Commun. Math. Phys. 269, 715–763 (2007)
Digital Object Identifier (DOI) 10.1007/s00220-006-0143-9

Communications in
**Mathematical
Physics**

Tunneling in Two Dimensions

G. Bellettini[1,2], A. De Masi[3], N. Dirr[4], E. Presutti[1,⋆]

[1] Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica 00133 Roma, Italy. E-mail: belletti@mat.uniroma2.it;presutti@mat.uniroma2.it
[2] INFN Laboratori Nazionali di Frascati, Frascati, Italy
[3] Dipartimento di Matematica, Università di L’Aquila, via Vetoio, loc. Coppito, 67100 l’Aquila, Italy. E-mail: demasi@univaq.it
[4] Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany. E-mail: ndirr@mis.mpg.de

Received: 2 November 2005 / Accepted: 20 July 2006
Published online: 8 November 2006 –

Abstract: Tunneling is studied here as a variational problem formulated in terms of a functional which approximates the rate function for large deviations in Ising systems with Glauber dynamics and Kac potentials, [9]. The spatial domain is a two-dimensional square of side L with reflecting boundary conditions. For L large enough the penalty for tunneling from the minus to the plus equilibrium states is determined. Minimizing sequences are fully characterized and shown to have approximately a planar symmetry at all times, thus departing from the Wulff shape in the initial and final stages of the tunneling. In a final section (Sect. 11), we extend the results to $d = 3$ but their validity in $d > 3$ is still open.

1. Introduction

Tunneling in the $d = 2$ ferromagnetic Ising model at low temperatures has been the object of many studies, mainly focused on metastability, namely the analysis of the Glauber dynamics when an external magnetic field $h > 0$ is present and the initial state is close to the minus Gibbs state at $h = 0$. We are instead interested here in studying a bistable equilibrium with “oscillations” between the two minimizers. Such a case has been considered by Martinelli, [26], in the n.n. ferromagnetic Ising model in a $d = 2$ square of side L, proving upper and lower bounds for the [random] transition time from the “plus” to the “minus” state (and vice versa) in the limit as $L \to \infty$. Much earlier Comets had attacked the problem in the context of Ising systems with Kac interactions. Supposing the side L of the square to be proportional to the range γ^{-1} of the Kac interaction, Comets [9] derived the large deviations rate function in the asymptotics of small γ. A “sharp” analysis of the path followed during the tunneling is however still an open problem in both models.

⋆ This research has been partially supported by MURST and NATO Grant PST.CLG.976552 and COFIN, Prin n.2004028108.

Tunneling is usually studied in two steps: the first one is based on a loss of memory property, namely that configurations close to one of the two stable states can be successfully coupled with large probability before leaving the neighborhood. Such estimates seem within the reach of the present techniques, as in [15] very strong properties of Glauber dynamics have been established. The second step for tunneling requires to solve a variational problem involving the large deviations rate function. In this paper we concentrate on the latter aspect and study tunneling in a purely variational setting. For simplicity we replace the Comets rate function by an "easier functional", already considered in [3] in the $d = 1$ version of the model. The extension to the true Comets functional and then to the Ising system may still require a non-trivial work, but we believe that the main physical features of the actual tunneling excursion are already captured by our results.

The extension from $d = 1$ to $d > 1$ is in general far from trivial. Large deviations and tunneling have been studied by Jona-Lasinio and Mitter, [22], for stochastic perturbations of the Allen-Cahn equation, partially extending the $d = 1$ work by Faris and Jona-Lasinio, [18] (see also [10]), but, as far as we know, a full analysis in $d = 2$ is open also for the Ginzburg-Landau action functional associated with the Allen-Cahn equation. Even more subtle is the analysis of tunneling under time constraints, namely when the excursion between the two stable states is required to occur within a given time interval. The picture in such a case may be dramatically different if time is short, and the optimal pattern may involve multiple nucleations. Results of this type are proved in $d = 1$ for the Ginzburg-Landau functional and Allen-Cahn equation, [23, 24], and for the non local interaction considered here, [11]; most of the proofs are still missing in the multi-dimensional case, but a clear picture of the phenomenon can at least be outlined, [24].

Geometric patterns are the main issues in a multi-dimensional analysis. In the sharp interface limit (i.e. when the spatial domain, a square of side L in our case, is observed in rescaled variables so that it always appears as a unit square as $L \to \infty$) the tunneling orbits are moving surfaces which describe the boundaries of the set where the plus phase is located. In $d = 1$ this is simply a point which moves from an endpoint of the unit interval to the other one (Neumann boundary conditions are responsible for the nucleation to start from the boundaries of the domain). To see geometrical effects we thus need to go to $d > 1$.

An important factor is then played by the Wulff shape. As it is well known (and briefly discussed in Sect. 2) in $d = 2$ dimensions the set with minimal perimeter for a given area θ is a quarter of a circle around a vertex of the unit square Q_1, or a rectangle with three sides lying on ∂Q_1 (again this is due to Neumann boundary conditions). Rectangles appear if their area and the area of the complement (in Q_1) are both larger than a critical value θ_{crit}, otherwise we observe a quarter of a circle. As Wulff shapes describe states with minimal free energies under the area constraint, one usually expects that if the process is "slow" and the transformation "adiabatic" then the tunneling patterns are determined by sequences of "equilibrium" Wulff shapes. It is however evident from the above description that tunneling orbits cannot always be close to Wulff shapes as there is a discontinuity at θ_{crit}. One possible scenario is depicted in (a) of Fig. 1 where the Wulff shape is deformed to interpolate around θ_{crit} between the two different regimes. We will prove instead that the optimal tunneling in our diffused interface model is all the way planar as in pattern (b) of Fig. 1, namely that it is convenient to nucleate initially in a less efficient way, the cost being recovered in the end. Our results hold whenever there exists a *stable* invariant manifold which connects a saddle point of minimal energy to the

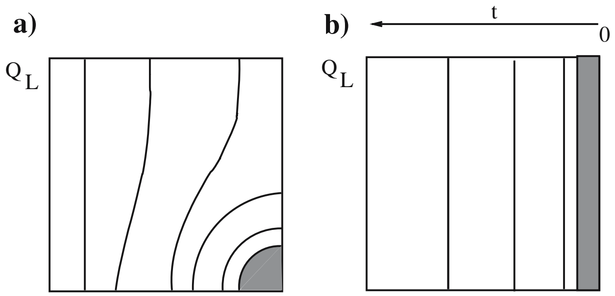

Fig. 1. In a) and b) we depict two possible tunneling paths in the sharp interface regime. In fig. a) a small droplet (Wulff shape) of the + phase (dark region) nucleates at a vertex of the square Q_L. It then invades Q_L as time increases, gradually changing its interface, and eventually becomes a rectangle. Our results, valid for the diffused interface model, show that a) is *not* minimizing, and that the minimizing path is the one corresponding to fig. b). In this path we have initially a nucleation of a flat interface (dark rectangular region), which smoothly invades Q_L

stable equilibria and which does not consist entirely of Wulff shapes. More discussions on this point can be found in Sect. 2.

The content of the paper is outlined in Sect. 2, Subsect. 2.9, after defining the model and stating the main results, the extension to $d = 3$ is discussed in Sect. 11. For reasons of brevity imposed by the journal we have written in a separate paper, [2], the analysis of the invariant manifolds for a non-local version of the Allen-Cahn equation, which is used here to characterize the optimal tunneling orbits.

2. Definitions and Results

We consider a continuum model of a two-dimensional magnet where states are functions $m \in L^\infty(Q_L, [-1, 1])$, $Q_L = \{r \in \mathbb{R}^2 : |r \cdot e_1| \le L/2, |r \cdot e_2| \le L/2\}$, $r \cdot e_1$ and $r \cdot e_2$ the x and y components of r. $m(r)$ is interpreted as a magnetization density which may be related, by a coarse graining procedure, to an underlying Ising spin configuration, hence the restriction to $[-1, 1]$. Time evolution is described by orbits which are smooth functions $u = u(r, t)$, $r \in Q_L$, t in $\mathbb{R}$ or in an interval of $\mathbb{R}$, $|u| \le 1$.

2.1. The penalty functional. The "action" of an orbit $u(\cdot)$ restricted to an interval $[t_0, t_1]$ of its domain of definition is

$$A_{L;t_0,t_1}(u) = F_L(u(\cdot, t_0)) + I_{L;t_0,t_1}(u),$$

where $F_L(m)$, the free energy of the state m, is

$$F_L(m) = \int_{Q_L} \phi_\beta(m)\, dr + \frac{1}{4} \int_{Q_L \times Q_L} J^{\mathrm{neum}}(r, r')[m(r) - m(r')]^2 dr\, dr'. \tag{2.1}$$

$J^{\mathrm{neum}}(r, r')$ is the interaction coupling constant (with Neumann boundary conditions), namely $J^{\mathrm{neum}}(r, r') = \sum_{r'' \simeq r'} J(r, r'')$, where $r'' \simeq r'$ means that r'' is equal to r' modulo reflections along the lines $\{y = \pm(2n + 1)L/2\}$ and $\{x = \pm(2n + 1)L/2\}$, $n \in \mathbb{Z}$. We suppose $J(r, r') = J(0, r' - r)$ and make the following "technical assumptions" on $J(0, r)$: $J(0, r)$ only depends on $|r|$; it is a smooth non-negative function supported in the unit ball; $\displaystyle\int J(0, r) = 1$;

$$j(0, x) = \int J((0, 0), (x, y))\, dy \tag{2.2}$$

is a non-increasing function of x when $x > 0$. In (2.1) we take $\beta > 1$,

$$\phi_\beta(m) = \tilde{\phi}_\beta(m) - \min_{|s|\le 1} \tilde{\phi}_\beta(s), \qquad \tilde{\phi}_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta}\mathcal{S}(m),$$

$$\mathcal{S}(m) = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}.$$

Finally,

$$I_{L;t_0,t_1}(u) = \frac{1}{4}\int_{t_0}^{t_1}\int_{Q_L} [u_t - f_L(u)]^2\, dr\, dt,$$

where u_t is the time derivative of u and

$$f_L(u) = -\frac{\delta F_L(u)}{\delta u} = J^{\mathrm{neum}} * u - a_\beta(u), \qquad a_\beta(u) = \frac{1}{\beta}\,\mathrm{arctanh}(u),$$
$$J^{\mathrm{neum}} * u(r) = \int_{Q_L} J^{\mathrm{neum}}(r, r')u(r')\, dr'. \tag{2.3}$$

As mentioned in the Introduction, $A_{L;t_0,t_1}(u)$ is a simplified version of the Comets large deviations rate function for Glauber dynamics in a ferromagnetic Ising system with Kac potential $J_\gamma(r, r') = \gamma^2 J(\gamma r, \gamma r')$.

Later on, in the course of the proofs, we will consider rectangles $Q_{L',L} = \{(x, y) : |x| \le L', |y| \le L/2\}$ with $L' \in (0, +\infty]$ and call channel the set $Q_{\infty,L}$. The definition of F_L in (2.1) naturally extends to domains $Q_{L',L}$ in which cases it will be denoted by $F_{Q_{L',L}}$, as a functional on $L^\infty(Q_{L',L}, [-1, 1])$.

2.2. Dynamics: the semigroups S_t and T_t. We denote by S_t the semi-group generated by the L^2-gradient dynamics, namely $S_t(u_0) = u(\cdot, t)$ is the solution to the non-local evolution equation

$$u_t = f_L(u) = -\frac{\delta F_L(u)}{\delta u}, \qquad u(\cdot, 0) = u_0. \tag{2.4}$$

The velocity field $f_L(u)$ is Lipschitz when restricted to sets of the form $\{\|u\|_\infty \le b\}$, $b < 1$. Then it is not difficult to prove global existence of $S_t(u_0)$ if $\|u_0\|_\infty < 1$, see [2] for details.

The semigroup $T_t(u_0)$ generated by the equation

$$u_t = -u + \tanh\{\beta J^{\mathrm{neum}} * u\}, \qquad u(\cdot, 0) = u_0, \tag{2.5}$$

has been much more studied in the literature, as (2.5) is the limit equation derived from Glauber dynamics with Kac potentials in a scaling limit, see [15]. Thus for ease of reference and in order to exploit results already existing in the literature we will often in the sequel consider $T_t(u_0)$, regarded either in Q_L or in $Q_{\infty,L}$. Observe that it also decreases the energy F_L, that its fixed points are the same as those of S_t, which are critical points of F_L.

