell_t(z)H(t)}{t}\Big\}\Big]$$

$$\approx \mathrm{e}^{H(t)}\mathbb{E}_0\Big[\exp\Big\{tJ\big(\tfrac{1}{t}\ell_t\big)\Big\}\Big],$$

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where $J(\mu) = \sum_{z\in\mathbb{Z}^d} \mu(z) \log \mu(z)$.

The Donsker-Varadhan-Gärtner LDP says that

$$\log \mathbb{P}_0\big(\tfrac{1}{t}\ell_t \approx \mu\big) \approx -t \big\| \nabla \sqrt{\mu} \big\|_2^2, \qquad \mu \in \mathcal{M}_1(\mathbb{Z}^d).$$

This and Varadhan's lemma then give that $\langle U(t) \rangle = \mathrm{e}^{H(t)} \mathrm{e}^{-t\chi}$ with $\chi = \inf_\mu (\|\nabla\sqrt{\mu}\|_2^2 - J(\mu))$.

Hence, for analysing the PAM for RWRC instead of simple random walk, we need to know about the large deviations of the local times.

Replace the Laplace operator Δ by the randomized Laplacian,

$$\Delta^\omega f(x) = \sum_{y\in\mathbb{Z}^d \colon\, y\sim x} \omega_{xy}(f(y) - f(x)),$$

where $\omega = (\omega_{xy})_{x\sim y}$ is an i.i.d. field of positive weights on the bonds, the conductances.

Δ^ω generates the RWRC $(X_s)_{s\in[0,\infty)}$, a random walk in random environment.

Random walk among random conductances (RWRC)

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Long-term objective: Understand the Cauchy problem for $\Delta^\omega + \xi$ for various potentials ξ.

Goal today: Understand annealed LDPs for the local times of the RWRC, ℓ_t.

In particular: Understand the long-time non-exit probability from a bounded set $B \subset \mathbb{Z}^d$:

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$$\log \mathtt{E}[\mathbb{P}_0^\omega(X_{[0,t]} \subset B)] \sim ?$$

Main assumption on the conductances: $\omega_e > 0$ a.s., but $\mathrm{essinf}\,\omega_e = 0$. More precisely,

Main Assumption:

for some $D, \eta \in (0,\infty)$, $\qquad \log \mathtt{P}(\omega_e \le \varepsilon) \sim -D\varepsilon^{-\eta}, \qquad \varepsilon \downarrow 0.$

Then the conductances can 'help' the RWRC to stay in B by assuming very small values.

Main result: the LDP

Fix a finite connected set $B \subset \mathbb{Z}^d$ and put $E_B = \{\{x,y\} \colon x \in B, y \in \mathbb{Z}^d, y \sim x\}$.

Theorem [K., Salvi, Wolff 2012]

The process of normalized local times, $(\frac{1}{t}\ell_t)_{t>0}$, under the annealed sub-probability law $\mathtt{E}[\mathbb{P}_0^\omega(\,\cdot\,\cap\{X_{[0,t]} \subset B\})]$ satisfies an LDP on the space of probability measures on B, with speed $t^{\frac{\eta}{\eta+1}}$ and rate function

$$J_{\rm d}(g^2) := K_{\eta,D} \sum_{\{x,y\}\in E_B} |g(y)-g(x)|^{\frac{2\eta}{\eta+1}}, \qquad g \in \ell^2(\mathbb{Z}^d), \mathrm{supp}(g) \subset B, \|g\|_2 = 1,$$

where $K_{\eta,D} = \big(1+\frac{1}{\eta}\big)(D\eta)^{\frac{1}{\eta+1}}$.

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Simplifying a bit, this means that

$$\log \mathtt{E}\Big[\mathbb{P}_0^\omega\Big(\tfrac{1}{t}\ell_t \approx g^2, X_{[0,t]} \subset B\Big)\Big] \approx -t^{\frac{\eta}{\eta+1}} J_{\mathrm{d}}(g^2) \qquad \text{for } g^2 \in \mathcal{M}_1(B).$$

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- $\eta \approx \infty$: $\approx$ conductances bounded away from zero $\Longrightarrow$ $\approx$ simple random walk,
- $\eta \approx 0$: $\approx$ heavy-tailed conductances $\Longrightarrow$ trapping model.