2.3. The cost of tunneling. The action $A_{L;t_0,t_1}(u)$ is always non-negative, as the integrands in F_L and $I_{L;t_0,t_1}$ are non-negative. Actually $A_{L;t_0,t_1}(u) > 0$ unless $u(r,t) \equiv \pm m_\beta$, where $m_\beta > 0$ is such that $m_\beta = \tanh\{\beta m_\beta\}$ (recall the assumption $\beta > 1$). Therefore $m^{(\pm)}(r) \equiv \pm m_\beta$ have the interpretation of the [only] two equilibrium states of the system and tunneling describes orbits which connect such states. Thus the space of tunneling orbits in a time $T > 0$ is

$$\mathcal{U}_{L,T} = \left\{u \in C^\infty(Q_L \times [0,T]) : u(r,0) = -m_\beta,\ u(r,T) = m_\beta \text{ for all } r \in Q_L\right\}$$

and, calling $I_{L,T}(u) = I_{L;0,T}(u)$, we define the cost of tunneling as

$$P_L := \inf_{T>0} \inf_{u \in \mathcal{U}_{L,T}} I_{L,T}(u), \tag{2.6}$$

noticing that since $F_L(m^{(-)}) = 0$, $A_{L,T}(u) = A_{L;0,T}(u) = I_{L,T}(u)$ when $u \in \mathcal{U}_{L,T}$.

As mentioned in the Introduction the problem is completely different if restrictions on T are imposed, but in this paper we will only study problem (2.6). To motivate our results let us first describe some properties of $A_{L;t_0,t_1}$.

2.4. Reversibility. First notice that $I_{L;t_0,t_1}(u) = 0$ if $u(\cdot,t) = S_{t-t_0}(u(\cdot,t_0))$, S_t being defined in Subsect. 2.2. Given $u(\cdot,t)$, $t \in [t_0,t_1]$, call $u^{\text{rev}}(\cdot, t_0+s) = u(\cdot, t_1 - s)$, $s \in [0, t_1 - t_0]$. Then

$$A_{L;t_0,t_1}(u) = A_{L;t_0,t_1}(u^{\text{rev}}). \tag{2.7}$$

To show (2.7), which is proved in [3], it suffices to expand the square in the integral defining $I_{L;t_0,t_1}$ and recall that $f_L(u) = -\delta F_L(u)/\delta u$. As a consequence of (2.7),

$$I_{L;t_0,t_1}(u) \geq F_L(u(\cdot,t_1)) - F_L(u(\cdot,t_0)), \tag{2.8}$$

$$I_{L;t_0,t_1}(u) = F_L(u(\cdot,t_1)) - F_L(u(\cdot,t_0)) \quad \text{if } u(\cdot,t_0) = S_{t_1-t_0}(u(\cdot,t_1)). \tag{2.9}$$

Remark 2.1. Note that $I_{L,T}(u) \geq F_L(u(\cdot,t))$ for any $t \in [0,T]$.

2.5. The Wulff shape. Given any tunneling orbit $u \in \mathcal{U}_{L,T}$ and $\alpha \in (-m_\beta, m_\beta)$, by continuity there must be a time $t \in (0,T)$ when $u(\cdot,t) \in \Sigma_\alpha$,

$$\Sigma_\alpha = \left\{ m \in L^\infty(Q_L, [-1,1]) : \fint_{Q_L} m = \alpha \right\}.$$

Thus from (2.8),

$$I_{L,T}(u) \geq \inf\left\{F_L(m) : m \in \Sigma_\alpha\right\} \quad \text{for any } \alpha \in (-m_\beta, m_\beta), \tag{2.10}$$

hence the intuition that optimality in tunneling requires closeness to the Wulff shape, namely the minimizer on the r.h.s. of (2.10). The Wulff problem is well understood in the limit $L \to \infty$. As the infimum on the r.h.s. of (2.10) grows proportionally to L, (L^{d-1} in d dimensions), it is natural to renormalize the free energy by dividing by L and have (see [15–17])

$$\lim_{L\to\infty} \inf\left\{\frac{F_L(m)}{L} : m \in \Sigma_\alpha\right\} = c_\beta \inf\left\{P(E, \mathrm{int}(Q_1)) : E \subseteq Q_1, |E| = \frac{1}{2} - \vartheta_\alpha\right\},$$

where $P(E, \mathrm{int}(Q_1))$ denotes the perimeter of the BV set E in the interior of Q_1 (namely the intersection with ∂Q_1 does not contribute); $|E|$ is the Lebesgue measure of E; $\vartheta_\alpha \in (-1/2, 1/2)$ is defined by

$$\left(\frac{1}{2} - \vartheta_\alpha\right) m_\beta - \left[1 - \left(\frac{1}{2} - \vartheta_\alpha\right)\right] m_\beta = \alpha. \tag{2.11}$$

Equation (2.11) has a clear geometrical interpretation, the magnetization α being realized by putting m_β in the rectangle $\{(x, y) \in Q_1 : x \geq \vartheta_\alpha\}$ and $-m_\beta$ in its complement. c_β, the surface tension, is equal to $c_\beta = F^{(1)}(\bar{m})$. Namely c_β is the one-dimensional free energy $F^{(1)}$ of the one-dimensional instanton $\bar{m}(x)$, $x \in \mathbb{R}$, where

$$F^{(1)}(m) = \int_{\mathbb{R}} \phi_\beta(m)\, dx + \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} j(x, x')[m(x) - m(x')]^2\, dx dx', \tag{2.12}$$

with $j(x, x')$ as in (2.2) and $\bar{m}$ the non-zero, antisymmetric solution of

$$\bar{m} = \tanh\{j * \bar{m}). \tag{2.13}$$

The limit Wulff problem

$$\inf\left\{P(E, \mathrm{int}(Q_1)) : E \subseteq Q_1,\ |E| = \frac{1}{2} - \theta\right\} \tag{2.14}$$

of minimizing the perimeter functional $P(E, \mathrm{int}(Q_1))$ is explicitly solved. Indeed (2.14) admits a solution and any solution E_θ is such that $Q_1 \cap \partial E_\theta$ is smooth and thus a critical point, [25]. Moreover $Q_1 \cap \partial E_\theta$ is connected and has constant curvature. Hence it is contained either in a circle or in a line. In addition the contact between ∂E_θ and ∂Q_1 is orthogonal. Let θ_{crit} be defined by

$$\frac{1}{2} - \theta_{\mathrm{crit}} = \frac{\pi R^2}{4}, \quad \text{where } \frac{2\pi R}{4} = 1.$$

Then the following result holds.

Proposition 2.1 [25]. *If $|\theta| \leq \theta_{\mathrm{crit}}$ then $Q_1 \cap \partial E_\theta$ is a segment parallel to one of the coordinate axes and intersecting two of the opposite sides of ∂Q_1. If $|\theta| \geq \theta_{\mathrm{crit}}$ then $Q_1 \cap \partial E_\theta$ is a quarter of a circle of radius $\dfrac{2}{\sqrt{\pi}}\left(\dfrac{1}{2} - \theta\right)^{1/2}$ centered at one of the four corners of Q_1.*

Remark 2.2. As already remarked in the Introduction, for L large enough a tunneling orbit cannot always be close to the Wulff shape, as the Wulff shape varies discontinuously when α crosses the critical value at which $\vartheta_\alpha = \theta_{\text{crit}}$. When $\alpha = 0$ the Wulff shape is planar and this may suggest that optimal orbits become eventually (approximately) planar. Two scenarios are then conceivable: (a) the plus phase grows initially as a quarter of circle around a corner and then progressively deforms to end up into a planar wave as $\alpha \to 0$; (b) the plus phase starts from the very beginning planar, so that in the limit picture the perimeter is discontinuous at time 0, jumping from 0 to its maximal value. In any case, both scenarios evidently contradict the intuitive idea that optimal orbits follow Wulff shapes. A discussion on this issue can be found in [27] in the context of statistical mechanics.

Planar symmetry suggests relevance of $d = 1$ tunneling, which is the argument of the next subsection.

2.6. Tunneling in one dimension. Let $F_L^{(1)}$ be defined by (2.12) with $\mathbb{R}$ replaced by $[-L/2, L/2]$ and with Neumann boundary conditions. Let $m \in L^\infty([-L/2, L/2], [-1, 1])$ and set

$$m^e(r) = m(r \cdot e_1), \qquad r \in Q_L.$$

Then

$$F_L(m^e) = LF_L^{(1)}(m). \tag{2.15}$$

Let $\mathcal{U}_{L,T}^{(1)}$ be the $d = 1$ tunneling orbits in a time T and $P_L^{(1)}$ the $d = 1$ tunneling cost associated with the functional $F^{(1)}$. We then have from (2.15),

$$P_L \le LP_L^{(1)}. \tag{2.16}$$

In [3, 4] it is proved that

$$P_L^{(1)} = F_L^{(1)}(\hat{m}_L), \tag{2.17}$$

where $\hat{m}_L$ is the unique non-zero, strictly monotone antisymmetric function of x which solves the equation

$$\hat{m}_L(x) = \tanh\{j^{\text{neum}} * \hat{m}_L(x)\}, \quad |x| \le L/2, \tag{2.18}$$

with j^{neum} obtained from j, see (2.2), by reflections at $\pm L/2$ (thus $\hat{m}_L$ is a critical point of $F_L^{(1)}$).

2.7. S_t-invariant manifolds. It is proved in [2] that $\hat{m}_L^e := (\hat{m}_L)^e$ is "dynamically connected" to $m^{(\pm)}$ in the sense that there are two S_t-invariant, one-dimensional manifolds, $\mathcal{W}_\pm = \{v_L^{(\pm)}(\cdot, s), s \in \mathbb{R}\}$, which connect $\hat{m}_L^e$ to $m^{(-)}$ and, respectively, to $m^{(+)}$. $v_L^{(\pm)}(\cdot, s)$ are planar functions (i.e. constant in the vertical direction) which satisfy the following two properties:

$$\lim_{s\to-\infty} \|v_L^{(\pm)}(\cdot, s) - \hat{m}_L^e\|_2 = 0, \quad \lim_{s\to\infty} \|v_L^{(\pm)}(\cdot, s) - m^{(\pm)}\|_2 = 0, \tag{2.19}$$

where $\|\cdot\|_2$ is the L^2 norm in Q_L, and

$$S_t(v_L^{(\pm)}(\cdot, s)) = v_L^{(\pm)}(\cdot, s + t) \quad \text{for all } s \in \mathbb{R} \text{ and all } t \ge 0.$$

Moreover $F_L(v_L^{(\pm)}(\cdot, s)) < LF_L^{(1)}(\hat{m}_L)$ for any $s \in \mathbb{R}$.

2.8 Main results

Theorem 2.3. *For L large enough*

$$P_L = LF_L^{(1)}(\hat{m}_L). \tag{2.20}$$

Theorem 2.3 will be proved starting from Sect. 5. It suggests that the best strategy for tunneling is to use orbits with planar symmetry, a statement made precise in Theorem 2.4 below which will be proved in Sect. 4 using heavily results from [2].

Theorem 2.4. *For all L large enough, if* $\{T_n, u_n\}$ *is a minimizing sequence for* (2.6), *then* $\lim_{n\to+\infty} T_n = +\infty$ *and, given any* $\epsilon > 0$ *there exists a positive integer* n_0 *such that for any* $n \ge n_0$, u_n *(or its image under a rotation by an integer multiple of* $\pi/2$*) has the following properties. There is* $s \in (0, T_n)$ *so that* $\|u_n(\cdot, s) - \hat{m}_L^e\|_2 \le \epsilon$ *and there are* τ' *and* τ'' *positive so that*

$$\|u_n(\cdot, t) - v_L^{(-)}(\cdot, \tau' - t)\|_2 \le \epsilon, \quad t \in [0, s], \tag{2.21}$$

$$\|u_n(\cdot, t) - v_L^{(+)}(\cdot, -\tau'' + (t - s))\|_2 \le \epsilon, \quad t \in [s, T_n]. \tag{2.22}$$

Theorem 2.4 proves that the best tunneling is obtained by orbits which have (approximately) a planar symmetry and which (approximately) follow the one-dimensional manifolds connecting saddle and stable points, first in the time reverse direction and then, after crossing the saddle, along the forward time direction. Initially the orbits look far from optimal, in the sense that it would be cheaper to gain the same value of total magnetization by following a different pattern, closer to the corresponding Wulff shape; but overall such an initial cost is recovered by smaller costs afterwards. In the limit $L \to \infty$ and rescaling penalties by dividing by L, we see that in optimal orbits the free energy jumps at time 0 to a value which then remains constant: in the limit the whole penalty is paid at time 0^+ . Thus pattern b) in Fig. 1 rather than a) is what we actually observe in tunneling events.

2.9. Content of the paper. In Sect. 3 we reduce the proof of Theorem 2.3 to the proof that • when $u(\cdot, t) \in \Sigma_\alpha$ with $|\alpha|$ small then $u(\cdot, t)$ is very close to a planar instanton, Theorem 3.1; • calling $m = u(\cdot, t)$, t as above, then either $T_s(m) \to \hat{m}_L^e$ as $s \to \infty$, or else $T_s(m)$ at some time s is close to a planar instanton suitably shifted away from the origin, Theorem 3.2; • if m is close to a planar instanton suitably shifted away to the right or to the left of the origin, then $T_s(m)$ is attracted by $m^{(-)}$ or respectively by $m^{(+)}$, Theorem 3.3. We conclude Sect. 3 by showing that indeed Theorem 2.3 follows from Theorems 3.1–3.3.

In Section 4 we prove Theorem 2.4 as a consequence of Theorem 2.3 and of existence and stability of the invariant manifolds $\mathcal{W}_\pm$, properties which are proved in a companion paper, [2].

In Sects. 5–7 we prove Theorem 3.1: in Sect. 5 we quote from the literature lower bounds on the free energy cost of deviations from equilibrium (Peierls estimates). In Sect. 6 we prove that the distance from an instanton can be controlled in terms of the free energy, Theorem 6.1, and in Sect.. 7 we conclude the proof of Theorem 3.1.

In Sect. 8 we prove Theorems 3.2 and 3.3, relying again on the companion paper [2], thus concluding the proof of Theorem 2.3.

In the Appendix, Sects. 9 and 10, we prove some spectral properties of operators obtained by linearizing the flows T_t and S_t which have been used in the proofs of Theorems 3.1–3.3. The extension of the results to $d = 3$ is sketched in Sect. 11.

3. Scheme of Proof of Theorem 2.3

By (2.16) and (2.17), $P_L \le L F_L^{(1)}(\hat{m}_L)$, so that Theorem 2.3 will be proved once we show that for L large enough

$$P_L \ge L F_L^{(1)}(\hat{m}_L). \tag{3.1}$$

Thus we may take arbitrarily $\epsilon > 0$, restrict to $T > 0$ and $u \in \mathcal{U}_{L,T}$ such that

$$I_{L,T}(u) \le P_L + \epsilon \le L F_L^{(1)}(\hat{m}_L) + \epsilon \tag{3.2}$$

and show that if L is large enough for any such u,

$$I_{L,T}(u) \ge L F_L^{(1)}(\hat{m}_L). \tag{3.3}$$

The main point is an a-priori characterization of the tunneling orbits which satisfy (3.2) at times t when $u(\cdot, t) \in \Sigma_\alpha$ with $|\vartheta_\alpha| < \theta_0$ (ϑ_α as in (2.11)) where θ_0 is fixed arbitrarily with the only requirement that

$$0 < \theta_0 < \theta_{\text{crit}} \tag{3.4}$$

(how large L is in our analysis will depend also on the value of θ_0). As we will see in Sect. 7, the proof of convergence to the Wulff shape as $L \to \infty$, see Proposition 7.1, essentially contains closeness to the instanton in the following sense:

For any $\delta > 0$ there are $\epsilon(\delta) > 0$ and $L(\delta)$ so that if $0 < \epsilon < \epsilon(\delta)$, $L > L(\delta)$, $m \in \Sigma_\alpha$ with $|\vartheta_\alpha| \le \theta_0$ and $F_L(m) < L F_L^{(1)}(\hat{m}_L) + \epsilon$, then, modulo a rotation of an integer multiple of $\pi/2$, there is $\xi \in (-L/2, L/2)$ so that $\|m - \bar{m}_{\xi,L}\|_1 \le \delta L^2$, where

$$\bar{m}_{\xi,L}(r) = \bar{m}(r \cdot e_1 - \xi), \quad r \in Q_L. \tag{3.5}$$

The bound $\|m - \bar{m}_{\xi,L}\|_1 \le \delta L^2$ is still far from what is needed in our proof of (3.1), but it is an important ingredient in the proof of a much sharper estimate, where "the error" $\|m - \bar{m}_{\xi,L}\|_2$ vanishes instead of growing as $L \to \infty$. This is the main technical point in the paper, its precise statement is the content of:

Theorem 3.1. *There are L_0 and $\epsilon_0(L) \in (0, L^{-100})$, so that for any $L \ge L_0$ if*

$$m \in \Sigma_\alpha \text{ with } |\vartheta_\alpha| \le \theta_0 \text{ and } F_L(m) < L F_L^{(1)}(\hat{m}_L) + \epsilon,\ \epsilon \in (0, \epsilon_0(L)) \tag{3.6}$$

then there exists $\xi \in (-\theta_0 L - 1, \theta_0 L + 1)$ such that, modulo a rotation of an integer multiple of $\pi/2$,

$$\|m - \bar{m}_{\xi,L}\|_2 < L^{-100}. \tag{3.7}$$

Remarks. Theorem 3.1 as well as Theorems 3.2 and 3.3 below, are proved in the next sections and in an appendix. The bound L^{-100} is not optimal. Analogously to (8.9), it can be proved that $|\vartheta_\alpha + \xi/L| < L^{-100}$. Our proof of Theorem 3.1 uses in an essential way two dimensions but it extends to $d > 3$ using an argument due to Bodineau and Ioffe, [5], and an extension of the theory of Wulff shapes to $d = 3$, [29], see Sect. 11.

The proof of (3.1) proceeds with a characterization of the critical points of $F_L(\cdot)$. For this purpose we use the dynamics with semigroup T_t defined by Eq. (2.5).

Theorem 3.2. *There exists $L_1 \geq L_0$ such that for any $L \geq L_1$ the following holds. If m satisfies* (3.6) *then either there is a time t when $T_t(m) \in \Sigma_{\alpha'}$, α' such that $|\vartheta_{\alpha'}| = \theta_0$, or else $\lim_{t\to\infty} T_t(m) = \hat{m}_L^e$ in $L^2(Q_L)$ (modulo a rotation of an integer multiple of $\pi/2$).*

Theorem 3.3. *There exists $L_2 \geq L_1$ such that for any $L \geq L_2$ the following holds. If $m \in \Sigma_{\alpha'}$ for some α' such that $\vartheta_{\alpha'} = \pm\theta_0$ and if there exists ξ such that $\|m - \bar{m}_{\xi,L}\|_2 < L^{-100}$, then*

$$\lim_{t\to\infty} T_t(m) = m^{(\mp)} \qquad \text{in } L^2(Q_L). \tag{3.8}$$

The proof of (3.1), giving the proofs of Theorems 3.1, 3.2 and 3.3, is then concluded using the following corollary:

Corollary 3.4. *Let L_2 be as in Theorem 3.3 and $L > L_2$. Then for any $u \in \mathcal{U}_{L,T}$ which satisfies* (3.2) *with ϵ as in* (3.6)*, there exists $t^* \in (0, T)$ so that*

$$F_L(u(\cdot, t^*)) \geq L F_L^{(1)}(\hat{m}_L), \tag{3.9}$$

and

$$\lim_{t\to\infty} T_t(u(\cdot, t^*)) = \hat{m}_L^e \qquad \text{in } L^2(Q_L). \tag{3.10}$$

Proof. Let $u \in \mathcal{U}_{L,T}$ be as in the statement, $\alpha(t)$ such that $u(\cdot, t) \in \Sigma_{\alpha(t)}$ and $I = \{t \in [0, T] : |\vartheta_{\alpha(t)}| \leq \theta_0\}$. Since $\vartheta_{\alpha(0)} = 1/2$, $\vartheta_{\alpha(T)} = -1/2$ and $\theta_0 < 1/2$, by continuity there is an interval $[t_0, t_1] \subset I$, where $\vartheta_{\alpha(t_0)} = \theta_0$ and $\vartheta_{\alpha(t_1)} = -\theta_0$. By Theorems 3.1, 3.2 and 3.3, $[t_0, t_1]$ is the disjoint union of the intervals I_+, I_- and $\hat{I}$, respectively where $u(\cdot, t)$ is attracted by $m^{(+)}$, $m^{(-)}$ and $\hat{m}_L^e$. By Theorem 3.3 $I_+ \ni t_1$ and $I_- \ni t_0$, thus $I_\pm$ are both non-empty. Moreover, since the equilibria $\pm m_\beta$ are stable, by the continuity of motion I_+ and I_- are open in I. Then necessarily also $\hat{I} \neq \emptyset$ and hence there is a time $t^* \in (t_0, t_1)$ so that (3.10) holds. Since T_t decreases the energy F_L, see (6.2), (3.9) follows from (3.10), (6.2) and (2.15). □

Conclusion of the proof of Theorem 2.3. From (2.8) and (3.9) it follows that, if L is large enough,

$$I_{L,T}(u) = A_{L,T}(u) \geq A_{L;t^*}(u) = F_L(u(\cdot, 0)) + I_{L,t^*}(u) \geq F_L(u(\cdot, t^*)) \geq L F^{(1)}(\hat{m}_L),$$

hence (3.3). Theorem 2.3 is proved. □

In $d = 1$, see [3, 4] (and [18] for Allen-Cahn), it is proved that for L large enough if $F_L^{(1)}(m) \leq F_L^{(1)}(\hat{m}_L) + \epsilon$ and m is a critical point, then $m \in \{m^+, m^-, \hat{m}_L\}$. (The statement is evident in Allen-Cahn once formulated in terms of a one dimensional mechanical point in a conservative force field.) It then follows that if $P_L^{(1)}(u) \leq F_L^{(1)}(\hat{m}_L) + \epsilon$ then at all t, $u(\cdot, t)$ is attracted by $\{m^+, m^-, \hat{m}_L\}$. In $d = 2$ we know that such a property is valid only at times t when $u(\cdot, t) \in \Sigma_\alpha$ with α such that $|\vartheta_\alpha| \leq \theta_0$. As shown above the proof of Theorem 2.3 can be worked out also with such a weaker statement, but there could be problems when extending the analysis to Glauber dynamics in Ising models with Kac potentials.

We have shown that the proof of Theorem 2.3 reduces to the proof of Theorems 3.1, 3.2 and 3.3, which is given in the next sections. While the proof of Theorems 3.2 and 3.3 is an extension of the proof of analogous statements in $d = 1$, see [3], the proof of Theorem 3.1 requires really new considerations, due to the geometrical complexities of a higher dimension and it will take most of the paper.

4. Proof of Theorem 2.4

In this section we prove Theorem 2.4 using Theorem 2.3 which is thus taken for proved. Let $\{u_n, T_n\}$ be a minimizing sequence for (2.6), i.e., $u_n \in \mathcal{U}_{L,T_n}$ and $\lim_{n\to\infty} I_{L,T_n}(u_n) = P_L = LF_L^{(1)}(\hat{m}_L)$, where the last equality follows from Theorem 2.3. Then for any $\epsilon > 0$ there exists n_ϵ so that for any $n \geq n_\epsilon$,

$$I_{L,T_n}(u_n) \leq LF_L^{(1)}(\hat{m}_L) + \epsilon, \quad LF^{(1)}(\hat{m}_L) = F_L(\hat{m}_L^e). \tag{4.1}$$

By Corollary 3.4 if $L > L_2$ and ϵ is as in (3.6), then for any $n \geq n_\epsilon$ there is a time $s_n \in (0, T_n)$ (s_n will be the time s in Theorem 2.4) so that

$$\lim_{t\to\infty} T_t\big(u(\cdot, s_n)\big) = \hat{m}_L^e, \quad F_L(u_n(\cdot, s_n)) \geq LF_L^{(1)}(\hat{m}_L). \tag{4.2}$$

By (2.8), $I_{L,T_n}(u_n) \geq I_{L,s_n}(u_n) \geq F_L(u_n(\cdot, s_n))$, then, using (4.1),

$$0 \leq F_L(u_n(\cdot, s_n)) - F_L(\hat{m}_L^e) \leq \epsilon. \tag{4.3}$$

The function

$$w_n(\cdot, t) = u_n(\cdot, s_n - t), \qquad t \in (0, s_n) \tag{4.4}$$

satisfies the identity

$$\frac{dw_n}{dt} = J^{\text{neum}} * w_n - \frac{1}{\beta} \operatorname{arctanh}(w_n) + K_n, \tag{4.5}$$

where K_n is defined by (4.5) itself. We then consider (4.5) as an equation in w_n, regarding K_n as a "known term". In the next lemma we will prove that K_n is "small" and then as a consequence and relying heavily on [2] that w_n follows closely the S_t-invariant manifold $\mathcal{W}_-$.