Corollary 1: Non-exit probability from B

$$\lim_{t\to\infty} t^{-\frac{\eta}{\eta+1}} \log \mathtt{E}\Big[\mathbb{P}_0^\omega\big(X_{[0,t]} \subset B\big)\Big] = -K_{\eta,D}\chi_{\mathrm{d}}(B),$$

where

$$\chi_{\mathrm{d}}(B) = \inf_{g^2\in\mathcal{M}_1(B)} \sum_{\{x,y\}\in E_B} |g(y)-g(x)|^{\frac{2\eta}{\eta+1}}.$$

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Corollary 2: Lower tails for the bottom of the spectrum of $-\Delta^\omega$, Lifshitz tails

Denote by $\lambda^\omega(B)$ the spectral radius of $-\Delta^\omega$ in B with zero boundary condition, then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^\eta \log \mathtt{P}(\lambda^\omega(B) \le \varepsilon) = -D\chi_{\mathrm{d}}(B)^{\eta+1}.$$

(See also [Exner/Helm/Stollmann 2007] for Anderson localisation properties of $-\Delta^\omega$.)

Heuristic derivation (I)

For any fixed conductance shape $\varphi\colon E_B \to (0,\infty)$, the Donsker-Varadhan-Gärtner LDP gives

$$\mathbb{P}_0^{\varphi}\Big(\tfrac{1}{t}\ell_t \approx g^2\Big) \approx \exp\big\{-tI_\varphi(g^2)\big\},$$

with rate function

$$I_\varphi(g^2) = \big(-\Delta^\varphi g, g\big) = \sum_{\{x,y\}\in E_B} \varphi_{xy}|g(x)-g(y)|^2.$$

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Rescaling by t^r (to be determined) and using our Main Assumption on ω gives

$$\mathtt{P}\big(t^r\omega \approx \varphi\big) = \prod_{\{x,y\}\in E_B} \mathtt{P}\big(\omega_{xy} \approx t^{-r}\varphi_{xy}\big) \approx \exp\big\{-t^{r\eta}H(\varphi)\big\},$$

where the rate function for the conductances is given by

$$H(\varphi) = D\sum_{\{x,y\}\in E_B} \varphi_{xy}^{-\eta}.$$

Combining the two gives

$$\begin{aligned}\mathtt{E}\Big[\mathbb{P}_0^\omega\big(\tfrac{1}{t}\ell_t \approx g^2\big)1\!\!1_{\{t^r\omega\approx\varphi\}}\Big] &\approx \mathbb{P}_0^{t^{-r}\varphi}\big(\tfrac{1}{t}\ell_t \approx g^2\big)\mathtt{P}\big(\omega \approx t^{-r}\varphi\big)\\ &\approx \exp\Big\{-tI_{t^{-r}\varphi}(g^2) - t^{r\eta}H(\varphi)\Big\}\\ &\approx \exp\Big\{-\sum_{\{x,y\}\in E_B}\Big(t^{1-r}\varphi_{xy}\big(g(x)-g(y)\big)^2 + t^{r\eta}D\varphi_{xy}^{-\eta}\Big)\Big\}.\end{aligned}$$

Heuristic derivation (II)

Combining the two gives

$$\mathtt{E}\Big[\mathbb{P}_0^{\omega}\big(\tfrac{1}{t}\ell_t \approx g^2\big)\mathbb{1}_{\{t^r\omega\approx\varphi\}}\Big] \approx \mathbb{P}_0^{t^{-r}\varphi}\big(\tfrac{1}{t}\ell_t \approx g^2\big)\mathtt{P}\big(\omega \approx t^{-r}\varphi\big)$$

$$\approx \exp\Big\{-tI_{t^{-r}\varphi}(g^2) - t^{r\eta}H(\varphi)\Big\}$$

$$\approx \exp\Big\{-\sum_{\{x,y\}\in E_B}\Big(t^{1-r}\varphi_{xy}\big(g(x)-g(y)\big)^2 + t^{r\eta}D\varphi_{xy}^{-\eta}\Big)\Big\}.$$

We obtain the slowest decay with $t^{1-r} = t^{r\eta}$, i.e., $r = (1+\eta)^{-1}$.