Lemma 4.1. *Let $\epsilon > 0$, u_n satisfy* (4.1)*, s_n as in* (4.2)*, w_n as in* (4.4) *and K_n as in* (4.5)*. Then for n sufficiently large,*

$$\|K_n\|^2 := \int_0^{s_n} \int_{Q_L} K_n(r, t)^2 \, dr dt < \epsilon. \tag{4.6}$$

Furthermore there exists $c > 0$ independent of n so that

$$\|w_n(\cdot, 0) - \hat{m}_L^e\|_2^2 \leq c\epsilon. \tag{4.7}$$

Proof. From (4.1) and (2.15) it follows that

$$F_L(\hat{m}_L^e) + \epsilon \geq \int_0^{s_n} \int_{Q_L} [(u_n)_t - f_L(u_n)]^2 = I_{L,s_n}(u_n). \tag{4.8}$$

By (2.7) and recalling that $F_L(u_n(\cdot, 0)) = 0$,

$$I_{L,s_n}(u_n) = A_{L;s_n}(u_n) = A_{L,s_n}(w_n) = I_{L,s_n}(w_n) + F_L(u_n(\cdot, s_n)) = \|K_n\|^2 + F_L(u_n(\cdot, s_n)),$$

which, together with (4.8) and (4.2), implies (4.6).

Let $\Sigma := \{m \in L^\infty(Q_L, (-1,1)) : \lim_{t\to\infty} \|T_t(m) - \hat{m}^e_L\|_2 = 0\}$. In [2, Theorem 7.2] it is proved that there is c so that

$$\|m - \hat{m}^e_L\|_2^2 \le c[F_L(m) - F_L(\hat{m}^e_L)] \qquad \text{for all } m \in \Sigma. \tag{4.9}$$

By (4.2) $u_n(\cdot, s_n) = w_n(\cdot, 0) \in \Sigma$, therefore (4.7) follows from (4.9) and (4.3). □

We will prove the properties of w_n stated in Theorem 2.4 by investigating the evolution equation (4.5) and exploiting that K_n is small. Smallness of K_n is however not enough: if we only knew the bounds on K_n from Lemma 4.1 we could not predict (even approximately) the evolution of w_n. Recall in fact that $\hat{m}^e_L$ is a stationary solution of the unperturbed evolution so that, no matter how small K_n is, it would nonetheless be larger than the unperturbed force in a correspondingly small neighborhood of $\hat{m}^e_L$. In other words, when close to $\hat{m}^e_L$ the evolution is essentially ruled by K_n. Besides this, the initial datum $w_n(\cdot, 0)$ is in the domain of attraction of $\hat{m}^e_L$ with "the wrong dynamics" T_t, under the "right evolution" S_t it may no longer converge to $\hat{m}^e_L$ but rather to m^- or even m^+. In conclusion the evolution of $w_n(\cdot, 0)$ may have completely different behavior if we only had the information in Lemma 4.1 concerning smallness of K_n and closeness of $w_n(\cdot, 0)$ to $\hat{m}^e_L$.

Let us now recall what is proved in [2], in particular Theorem 7.3 of [2]. Call $S^K_t(m)$ the flow generated by the equation $u_t = J^{\text{neum}} * u - \beta^{-1}\text{arctanh}(u) + K$, $u(\cdot, 0) = m$, where $K = K(r,t)$, $(r,t) \in Q_L \times \mathbb{R}_+$, is a smooth space-time dependent force. Then for any ζ and τ positive there is $\epsilon' > 0$ so that if $\|K\| < \epsilon'$ and $\|m - \hat{m}^e_L\|_2 < \epsilon'$ only the following two alternatives hold:

- For all times $t \ge 0$, $\|S^K_t(m) - \hat{m}^e_L\|_2 < \zeta$.
- There are $t^* > 0$ and $\sigma \in \{-,+\}$ so that $\|S^K_t(m) - \hat{m}^e_L\|_2 < 2\|v^{(\sigma)}_L(\cdot, -\tau) - \hat{m}^e_L\|_2$ for all $t \le t^*$ while $\|S^K_t(m) - v^{(\sigma)}_L(\cdot, -\tau + (t - t^*)\|_2 < \zeta$ for all $t \ge t^*$.

Let us now prove the statements in Theorem 2.4 referring to $\mathcal{W}_-$, calling ϵ^* the parameter ϵ in Theorem 2.4 to avoid confusion with the ϵ in (4.1) and identifying $s = s_n$. Recall that $u_n(s_n - t) = S^{K_n}_t(w_n(0))$, $t \in [0, s_n]$; we are only writing the time variable in the argument of the functions.

We choose: τ such that $\sup_{s\le -\tau} \|v^{(-)}_L(s) - \hat{m}^e_L\|_2 \le \epsilon^*/10$; $\zeta < \epsilon^*/10$; ϵ' is determined by τ and ζ as above; ϵ in (4.1) so that $\epsilon < \epsilon'$ and $c\epsilon < \epsilon^*$, $c\epsilon$ as in (4.7), so that the inequality $\|u_n(\cdot, s_n) - \hat{m}^e_L\|_2 \le \epsilon^*$ in Theorem 2.4 follows from (4.7).

Since $u_n(0) = m^{(-)}$ the first alternative above is excluded and in the second alternative $\sigma = -$. Let t^* be as in the second alternative. We then have $\|u_n(s_n - t) - v^{(-)}_L(-\tau + t - t^*)\|_2 < \epsilon^*$ for $t \in [t^*, s_n]$. For $t \in [0, t^*]$ we write

$$\|u_n(s_n - t) - v^{(-)}_L(-\tau + t - t^*)\|_2 \le \|u_n(s_n - t) - \hat{m}^e_L\|_2 + \|v^{(-)}_L(-\tau + t - t^*) - \hat{m}^e_L\|_2$$

which is $\le 3 \sup_{s\le -\tau} \|v^{(-)}_L(s) - \hat{m}^e_L\|_2 \le 3\epsilon^*/10$. Equation (2.21) is thus proved with $\tau' = -\tau + (s_n - t^*)$.

The proof of (2.22) is analogous. We now take

$$w^+_n(\cdot, t) = u_n(\cdot, s_n + t), \qquad t \in [0, T_n - s_n],$$

so that w_n^+ satisfies the "equation"

$$\frac{dw_n^+}{dt} = J^{\text{neum}} * w_n^+ - \frac{1}{\beta}\,\text{arctanh}(w_n^+) + K_n^+ \tag{4.10}$$

with K_n^+ defined by (4.10). Analogously to Lemma 4.1, $\|K_n^+\| < \epsilon$ for n large enough. Since $S^{K^+}_{T_n-t_n}(w_n^+) = m^{(+)}$, the first alternative is again excluded and the second one is followed with $\sigma = +$. Again we require τ so large and ζ so small (and ϵ correspondingly small) that $\|v_L^{(+)}(\cdot, -\tau) - \hat{m}_L^e\|_2 \le \epsilon^*/10$ and $\zeta < \epsilon^*$. Then (2.22) follows with $\tau'' = \tau' + t^*$ (t^* the time appearing in the second alternative applied to the present case). Notice finally that if $\epsilon^* \to 0$ the time τ in the above construction diverges and then we need also $T_n \to \infty$ as stated in Theorem 2.4.

5. Local Equilibrium and Peierls Estimates

The heuristics behind the proof of Theorem 3.1 goes as follows. The Wulff theorem and the limit Wulff shape suggest that if $u(\cdot,\cdot)$ satisfies (3.2), at times t when $u(\cdot, t) \in \Sigma_\alpha$ with $|\vartheta_\alpha| \le \theta_0$, to "zero order" $u(\cdot, t)$ looks like

$$W_{\alpha,L} := m_\beta 1_{\{(x,y): x \ge L\vartheta_\alpha\}} - m_\beta 1_{\{(x,y): x < L\vartheta_\alpha\}}. \tag{5.1}$$

To a next approximation we expect $u(\cdot, t)$ close to $\bar{m}_{\xi,L}$ with ξ such that $\bar{m}_{\xi,L} \in \Sigma_\alpha$. Behind this picture is the intuition that it does not pay to have deviations from $+m_\beta$ and $-m_\beta$ away from the interface and that the actual profile at the interface is not exactly as sharp as in $W_{\alpha,L}$ but rather the diffuse interface defined by the $d = 1$ instanton $\bar{m}$ shifted by ξ.

In this section we quote from the literature lower bounds on the free energy due to deviations from equilibrium [Peierls estimates], in the next one we prove lower bounds due to deviations from the instanton shape and in Sect. 7 we use all that to prove Theorem 3.1.

Local equilibrium and deviations from equilibrium as usual in statistical mechanics are defined in terms of "averages" and of "coarse grained" variables. We briefly recall the main notion adapted to the present context.

Definition 5.1 (*Coarse graining*). *We denote by $\mathcal{D}^{(\ell)}$, $\ell > 0$, the partition of $\mathbb{R}^2$ into the squares $\{(x, y) : x \in [n\ell, (n+1)\ell), y \in [n'\ell, (n'+1)\ell)\}$, n, n' integers, and by $C_r^{(\ell)}$ the square of $\mathcal{D}^{(\ell)}$ which contains r. Then the ℓ-coarse grained image $m^{(\ell)}$ of a function $m \in L^\infty(\mathbb{R}^2)$ is*

$$m^{(\ell)}(r) := \fint_{C_r^{(\ell)}} m(r').$$

Definition 5.2 (*Geometrical notions*). *A set is $\mathcal{D}^{(\ell)}$-measurable if it is union of squares in $\mathcal{D}^{(\ell)}$, two sets are connected if their closures have non-empty intersection and B is a vertical connection if it is a $\mathcal{D}^{(\ell_+)}$-measurable, connected set which is connected to both lines $\{y = \pm L/2\}$. Given a $\mathcal{D}^{(\ell_+)}$-measurable region $\Lambda \subset Q_L$ we call $\delta_{\text{out}}^{\ell_+}[\Lambda]$ the union of all squares of $\mathcal{D}^{(\ell_+)}$ in $Q_L \setminus \Lambda$ which are connected to Λ.*

Definition 5.3 (*Phase indicators*)**.** *Given an "accuracy parameter" $\zeta > 0$ and $m \in L^\infty(\mathbb{R}^2, [-1, 1])$, we define the "local phase indicator"*

$$\eta^{(\zeta,\ell)}(m; r) = \begin{cases} \pm 1 & \text{if } |m^{(\ell)}(r) \mp m_\beta| \le \zeta, \\ 0 & \text{otherwise.} \end{cases}$$

Given $\ell_- > 0$, ℓ_+ an integer multiple of ℓ_-, $\mathcal{D}^{(\ell_+)}$ a coarser partition of $\mathcal{D}^{(\ell_-)}$, we define the "global phase indicator"

$$\Theta^{(\zeta,\ell_-,\ell_+)}(m; r) = \begin{cases} \pm 1 & \text{if } \eta^{(\zeta,\ell_-)}(m; \cdot) = \pm 1 \text{ in } C_r^{(\ell_+)} \cup \delta_{\text{out}}^{\ell_+}[C_r^{(\ell_+)}], \\ 0 & \text{otherwise.} \end{cases}$$

$\eta^{(\zeta,\ell)}(m; r)$ and $\Theta^{(\zeta,\ell_-,\ell_+)}(m; r)$ are defined also for functions $m \in L^\infty(Q_L, [-1, 1])$ by simply extending m to $\mathbb{R}^2$ by reflections along the lines $\{y = (2n + 1)L/2\}$ and $\{x = (2n + 1)L/2\}$, $n \in \mathbb{Z}$.

Definition 5.3 introduces the notion of "local equilibrium": a point r is attributed to the plus phase if $\Theta^{(\zeta,\ell_-,\ell_+)}(m; r) = 1$, to the minus phase if $\Theta^{(\zeta,\ell_-,\ell_+)}(m; r) = -1$ while, if $\Theta^{(\zeta,\ell_-,\ell_+)}(m; r) = 0$, r belongs to a contour, contours being the maximal connected components of $\{r : \Theta^{(\zeta,\ell_-,\ell_+)}(m; r) = 0\}$. Local equilibrium in r requires closeness to m_β in a large region, the 9 squares in Fig. 2. By choosing ℓ_- small we try to approximate point-wise closeness (which would be too strong a request as the energy is defined by integrals) while by taking ℓ_+ large we try to approximate global equilibrium. Very little is needed for local equilibrium to fail as exemplified in Fig. 2.