The optimal conductance shape φ is

$$\varphi_{xy} = (D\eta)^{\frac{1}{\eta+1}}|g(y)-g(x)|^{-\frac{2}{\eta+1}}.$$

This leads to the rate function

$$J(g^2) = \inf_{\varphi}\big[I_\varphi(g^2) + H(\varphi)\big] = K_{\eta,D}\sum_{\{x,y\}\in E_B}|g(y)-g(x)|^{\frac{2\eta}{\eta+1}}.$$

Time-depending, large box

Replace B by $B_t = \alpha_t G \cap \mathbb{Z}^d$ with $G \subset \mathbb{R}^d$ bounded and $1 \ll \alpha_t \ll t^{1/(d+2)}$ a scale function. (Very relevant for future study of PAM with RWRC.)

Rescaled local times:

$$L_t(x) = \frac{\alpha_t^d}{t} \ell_t\big(\lfloor \alpha_t x \rfloor\big), \qquad x \in \mathbb{R}^d.$$

For simple random walk, there is an LDP for L_t in B_t in the spirit of Donsker-Varadhan-Gärtner

[Gantert, K., Shi (2007)]

For L^2-normalized functions $f \in H_0^1(G)$,

$$\log \mathbb{P}_0\Big(L_t \approx f^2, X_{[0,t]} \subset \alpha_t G\Big)\Big] \approx -\frac{t}{\alpha_t^2} \sum_{i=1}^d \int_G |\partial_i f(x)|^2 \, \mathrm{d}x.$$

This can be easily heuristically derived by a proper combination of Donsker's invariance principle with the LDP for Brownian occupation measures.

LDP for local times of RWRC in large boxes

Now we extend this to the RWRC, again under the Main Assumption. It turns out that we have to assume that the tails of the conductances are not too thin.

[K. and Wolff (2013)]

Assume that $\eta > d/2$. Then, L_t satisfies on $\{X_{[0,t]} \subset \alpha_t G\}$ an LDP. More explicitly, for L^2-normalized functions $f \in H_0^1(G)$,

$$\log \mathtt{E}\Big[\mathbb{P}_0^\omega\Big(L_t \approx f^2, X_{[0,t]} \subset \alpha_t G\Big)\Big] \approx -\alpha_t^{\frac{d-2\eta}{\eta+1}} t^{\frac{\eta}{\eta+1}} J_{\mathrm{c}}(f^2),$$

where

$$J_{\mathrm{c}}(f^2) = K_{\eta,D} \sum_{i=1}^d \int_G |\partial_i f(x)|^{2\frac{\eta}{\eta+1}} \, \mathrm{d}x.$$

- In particular, J_{c} has compact level sets, and its minimum is attained; standard compactness arguments apply. However, there is no reason to believe that it is convex.
- A heuristic derivation goes along the above lines, leading to the scale $(t/\alpha_t^2)^{\eta/(1+\eta)}$, with additional term $\alpha_t^{d/(1+\eta)}$, coming from spatial rescaling.
- Proof of lower bound uses an extension of the above rescaled LDP; the proof of the upper bound relies on an explicit formula for a density of the local times [Brydges, van der Hofstad, K. (2007)].

Non-exit probability

For $\eta > d/2$,

$$\log \mathtt{E}\Big[\mathbb{P}_0^\omega\Big(X_{[0,t]} \subset \alpha_t G\Big)\Big] \approx -\alpha_t^{\frac{d-2\eta}{\eta+1}}\, t^{\frac{\eta}{\eta+1}} K_{\eta,D}\chi_{\rm c}(G),$$

where

$$\chi_{\rm c}(G) = \inf_{f^2\in H_0^1(G)\colon\, \|f\|_2=1} \sum_{i=1}^d \int_G |\partial_i f(x)|^{2\frac{\eta}{\eta+1}}\,{\rm d}x > 0,$$

and this variational problem has a minimizer.