Definition 5.4 (*Choice of parameters*)**.** *We choose ℓ_- and ℓ_+ as functions of ζ and L. The definition is used only when ζ is small and L much larger than ℓ_+, and the dependence on L is only through the requirement that $L\ell_\pm^{-1}$ is an integer. We require that for ζ small enough: $\ell_- \in [\zeta^2/2, \zeta^2]$; $\ell_+ \in [\zeta^{-4}/2, \zeta^{-4}]$ with ℓ_+ an integer multiple of ℓ_-; Q_L to be the closure of union of squares of $\mathcal{D}^{(\ell_+)}$; each square of $\mathcal{D}^{(\ell_+)}$ to be the union of squares of $\mathcal{D}^{(\ell_-)}$.*

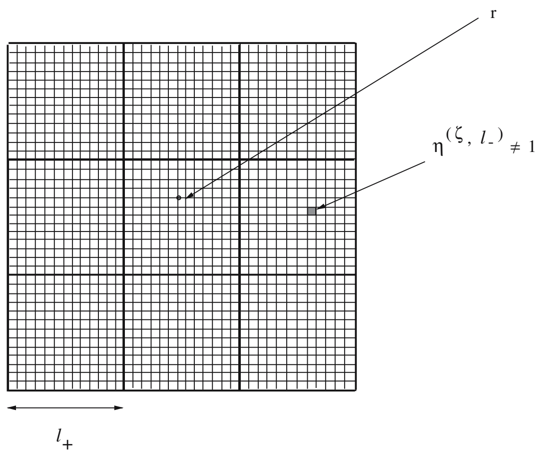

Fig. 2. Nine large squares belonging to $\mathcal{D}^{(\ell_+)}$. The small squares are instead elements of $\mathcal{D}^{(\ell_-)}$. Even if $\eta^{(\zeta,\ell_-)}(m; \cdot) = 1$ in all small squares except the one in grey, nonetheless $\Theta^{(\zeta,\ell_-,\ell_+)}(m; r) = 0$

We have the following two theorems, whose proof is (essentially) contained in [28]:

Theorem 5.5. *There exist $c > 0$ and $\omega > 0$ such that if $\zeta > 0$ is small enough and ℓ_-, ℓ_+ and L are as above, the following assertions hold. Let $\Lambda \subset Q_L$ be $\mathcal{D}^{(\ell_-)}$-measurable, and let m be such that $\eta^{(\zeta,\ell_-)}(m; r) = 1$ for all $r \in Q_L$ at distance ≤ 1 from Λ. Then there exists a function ψ satisfying*

$$\begin{aligned}
&\psi = m \text{ outside } \Lambda,\\
&\eta^{(\zeta,\ell_-)}(\psi;\cdot) = 1 \text{ in } \Lambda,\\
&\psi(r) = \tanh\{\beta J^{\text{neum}} * \psi(r)\}, \qquad r \in \Lambda,\\
&|\psi(r) - m_\beta| \le c e^{-\omega \text{dist}(r, Q_L \setminus \Lambda)}, \qquad r \in \Lambda,\\
&F_L(\psi) \le F_L(m).
\end{aligned}$$

The analogous statement holds if $\eta^{(\zeta,\ell_-)}(m;\cdot) = -1$, provided m_β is replaced by $-m_\beta$.

Theorem 5.6. *There exists $c_1 > 0$ such that if ζ is small enough and ℓ_-, ℓ_+ and L are as above, the following assertions hold. Let $\Lambda \subset Q_L$ be $\mathcal{D}^{(\ell_+)}$-measurable and let m be such that $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot) = 1$ in $\delta_{\text{out}}^{\ell_+}[\Lambda]$. Then there exists a function ψ satisfying*

$$\begin{aligned}
&\psi = m \text{ outside } \Lambda,\\
&\eta^{(\zeta,\ell_-)}(\psi;\cdot) = 1 \text{ in } \Lambda,\\
&F_L(m) \ge F_L(\psi) + c_1 \zeta^2 (\ell_-)^2 N_0,
\end{aligned} \tag{5.2}$$

where N_0 denotes the number of squares of $\mathcal{D}^{(\ell_+)} \cap \Lambda$, where $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot) = 0$. The analogous statement holds if $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot) = -1$ in $\delta_{\text{out}}^{\ell_+}[\Lambda]$.

6. Free Energy Bounds in the Channel

This section continues the "preparation" to the proof of Theorem 3.1. We will estimate here the cost of deviations from the instanton shape. The natural setup for the problem is the channel $Q_{\infty,L} = \{(x, y) : |y| \le L/2\}$; in the next section we will in fact eventually reduce from Q_L to $Q_{\infty,L}$. Our main result is an extension to $Q_{\infty,L}$ of a $d = 1$ result in [28]:

Theorem 6.1. *There is c so that for any L large enough and for any $m \in L^\infty(Q_{\infty,L}, [-1, 1])$ such that uniformly in y $\liminf_{x\to\infty} m(x, y) > 0$ and $\limsup_{x\to-\infty} m(x, y) < 0$ and such that for some $\xi \in \mathbb{R}$, $\|m - \bar m^e_\xi\|_2^2 < \infty$,*

$$F_{Q_{\infty,L}}(m) - F_{Q_{\infty,L}}(\bar m^e) \ge \begin{cases} cL^{-[22+36\beta]}, & \text{if } \inf_{\xi\in\mathbb{R}} \|m - \bar m^e_\xi\|_2^2 > L^{-24\beta-8} \\ cL^{-[2+12(\beta+1)]} \inf_{\xi\in\mathbb{R}} \|m - \bar m^e_\xi\|_2^2, & \text{if } \inf_{\xi\in\mathbb{R}} \|m - \bar m^e_\xi\|_2^2 \le L^{-24\beta-8} \end{cases}. \tag{6.1}$$

The dependence on L in (6.1) is not optimal. We cannot possibly have

$$F_{Q_{\infty,L}}(m) - F_{Q_{\infty,L}}(\bar m^e) \ge c \inf_{\xi\in\mathbb{R}} \|m - \bar m^e_\xi\|_2^2, \quad c > 0$$

because $\inf_{\xi\in\mathbb{R}} \|m - \bar m^e_\xi\|_2^2$ can be made arbitrarily large while keeping the free energy bounded: just take m as a piecewise constant function of x which as x increases from $-\infty$ to $+\infty$ has values $-m_\beta$, m_β, $-m_\beta$ and m_β. Then the L^2 norm increases to ∞ as the two intermediate intervals are made long enough while the free energy is bounded by $4cL$, $c = \int_{x\le 0}\int_{x'\ge 0} j(x,x')$. Thus the lower bound can hold only if $\inf_{\xi\in\mathbb{R}} \|m - \bar m^e_\xi\|_2^2$ is small enough. Theorem 6.1 is proved at the end of the section. Its proof, essentially perturbative, is obtained by expanding $F_{Q_{\infty,L}}(m)$ around $F_{Q_{\infty,L}}(\bar m^e)$. The linear term disappears because the instanton is a critical point; the quadratic term becomes then the leading one. Its analysis requires the study of the spectral properties of a linear operator, which is the second derivative of the functional and hence also the operator obtained by linearizing the time flow around $\bar m^e$. The spectral properties of such an operator are interesting in their own right, their analysis far from trivial and rather long. We have thus decided to just use in this section the outcome of the theory leaving details and proofs to an appendix, where the issue is presented in a self contained fashion.

Thus a spectral gap estimate will prove the desired lower bounds to a second order approximation, an analysis of the energy landscape away from the instanton shape where non-linear effects are dominant requires a different set of ideas. Both close and away from the instanton shape, dynamical properties of the flow $T_t(m)$ play a dominant role, as well as in the proofs of Theorems 5.5-5.6. We thus begin our analysis by quoting from the literature some basic properties of the time flow.

6.1. Monotonicity of energy. The semigroup T_t generated by (2.5) (either in $\mathbb{R}^d$ or else in Q_L or in $Q_{\infty,L}$ with $J \to J^{\mathrm{neum}}$, at the moment our notation does not distinguish among them) has the following properties (which explains why they are useful in proving energy bounds):

(i) T_t decreases the energy F (respectively in $\mathbb{R}^d$ or else in Q_L and in $Q_{\infty,L}$): $F(T_t(m)) \le F(T_s(m))$ for $s \le t$ and if $\lim_{t\to\infty} T_t(m) \to m^*$ uniformly on the compacts then

$$\liminf_{t\to\infty} F(T_t(m)) \ge F(m^*). \tag{6.2}$$

(ii) As $t \to \infty$, $T_t(m)$ converges by subsequences uniformly on the compacts to a solution of the stationary equation $m = \tanh\{\beta J * m\}$ (with $J \to J^{\mathrm{neum}}$ in Q_L or $Q_{\infty,L}$).

6.2. Properties of the instanton. In [17] it is proved that there exists $a > 0$ so that

$$\lim_{x\to\infty} e^{\alpha x}\bar m'(x) = a, \tag{6.3}$$

where $\alpha > 0$ is such that

$$p_- \int_{\mathbb{R}} j(0,x)e^{\alpha x} = 1, \quad p_- = \lim_{x\to\infty} p(x) = \beta(1-m_\beta^2) < 1.$$

The finite volume instanton $\hat{m}_L$ is close to $\bar{m}$ restricted to $[-L/2, L/2]$; we will just need here that their energies are exponentially close: there are $c > 0$ and $\omega > 0$ so that for all L,

$$|F_L^{(1)}(\hat{m}_L) - F^{(1)}(\bar{m})| \le ce^{-\omega L}. \tag{6.4}$$

A function m on $\mathbb{R}$ "is close in shape to an instanton" if m is close to a translate $\bar{m}_\xi$ of $\bar{m}$, $\bar{m}_\xi(x) = \bar{m}(x-\xi)$, $\xi \in \mathbb{R}$. Usually ξ is chosen by minimizing a weighted L^2 distance of m from the instanton manifold $\{\bar{m}_\xi, \xi \in \mathbb{R}\}$. We will use here the notion of center of m: ξ_m is a "center of m" if $\int_{\mathbb{R}} m\bar{m}'_{\xi_m} p_{\xi_m}^{-1} = 0$, $p_\xi = \beta(1-\bar{m}_\xi^2)$. ξ_m is then a critical point of $\xi \to \|m - \bar{m}_\xi\|_\xi^2 := \int_{\mathbb{R}} (m-\bar{m}_\xi)^2 p_\xi^{-1}$, i.e. the p_ξ^{-1}-weighted L^2 distance. Existence and uniqueness of the center are proved when either $\inf_\xi \|m-\bar{m}_\xi\|_2$ or $\inf_\xi \|m-\bar{m}_\xi\|_\infty$ are small enough, see [28]. The proof extends straightforwardly to the case of the channel, $Q_{\infty,L}$, where ξ_m is defined as

$$\int_{Q_{\infty,L}} m(r)\bar{m}'_{\xi_m}(r\cdot e_1) p_{\xi_m}^{-1}(r\cdot e_1) = 0, \qquad p_\xi(x) = \beta[1-\bar{m}_\xi^2(x)]. \tag{6.5}$$

The precise statement (in the L^2 case) is given in the lemma below where we show that the center of m is related to the minimization of the usual L^2 distance from the instanton manifold.

Lemma 6.1. *There are c and ϵ positive so that if $m \in L^\infty(Q_{\infty,L}, [-1,1])$ and $\inf_{\xi\in\mathbb{R}} \|m-\bar{m}_\xi^e\|_2 < \epsilon$ then there is a unique ξ_m such that* (6.5) *holds, $\|m-\bar{m}_{\xi_m}^e\|_2 \le c\epsilon$ and*

$$\|m-\bar{m}_{\xi_m}^e\|_2^2 \le \frac{1}{1-m_\beta^2} \inf_{\xi\in\mathbb{R}} \|m-\bar{m}_\xi^e\|_2^2. \tag{6.6}$$

Proof. The proof of existence and uniqueness of the center and that $\|m-\bar{m}_{\xi_m}^e\|_2 \le c\epsilon$ are a simple extension of their proof in $d=1$, [28], and are omitted. It remains to prove (6.6). Without loss of generality we may suppose $\xi_m = 0$. Let then

$$f(\xi) := \int_{Q_{\infty,L}} \frac{[m(r)-\bar{m}_\xi(r\cdot e_1)]^2}{p(r\cdot e_1)} dr, \quad b := \sqrt{f(0)}.$$

Since $p_{\xi_m}^{-1} \le [\beta(1-m_\beta^2)]^{-1} < (1-m_\beta^2)^{-1}$ and $\|m-\bar{m}^e\|_2 \le c\epsilon$ then $b^2 \le (1-m_\beta^2)^{-1}(c\epsilon)^2$ for all m such that $\inf_{\xi\in\mathbb{R}} \|m-\bar{m}_\xi^e\|_2 < \epsilon$.

We claim that $f(\xi)$ has a unique minimum at $\xi=0$ if ϵ is small enough. Call f' and f'' the first and second derivatives of f w.r.t. ξ. By an explicit computation, $f'(0)=0$ and

$$f''(0) \ge 2\int \frac{(\bar{m}')^2}{p} dr - 2b\Big[\int \frac{(\bar{m}'')^2}{p} dr\Big]^{1/2}.$$

We suppose ϵ so small that $2b\Big[\int \frac{(\bar{m}'')^2}{p} dr\Big]^{1/2} < 2^{-1} \int \frac{(\bar{m}')^2}{p} dr$. Then there is $\epsilon^* > 0$ so that

$$f''(\xi) \ge \int \frac{(\bar{m}')^2}{p} dr, \quad |\xi| \le \epsilon^*,$$

which proves that $f(\xi)$ has a unique minimum at $\xi = 0$ when $\xi \in [-\epsilon^*, \epsilon^*]$. Call

$$A(\xi)^2 = \int \frac{(\bar m - \bar m_\xi)^2}{p} dr, \quad A^2 = \inf_{|\xi| \ge \epsilon^*} A(\xi)^2 > 0$$

and suppose ϵ so small that $b < A/2$. Write $f(\xi) = \int [\{m - \bar m\} - \{\bar m_\xi - \bar m\}]^2 p^{-1} dr = b^2 + A(\xi)^2 - 2\int (m - \bar m)(\bar m_\xi - \bar m) p^{-1} dr$, hence $f(\xi) \ge A(\xi)^2 \big(1 - \frac{b}{A(\xi)}\big)^2 \ge A^2/4$. Then

$$f(0) = b^2 < A^2/4 \le \inf_{|\xi| \ge \epsilon^*} f(\xi)$$

thus proving the claim that 0 is the unique minimizer of f.

Using that $p_{\xi_m} < \beta$ and that $1 - \bar m_\xi^2 > 1 - m_\beta^2$ we then have

$$\begin{aligned} \|m - \bar m^e_{\xi_m}\|_2^2 &\le \int\limits_{Q_{\infty,L}} \frac{\beta}{p^e_{\xi_m}} [m - \bar m^e_{\xi_m}]^2 = \inf_{\xi \in \mathbb{R}} \beta \int\limits_{Q_{\infty,L}} \frac{[m - \bar m^e_\xi]^2}{p^e_{\xi_m}} \\ &\le \frac{1}{1 - m_\beta^2} \inf_{\xi \in \mathbb{R}} \|m - \bar m^e_\xi\|_2^2. \end{aligned}$$

□

6.3. Spectral estimates. There are several results on the linear stability of the instanton shape in $d = 1$ which extend to the case of the channel $Q_{\infty,L}$. In an Appendix (Sects. 9 and 10) we will prove the following statements where, to simplify notation, we drop the superscript "e"to denote the extension of a function on $\mathbb{R}$ to the channel $Q_{\infty,L}$. Recalling that $g_L(m) := -m + \tanh\{\beta J^{\text{neum}} * m\}$ and that $g_L(\bar m_\xi) = 0$, the first order term in the expansion of $g_L(\bar m_\xi + \psi)$, $\psi = m - \bar m_\xi$, gives

$$\Omega_\xi \psi = -\psi + p_\xi \; J^{\text{neum}} * \psi, \qquad p_\xi = \beta(1 - \bar m_\xi^2). \tag{6.7}$$

We will regard Ω_ξ as an operator on L^∞ and/or L^2. It is easily checked that Ω_ξ has an eigenvalue 0 with eigenvector $\bar m'_\xi$ and that Ω_ξ is self-adjoint on $L^2(Q_L, p_\xi^{-1})$. In Sect. 10 we will prove the existence of a L^2 spectral gap: there is a positive number κ (called a in Theorem 10.1) so that

$$\langle \psi, \Omega_\xi \psi \rangle_\xi \le -\frac{\kappa}{L^2} \langle \psi, \psi \rangle_\xi, \quad \langle \psi, \bar m'_\xi \rangle_\xi = 0, \tag{6.8}$$

where $\langle \cdot, \cdot \rangle_\xi$ is the scalar product in $L^2(Q_L, p_\xi^{-1})$. A spectral gap in L^∞ is proved in Sect. 9: there is $c > 0$ so that (see Theorem 9.4)

$$\|e^{\Omega_\xi t} \psi\|_\infty \le c e^{-(\kappa/L^2)t} \|\psi\|_\infty, \quad \langle \psi, \bar m'_\xi \rangle_\xi = 0. \tag{6.9}$$

The orthogonality condition in (6.8)–(6.9) is behind the definition of center of a function in (6.5). Indeed if $\xi = \xi_m$ then $\psi = m - \bar m_\xi$ fulfills the requirement in (6.8)–(6.9) as

$$\langle (m - \bar m_\xi), \bar m'_\xi \rangle_\xi = 0. \tag{6.10}$$

Recall in fact that, $\langle \bar m_\xi, \bar m'_\xi \rangle_\xi = 0$ because $\bar m$ is antisymmetric and $\bar m'$ symmetric, then (6.10) with $\xi = \xi_m$ follows from (6.5).

6.4. Stability of the instanton. We start by proving a weaker version of (6.1):

Theorem 6.2. *Let $m \in L^\infty(Q_{\infty,L}, [-1,1])$ be such that $\liminf_{x\to\pm\infty} m(x,y) \gtrless 0$ uniformly in y. Then there exists $\widehat{\xi}$ such that*

$$\lim_{t\to\infty} T_t(m) = \bar m^e_{\widehat{\xi}} \qquad \text{in } L^\infty(Q_{\infty,L}) \tag{6.11}$$

and

$$F_{Q_{\infty,L}}(m) \geq F_{Q_{\infty,L}}(\bar m^e_{\widehat{\xi}}) = F_{Q_{\infty,L}}(\bar m^e) = c_\beta L. \tag{6.12}$$

Proof. The analogous statements hold for the instanton $\bar m$ in $d = 1$ and indeed the proof of the theorem is a simple adaptation of the $d = 1$ proof. We just sketch the main lines. The first step uses in an essential way the spectral gap property of the previous subsection to extend from linear to local stability. The argument is standard and allows to conclude that if $\|m - \bar m^e_\xi\|_\infty \leq \epsilon$ with $\epsilon > 0$ small enough, then (6.11) is verified. The global stability statement in the theorem follows from the above local stability using more involved arguments based on a comparison theorem (ferromagnetic inequalities). The proof is however essentially as in $d = 1$, see [12, 16, 17], and its details are omitted. Equation (6.12) follows from (6.11) and (i) of Subsect. 6.1. □

Lemma 6.3. *Let $m \in L^\infty(Q_{\infty,L}, [-1,1])$. Then*

$$\|\nabla(T_t(m) - e^{-t}m)\|_\infty \leq \beta \,\|\nabla J\|_\infty. \tag{6.13}$$

Moreover, there exists $\tau > 0$ so that for any $t \geq \tau$ and any $m \in L^\infty(Q_{\infty,L}, [-1,1])$,

$$\|T_t(m)\|_\infty \leq m_\beta + \frac{1 - m_\beta}{2}. \tag{6.14}$$

Proof. The integral version of (2.5) yields

$$T_t(m) - e^{-t}m = \int_0^t e^{-(t-s)} \tanh\{\beta J^{\text{neum}} * T_s(m)\},$$

hence (6.13). A comparison theorem holds for (2.5) so that $T_t(-1) \leq T_t(m) \leq T_t(1)$ which then gives (6.14). □

Lemma 6.4. *There exists a constant $c > 0$ such that if $m \in L^\infty(Q_{\infty,L}, [-1,1])$ and $\xi \in \mathbb{R}$ then*

$$\|T_t(m) - \bar m^e_\xi\|_\infty \leq 2e^{-t} + c\left(\|T_t(m) - \bar m^e_\xi\|_2 + e^{-t}\|m - \bar m^e_\xi\|_2\right)^{2/3}.$$

Proof. We may assume $\xi = 0$ and write simply $\bar m^e$. The function $\psi = T_t(m) - \bar m^e - e^{-t}(m - \bar m^e)$ has bounded derivative hence (see for instance [20]) there exists $c > 0$ (which depends on the L^∞ norm of the derivative) so that $\|\psi\|_\infty \leq c\|\psi\|_2^{2/3}$. Thus

$$\|T_t(m) - \bar m^e\|_\infty \leq \|\psi\|_\infty + 2e^{-t} \leq 2e^{-t} + c\left(\|T_t(m) - \bar m^e - e^{-t}(m - \bar m^e)\|_2\right)^{2/3}.$$

□

Lemma 6.5. *If $m \in L^\infty(Q_{\infty,L}, [-1,1])$ and $m - \bar m^e_\xi \in L^2(Q_{\infty,L})$, then for all $t \ge 0$,*

$$e^{-2(\beta+1)t}\|m-\bar m^e_\xi\|_2^2 \le \|T_t(m)-\bar m^e_\xi\|_2^2 \le e^{2(\beta-1)t}\|m-\bar m^e_\xi\|_2^2 \tag{6.15}$$

and

$$\Big|\frac{d}{dt}\|T_t(m)-\bar m^e_\xi\|_2^2\Big| \le 2(\beta-1)\|T_t(m)-\bar m^e_\xi\|_2^2. \tag{6.16}$$

Proof. Suppose again $\xi = 0$ and write simply $\bar m^e$. Let $v = T_t(m) - \bar m^e$, then $v_t = -v + \tanh\{\beta J^{\text{neum}} * T_t(m)\} - \tanh\{\beta J^{\text{neum}} * \bar m^e\}$. Since $|\tanh\{\beta A\} - \tanh\{\beta B\}| \le \beta|A-B|$ and since $J^{\text{neum}}(r,r')$ is a symmetric transition probability kernel,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \|v\|_2^2 \le \beta(|v|, J^{\text{neum}} * |v|)| \le \beta\|v\|_2^2,\\
&\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|v\|^2 \ge -\beta(|v|, J^{\text{neum}} * |v|)| \ge -\beta\|v\|_2^2.
\end{aligned}$$

Hence (6.16) which, by integration, yields (6.15). □

Proof of Theorem 6.1. To simplify notation we omit in this proof the superscript "e" to denote extensions to $Q_{\infty,L}$ and start by establishing some preliminary results. By Lemma 6.5 at any time t and for any ξ, $\|T_t(m) - \bar m_\xi\|_2^2 < \infty$, as this holds at time 0 by assumption. Moreover by (6.16) for any ξ, $\|T_t(m) - \bar m_\xi\|_2^2$ is a continuous function of t and for any t, $\|T_t(m) - \bar m_\xi\|_2^2$ is a continuous function of ξ which diverges as $|\xi| \to \infty$, (recall the properties of $\bar m$ in Subsect. 6.2). It then follows that $\inf_\xi \|T_t(m) - \bar m_\xi\|_2^2$ is a min and it is a continuous function of t.

We can now start the proof of Theorem 6.1 and consider first $\inf\limits_{\xi\in\mathbb{R}} \|m - \bar m_\xi\|_2^2 > L^{-24\beta-8}$. There are then two possible alternatives: (a) at all times $\inf_\xi \|T_t(m) - \bar m_\xi\|_2^2 > L^{-24\beta-8}$; (b) there is a time $t < \infty$ when $\inf_\xi \|T_t(m) - \bar m_\xi\|_2^2 \le L^{-24\beta-8}$. Case (b). Since $\inf_\xi \|T_t(m) - \bar m_\xi\|_2^2$ is a continuous function of t, there is a time t_0 when $\inf_\xi \|T_{t_0}(m) - \bar m_\xi\|_2^2 = L^{-24\beta-8}$ and since $F_{Q_{\infty,L}}(T_{t_0}(m)) \le F_{Q_{\infty,L}}(m)$, this case is actually contained in the case when $\inf\limits_{\xi\in\mathbb{R}} \|m - \bar m_\xi\|_2^2 \le L^{-24\beta-8}$, which is examined next (postponing the analysis of case (a)). We thus suppose

$$\inf_{\xi\in\mathbb{R}} \|m - \bar m_\xi\|_2^2 =: \|m - \bar m_{\hat\xi}\|_2^2 \le L^{-24\beta-8} \tag{6.17}$$

and set $e^{-\tau} := L^{-6}$ and $m^* := T_\tau(m)$. By Lemma 6.5, $\|m^* - \bar m_{\hat\xi}\|_2 \le L^{6(\beta-1)-12\beta-4} < \epsilon$, ϵ as in Lemma 6.1, for L large enough. Then there exists $\xi_{m*} =: \xi^*$ and (6.5)–(6.6) hold. By definition we have

$$\begin{aligned}
&F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_{\xi^*})\\
&= -\frac{1}{\beta}\int_{Q_{\infty,L}} \mathcal{S}(m^*) - \mathcal{S}(\bar m_{\xi^*})\\
&\quad -\frac{1}{2}\int_{Q_{\infty,L}\times Q_{\infty,L}} J^{\text{neum}}(r,r')\left\{m^*(r)m^*(r') - \bar m_{\xi^*}(r)\bar m_{\xi^*}(r')\right\}.
\end{aligned}$$