We also prove analytically that

$$\eta > d/2 \qquad \Longrightarrow \qquad \chi_{\rm c}(G) > 0 \qquad \text{and} \qquad \chi_{\rm d}(\mathbb{Z}^d) = 0,$$

which can be interpreted by saying that the RWRC 'homogeneously fills' the domain $\alpha_t G \cap \mathbb{Z}^d$.

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which can be interpreted by saying that the RWRC 'homogeneously fills' the domain $\alpha_t G \cap \mathbb{Z}^d$.

However, for $\eta \le d/2$, the power of α_t is nonnegative ($\Longrightarrow$ wrong monotonicity). Explanation:

$$\eta \le d/2 \qquad \Longrightarrow \qquad \chi_{\mathrm{c}}(G) = 0 \qquad \text{and} \qquad \chi_{\mathrm{d}}(\mathbb{Z}^d) > 0,$$

i.e., the RWRC even concentrates on a set that is not growing with t.

Analytic reason for this dichotomy: Sobolev inequalities for p-Norms with $p > \frac{d}{d+2}$ on $\mathbb{R}^d$ resp. $p \le \frac{d}{d+2}$ on $\mathbb{Z}^d$.

Here is a probabilistic version of this interpretation.

Non-exit probabilities for $\eta \le d/2$

Suppose $1 \ll \alpha_t \ll t^{\frac{\eta}{d(\eta+1)}}$. Then, for $\eta \le d/2$, for any finite and connected set $B \subset \mathbb{Z}^d$ containing the origin,

$$
\begin{aligned}
& -K_{\eta,D}\chi_{\mathrm{d}}(\mathbb{Z}^d)(1+o(1)) \\
& \qquad \geq t^{-\frac{\eta}{\eta+1}} \log \mathtt{E}\Big[\mathbb{P}_0^\omega\big(\mathrm{supp}(\ell_t) \subset \alpha_t G\big)\Big] \\
& \qquad\qquad \geq -K_{\eta,D}\chi_{\mathrm{d}}(B)(1+o(1)).
\end{aligned}
$$

and a lower bound with $\chi_{\mathrm{d}}(\mathbb{Z}^d)$ for $\eta = d/2$.

Hence, the non-exit probability from $\alpha_t G$ has the same asymptotics as the one from some set that does not depend on t.

We contrast with the case where $\lambda \leq \omega_e \leq \frac{1}{\lambda}$ a.s., for some $\lambda > 0$. Here, we have a quenched functional central limit theorem: there is an effective diffusion constant $\sigma^2 \in (0, \infty)$, such that, ω-almost surely,

$$\left(X_{tc}t^{-1/2}\right)_{c\in[0,\infty)} \Longrightarrow \sigma\mathrm{BM}, \qquad t \to \infty.$$

(see, e.g., [SZNITMAN, SIDORAVICIUS 2004]).

Quenched LDP for rescaled local times, [WOLFF 2012]

If $1 \ll \alpha_t \ll t^{1/(d+2)}$ is a scale function, then, almost surely, L_t satisfies an LDP under $\{\mathrm{supp}(\ell_t) \subset \alpha_t G\}$ with speed $t\alpha_t^{-2}$ and rate function $f^2 \mapsto \sigma^2\|\nabla f\|_2^2$.

- The proof uses homogenization for the principal eigenvalue of the Dirichlet Laplacian in $\alpha_t G \cap \mathbb{Z}^d$.

Open questions

Analytic questions:

- Does $\chi_{\mathrm{c}}(G)$ for $\eta > d/2$ respectively $\chi_{\mathrm{d}}(B)$ for $\eta \leq d/2$ have more than one minimiser? Is there some useful convexity?
- Does the latter minimiser become trivial for $B \uparrow \mathbb{Z}^d$? If not, what does it converge to?
- Are J_{c} or J_{d} linked with some interesting operator, like the p-Laplacian for $p = \eta/(1+\eta)$?

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Probabilistic questions:

- Behaviour of path in box $\alpha_t G$ for $\eta \leq d/2$?
- Annealed behaviour of conductances on $\{\mathrm{supp}(\ell_t) \subset \alpha_t G\}$?
- Almost-sure versions of the LDPs or of the non-exit probabilities?

And finally, of course, the PAM with additional random potential ξ ...