Calling $v = m^* - \bar m_{\xi^*}$ and $\alpha = \max(\|m^*\|_\infty, \|\bar m_{\xi^*}\|_\infty)$,

$$-\big(\mathcal{S}(m^*) - \mathcal{S}(\bar m_{\xi^*})\big) \ge -\mathcal{S}'(\bar m_{\xi^*})v + \frac{1}{2(1-\bar m_{\xi^*}^2)}v^2 - \frac{\alpha}{3(1-\alpha^2)^2}|v|^3.$$

By (6.14) if L is large enough, $\alpha \le (1+m_\beta)/2 < 1$. Calling

$$\mathcal{L}_\xi = p_\xi^{-1}\Omega_\xi, \quad \mathcal{L}_\xi v = J^{\text{neum}} * v - p_\xi^{-1}v, \quad p_\xi = \beta(1-\bar m_\xi^2),$$

where Ω_{ξ^*} is defined in (6.7). We denote by (v, w) the scalar product on $L^2(Q_{\infty,L})$ and regard $\mathcal{L}_\xi$ as an operator on $L^2(Q_{\infty,L})$. We have

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_{\xi^*}) \ge -\frac{1}{2}(v, \mathcal{L}_{\xi^*}v) - \frac{\alpha\|v\|_\infty}{3\beta(1-\alpha^2)^2}(v, v).$$

Since $(v, \mathcal{L}_{\xi^*}v) = \langle v, \Omega_{\xi^*}v\rangle_{\xi^*}$, recalling that $\langle v, \bar m'_{\xi^*}\rangle_{\xi^*} = 0$, by the L^2 spectral gap theorem, (6.8), and using that $p_{\xi^*} \ge \beta[1-m_\beta^2]$ we get

$$\begin{aligned} F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_{\xi^*}) &\ge \frac{\kappa}{2L^2}\ \langle v, v\rangle_{\xi^*} - (v, v)\frac{\alpha\|v\|_\infty}{3\beta(1-\alpha^2)^2} \\ &\ge (v, v)\left(\frac{\kappa}{2L^2\beta(1-m_\beta^2)} - \frac{\alpha\|v\|_\infty}{3\beta(1-\alpha^2)^2}\right). \end{aligned}$$

By Lemma 6.4 with $t = \tau$ and $\xi = \xi^*$ after recalling that $v = T_\tau(m) - \bar m_{\xi^*}$,

$$\|v\|_\infty \le 2L^{-6} + c\big(\|m^* - \bar m_{\xi^*}\|_2 + L^{-6}\|m - \bar m_{\xi^*}\|_2\big)^{2/3}.$$

Let $\hat\xi \in \mathbb{R}$ be as in (6.17), then by (6.6) and Lemma 6.5,

$$\begin{aligned} \|m^* - \bar m_{\xi^*}\|_2 &\le \frac{1}{\sqrt{1-m_\beta^2}}\|m^* - \bar m_{\hat\xi}\|_2 \le \frac{1}{\sqrt{1-m_\beta^2}}e^{(\beta-1)\tau}\|m - \bar m_{\hat\xi}\|_2 \\ &\le \frac{1}{\sqrt{1-m_\beta^2}}L^{6(\beta-1)-12\beta-4}. \end{aligned}$$

By Lemma 6.5, $e^{-(\beta+1)\tau}\|m - \bar m_{\xi^*}\|_2 \le \|m^* - \bar m_{\xi^*}\|_2$ so that, for L large enough,

$$\|v\|_\infty \le 2L^{-6} + c\left(L^{6(\beta-1)-12\beta-4} + L^{-6+6(\beta+1)+6(\beta-1)-12\beta-4}\right)^{2/3} \le 3L^{-6},$$

and

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_{\xi^*}) \ge \frac{\kappa}{4L^2\beta(1-m_\beta^2)}\ (v, v).$$

By (6.15), $\|v\|_2^2 \ge \|m - \bar m_{\xi^*}\|_2^2 e^{-2(\beta+1)\tau} \ge \inf_\xi \|m - \bar m_\xi\|_2^2 e^{-2(\beta+1)\tau}$. Recalling that $e^{-\tau} = L^{-6}$,

$$F_{Q_{\infty,L}}(m) - F_{Q_{\infty,L}}(\bar m_{\xi^*}) \ge \frac{\kappa}{4\beta(1-m_\beta^2)}\ L^{-[2+12(\beta+1)]}\ \inf_\xi \|m - \bar m_\xi\|_2^2.$$

Case (a), namely when at all times t, $\inf_\xi \|T_t(m) - \bar m_\xi\|_2^2 > L^{-24\beta-8}$. By Theorem 6.2 for any $\epsilon > 0$ there are t and ξ so that $\|T_t(m) - \bar m_\xi\|_\infty < \epsilon$. Call $m^* = T_t(m)$ and write

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_\xi) \ge \int\limits_{|x-\xi|>L} \phi_\beta(m^*)\,dr - c\left\{\epsilon L^2 + Le^{-\alpha L}\right\}$$

having used (6.3) to bound the contribution of $\{|x-\xi| > L\}$ to $F_{Q_{\infty,L}}(\bar m_\xi)$. As already remarked there is $c' > 0$ so that

$$\phi_\beta(m^*) = \phi_\beta(m^*) - \phi_\beta(m_\beta) \ge c'|m^* - m_\beta|^2, \quad m^* > 0,$$

hence

$$\int\limits_{|x-\xi|>L} \phi_\beta(m^*)\,dr \ge \frac{c'}{2} \int\limits_{|x-\xi|>L} |m^* - \bar m_\xi|^2 - c'' Le^{-\alpha L}.$$

Thus there is a new constant $c > 0$ so that

$$F_{Q_{\infty,L}}(m^*) - F_{Q_{\infty,L}}(\bar m_\xi) \ge \|m^* - \bar m_\xi\|_2^2 - c\left\{\epsilon L^2 + Le^{-\alpha L}\right\},$$

and since $\|m^* - \bar m_\xi\|_2^2 \ge L^{-24\beta-8}$ we obtain (6.1) for ϵ small enough. $\square$

7. Proof of Theorem 3.1

In this section we will prove that orbits whose penalty is close to optimal approximately have an instanton shape at times when the total magnetization is small. Given $\theta_0 \in (0, \theta_{\text{crit}})$, see (3.4), we fix once and for all θ_1 and θ_2 so that

$$\frac{1}{2} > \theta_2 > \theta_1 > \theta_0 \tag{7.1}$$

(the values of L for which our analysis applies depend on the actual choice of such parameters). Let $W_{\alpha,L}$ be as in (5.1) and for any $\delta > 0$ set

$$\mathcal{N}_{\delta,L} := \bigcup_{|\vartheta_\alpha|\le\theta_0} \left\{ m \in \Sigma_\alpha : \fint_{Q_L} |m - W_{\alpha,L}| < \delta \right\}, \tag{7.2}$$

and in the sequel we will restrict to functions m which satisfy

$$m \in \mathcal{N}_{\delta,L}, \quad F_L(m) < c_\beta L + \epsilon L, \qquad c_\beta = F^{(1)}(\bar m). \tag{7.3}$$

We will see that as $\epsilon > 0$ gets small (7.3) forces m progressively closer to $\bar m_\xi^e$, for a suitable value of ξ. Before entering into the whole issue we remark:

Lemma 7.1. *For every $a > 0$ there is L_a so that for all $L \ge L_a$ the following holds. Let (u_n, T_n) be an optimizing orbit, namely such that $\liminf_{n\to\infty} I_{L,T_n}(u_n) \le F_L^{(1)}(\hat m_L)L$. Then for all n large enough and all $t \in [0, T_n]$,*

$$F_L(u_n(\cdot,t)) < c_\beta L + \left(\frac{1}{L^a}\right) L, \quad L \ge L_a. \tag{7.4}$$

Proof. For any $\delta > 0$ if n is large enough, $F_L(u_n(\cdot,t)) < F_L^{(1)}(\hat{m}_L)L + \delta$. By (6.4)

$$F_L(u_n(\cdot,t)) < L(c_\beta + ce^{-\omega L}) + \delta.$$

Choose $\delta = (2L^a)^{-1}$ and L_a so that $ce^{-\omega L_a} < (2L^a)^{-1}$ and (7.4) follows. $\square$

Thus we can take in (7.3) $\epsilon = L^{-a}$ with a as large as desired, provided $L \geq L_a$ and that we restrict to optimizing sequences. Our first result is a corollary of the convergence theorem to the Wulff shape of Subsect. 2.5.

Proposition 7.1. *For any $\delta > 0$ there exist $\epsilon > 0$ and $\bar{L}$ such that if $L \geq \bar{L}$, $m \in \Sigma_\alpha$ with $|\vartheta_\alpha| \leq \theta_0$ and $F_L(m) < c_\beta L + \epsilon L$, then $m \in \mathcal{N}_{\delta,L}$ (modulo a rotation of an integer multiple of $\pi/2$).*

Proof. In the course of the proof we use the following notation: given a set $A \subset Q_1$ we call f_A the function equal to m_β in A and to $-m_\beta$ in $Q_1 \setminus A$ and $f_{A,L}$ its image as a function on Q_L, i.e. $f_{A,L}(Lr) = f_A(r)$. If $L = 1$ we simply write f_A. Let $E_{\vartheta_\alpha} \subseteq Q_1$ be a solution of (2.14) with $|E_{\vartheta_\alpha}| = 1/2 - \vartheta_\alpha$.

We argue by contradiction. Thus we suppose that there is $\delta > 0$ such that for any $\epsilon > 0$ and any $\bar{L}$ positive the following holds. There exist α such that $|\vartheta_\alpha| \leq \theta_0$, $L > \bar{L}$ and $m \in \Sigma_\alpha$ such that $F_L(m) < c_\beta L + \epsilon L$ and

$$\min_{E_{\vartheta_\alpha}} \fint_{Q_L} |m - f_{E_{\vartheta_\alpha},L}\, dr| \geq \delta.$$

We can then find an increasing sequence $\{L_h\}$ converging to $+\infty$ as $h \to +\infty$, α_h such that $|\vartheta_{\alpha_h}| \leq \theta_0$, and functions $m_h \in \Sigma_{\alpha_h}$ satisfying

$$\frac{F_{L_h}(m_h)}{L_h} < c_\beta + \frac{1}{h}, \qquad \min_{E_{\vartheta_{\alpha_h}}} \fint_{Q_{L_h}} |m_h - f_{E_{\vartheta_\alpha},L_h}\, dr| \geq \delta. \tag{7.5}$$

Rescale the functions m_h by defining $v_h(r) := m_h(L_h r)$, $r \in Q_1$. Then there is a (not relabelled) subsequence so that $\alpha_h \to \alpha$ as $h \to +\infty$ with $|\vartheta_\alpha| \leq \theta_0$ while $\{v_h\}$ converges in $L^1(Q_1)$ to a function f_A, i.e. equal to m_β in A and to $-m_\beta$ in $Q_1 \setminus A$, $A \in BV$, and $\fint f_A = \alpha$, [2]. Using the Γ-convergence of the rescaled sequence of functionals,

$$c_\beta \geq \liminf \frac{F_{L_h}(m_h)}{L_h} \geq c_\beta P(A, \mathrm{int}(Q_1)). \tag{7.6}$$

Since $|A| \in [\frac{1}{2} - \theta_0, \frac{1}{2} + \theta_0]$, $P(A, \mathrm{int}(Q_1)) \geq 1$ (1 being the minimal perimeter when the area is in $[\frac{1}{2} - \theta_0, \frac{1}{2} + \theta_0]$) hence from (7.6) $P(A, \mathrm{int}(Q_1)) = 1$ and A is a minimizer of the perimeter. By rescaling the second equation in (7.5),

$$\min_{E_{\vartheta_{\alpha_h}}} \fint_{Q_1} |v_h - f_{E_{\vartheta_{\alpha_h}}}| \geq \delta.$$

As $h \to \infty$ (along the converging subsequence)

$$\min_{E_{\vartheta_\alpha}} \fint_{Q_1} |f_A - f_{E_{\vartheta_\alpha}}| \geq \delta, \tag{7.7}$$

which gives the desired contradiction because the left-hand side on (7.7) vanishes. $\square$

All functions $W_{\alpha,L}$ in $\mathcal{N}_{\delta,L}$ have value $+m_\beta$ in $\{(x,y) : x \geq \theta_0 L\}$ and value $-m_\beta$ in $\{(x,y) : x \leq -\theta_0 L\}$, a property which evidently fails for the generic element of $\mathcal{N}_{\delta,L}$. A weaker property however holds, namely there are two vertical strips (a precise definition is given below), one in $A_+ = \{(x,y) : x \in [\theta_0 L, \theta_1 L]\}$ (see (7.1)) and the other one in $A_- = -A_+$, where on a large fraction of points $\Theta^{(\zeta,\ell_-,\ell_+)} = 1$, respectively $\Theta^{(\zeta,\ell_-,\ell_+)} = -1$. Under the additional assumption that (7.3) holds with ϵ small enough (yet independent of L) there are "vertical connections" (see Definition 5.2) where identically $\Theta^{(\zeta,\ell_-,\ell_+)} = 1$ and $\Theta^{(\zeta,\ell_-,\ell_+)} = -1$. If we further strengthen the assumption by supposing $\epsilon = L^{-2}$ and L large, then $\eta^{(\zeta,\ell_-)} = 1$ for $x \geq \theta_2 L$ and $\eta^{(\zeta,\ell_-)} = -1$ for $x \leq -\theta_2 L$.

We define the [vertical] strips $S(n)$ by

$$S(n) := [n\ell_+, (n+1)\ell_+) \times [-L/2, L/2)$$

and call $Z_L^\pm \subset \mathbb{Z}$ the set of all $n \in \mathbb{Z}$ such that $S(n) \subset A_\pm$ and $Z_L = Z_L^+ \cup Z_L^-$.

Proposition 7.2. *There exists a constant $c = c(\zeta, \ell_-, \ell_+)$ such that for any $m \in \mathcal{N}_{\delta,L}$ there are $n_\pm \in Z_L^\pm$ such that $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot) \neq \pm 1$ in at most $N_\delta := \dfrac{c\delta}{\theta_1 - \theta_0} L$ squares of $\mathcal{D}^{(\ell_+)}$ inside $S(n_\pm)$.*

Proof. The value of $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot)$ on $S(n)$ is determined by the value of $\eta^{(\zeta,\ell_-)}$ on a strip which is three times larger than $S(n)$. With reference to Fig. 2 in fact if the middle square is in $S(n)$ then all 9 squares are needed to determine the value of $\Theta^{(\zeta,\ell_-,\ell_+)}$ in the middle one. Set then $S^{(3)}(n) := [(n-1)\ell_+, (n+2)\ell_+) \times [-L/2, L/2)$. By definition of $\mathcal{N}_{\delta,L}$,

$$\sum_{n \in Z_L^-} \int_{S^{(3)}(n)} |m + m_\beta| + \sum_{n \in Z_L^+} \int_{S^{(3)}(n)} |m - m_\beta| \leq 3 \int_{Q_L} |m - W_{\alpha,L}| < 3\delta L^2.$$

Since the cardinality of Z_L^- is $\geq L((\theta_1 - \theta_0)/2$ (for L large), there are two strips $S^{(3)}(n_\pm)$ such that

$$\int_{S^{(3)}(n_\pm)} |m \mp m_\beta| \leq 3\delta L^2 \left(\frac{2\ell_+}{L(\theta_1 - \theta_0)} \right) = \frac{6\ell_+\delta}{\theta_1 - \theta_0} L$$

which implies that $\eta^{(\zeta,\ell^-)}(m,\cdot) \neq \pm 1$ on at most

$$N_\delta := \left(\frac{6\ell_+\delta L}{\theta_1 - \theta_0} \right) \frac{1}{\zeta(\ell_-)^2}$$

squares of $\mathcal{D}^{(\ell_-)}$ inside $S^{(3)}(n_\pm)$. Thus there are at most $9N_\delta$ squares in $S(n_+)$ (resp. in $S(n_-)$) where $\Theta^{(\zeta,\ell_-,\ell_+)} \neq 1$ (resp. $\Theta^{(\zeta,\ell_-,\ell_+)} \neq -1$). $\square$

Proposition 7.3. *There are δ, L^* and ϵ^* all positive so that if m satisfies* (7.3) *with $L \geq L^*$ and $\epsilon \in (0, \epsilon^*)$, then there are two vertical connections $B_\mp$, one in $\mathcal{B}_- = \{(x,y) : x \in [-L\theta_2, -L\theta_0]\}$ and the other one in $\mathcal{B}_+ = \{(x,y) : x \in [L\theta_0, L\theta_2]\}$, where $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot)$ is identically equal to -1 and respectively to* $+1$.

Proof. The proof by contradiction is based on successive modifications of m into new functions so that if the vertical connections were absent then the final function would have both energy smaller than the initial one and larger than $c_\beta L + \epsilon L$, which is the desired contradiction. We next outline the main steps postponing their proofs. By symmetry we may restrict to the case where the vertical connection is absent in $\mathcal{B}_-$ and it may or may not be absent in $\mathcal{B}_+$.

1. The absence of a vertical connection in $\mathcal{B}_-$ implies that the set
$$\left\{r \in Q_L : \Theta^{(\zeta,\ell_-,\ell_+)}(m; r) > -1\right\}$$
connects $S(n_-)$ to $\{(x, y) \in Q_L : x = -\theta_2 L\}$. From this it will follow that the number K_0 of $\mathcal{D}^{(\ell_+)}$-squares strictly to the left of $S^{(3)}(n_-)$, where $\Theta^{(\zeta,\ell_-,\ell_+)}(m; \cdot) = 0$ is $K_0 \geq c_0(\theta_2 - \theta_1)L$, c_0 a positive constant.
2. It is possible to modify m only in $S^{(3)}(n_-)$ in such a way that the new function $\tilde{m}$ verifies $\Theta^{(\zeta,\ell_-,\ell_+)}(\tilde{m}; r) = -1$, $r \in S(n_-)$ and $F_L(\tilde{m}) \leq F_L(m) + c'\ell_+^2\delta L$, c' a positive constant.
3. By Theorem 5.6 applied to $\tilde{m}$ with Λ the region strictly to the left of $S(n_-)$ there exists $m^* = \tilde{m}$ on Λ^c such that $\eta^{(\zeta,\ell_-)}(m^*; \cdot) = -1$ on Λ and $F_L(m^*) \leq F_L(\tilde{m}) - c_1\zeta^2\ell_-^2 K_0$.
4. By Theorem 5.5 we can further modify m^* into a new function ψ_- equal to m^* outside Λ in such a way that $\eta^{(\zeta,\ell_-)}(\psi_-; r) = -1, r \in \Lambda, \psi_-(x, y) = -m_\beta, x < -L\theta_2 - 1$ and $F_L(\psi_-) \leq F_L(m^*) + c''e^{-\omega(1/2-\theta_2)L}$, c'' a positive constant.

Conclusion of proof. Call ψ the function where the analogous modifications are made to the right of the origin, namely by repeating Steps 2-4 above (notice that a vertical connection in $\mathcal{B}_+$ may very well exist, in which case we do not have the lower bound for the corresponding K_0 as in Item 1). The "previous errors" occur therefore twice while the gain term occurs only once, in the worst case, then

$$F_L(m) \geq F_L(\psi) - 2c'\ell_+^2\delta L\ell_- - 2c''e^{-\omega(1/2-\theta_2)L} + c_1\zeta^2\ell_-^2\left\{c_0(\theta_2 - \theta_1)L\right\}. \tag{7.8}$$

Since $\psi(x, y) = \pm m_\beta$ in $x \geq L/2 - 1$ and respectively $x \leq -L/2 + 1$, $F_L(\psi) = F_{Q_{\infty,L}}(\tilde{\psi})$, where $\tilde{\psi}(x, y) = -m_\beta$ for $x \leq -L/2$ and $= m_\beta$ for $x \geq L/2$. Then by (6.12),

$$F_L(m) \geq c_\beta L - 2c'\ell_+^2\delta L - 2c''e^{-\omega(1/2-\theta_2)L} + c_1\zeta^2\ell_-^2\left\{c_0(\theta_2 - \theta_1)L\right\}, \tag{7.9}$$

which for δ small enough yields for all $L \geq L^*$, L^* large enough,

$$F_L(m) \geq c_\beta L + \frac{c_1}{2}\,\zeta^2\ell_-^2\left\{c_0(\theta_2 - \theta_1)L\right\}. \tag{7.10}$$

Choosing $\epsilon^* = \dfrac{c_1}{2}\,\zeta^2\ell_-^2\{c_0(\theta_2 - \theta_1)\}$,

$$F_L(m) \geq c_\beta L + \epsilon^* L \tag{7.11}$$

which contradicts $F_L(m) < c_\beta L + \epsilon L$ because $\epsilon < \epsilon^*$. □

Remarks. The above argument is strictly two dimensional. Indeed the lower bound on K_0 grows like L in all dimensions (the "thin fingers effect"), while the error in Item 2 grows as $c\delta L^{d-1}$ which in $d > 2$ wins against the "gain term". A different argument developed by Bodineau and Ioffe, [5], applies in $d > 2$ and since the theory of Wulff shape can be partially extended to $d = 3$ the result extends to $d = 3$ as sketched in Sect. 11.

While Items 3 and 4 are self explanatory, Items 1 and 2 do need a proof.:

Proof of Item 1. We call $\pm$ or 0 a $\mathcal{D}^{(\ell_+)}$-square where $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot)$ is ± 1 or 0, respectively and, given $C \in \mathcal{D}^{(\ell_+)}$, we set

$$\mathcal{S}_{\text{left}}(C) := \left\{\widehat{C} \in \mathcal{D}^{(\ell_+)} \cap Q_L : \text{ for any } (x,y) \in \widehat{C} \text{ there is } x' > x \text{ with } (x',y) \in C\right\},$$

$$\mathcal{S}_{\text{vert}}(C) := \left\{\widehat{C} \in \mathcal{D}^{(\ell_+)} \cap Q_L : \text{ for any } (x,y) \in \widehat{C} \text{ there is } y' \in \left(-\frac{L}{2}, \frac{L}{2}\right) \right.$$
$$\left. \text{with } (x,y') \in C\right\}.$$

Denote by K the number of 0-squares to the left of $S(n_-)$ (included). Item 1 then follows from the two alternatives below:
Case (i). Assume that there exists a $-$ square $C_0 \in \mathcal{D}^{(\ell_+)} \cap S(n_-)$ such that the strip $\mathcal{S}_{\text{left}}(C_0) \cap \{(x,y) \in Q_L : -\theta_2 L \le x \le n_-\ell_+\}$ contains only $-$ squares. For each C' in the strip we have that $\mathcal{S}_{\text{vert}}(C')$ contains at least one $-$ square, because $C' \subset \mathcal{S}_{\text{vert}}(C')$. On the other hand $\mathcal{S}_{\text{vert}}(C')$ cannot consist entirely of $-$ squares by our assumption that there is no vertical connection. Since the sets $\Theta^{(\zeta,\ell_-,\ell_+)} = 1$ and $\Theta^{(\zeta,\ell_-,\ell_+)} = -1$ are not connected, there must be at least one 0-square in $\mathcal{S}_{\text{vert}}(C')$. Thus $K \ge \dfrac{(\theta_2-\theta_1)L}{2\ell_+}$.
Case (ii). Any $-$ square $C_0 \in \mathcal{D}^{(\ell_+)} \cap S(n_-)$ is such that $\mathcal{S}_{\text{left}}(C_0)$ contains at least a 0-square. In this case $K \ge \dfrac{L}{\ell_+} - N_\delta$ by definition of $S(n_-)$. $\square$

Proof of Item 2. Call $\tilde{m}$ the function obtained from m by putting $-m_\beta$ on all squares connected to those in $S^{(3)}(n_-)$, where $\eta^{(\zeta,\ell_-)}(m,\cdot)$ is not identically -1. Then

$$F_L(m) \ge F_L(\tilde{m}) - c_J \ell_+^2 N_\delta,$$

where $c_J > 0$ is a constant depending only on J hence by Proposition 7.2 Item 2 is proved. $\square$

As already remarked Items 3 and 4 are self explanatory and the proposition is therefore proved. $\square$

Corollary 7.4. *In the same context as in Proposition 7.3 assume in addition that* (7.3) *is verified with* $\epsilon = L^{-2}$. *Then* $\eta^{(\zeta,\ell_-)}(m,(x,y)) = \pm 1$ *for all* $x \ge \theta_2 L - 1$ *and respectively* $x \le -\theta_2 L + 1$.

Proof. By Proposition 7.3 there are two vertical connections $B_\mp$ respectively to the right of $x = -\theta_2 L$ and to the left of $x = \theta_2 L$, where $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot) = \mp 1$. Arguing again by contradiction and referring for definiteness to what happens to the left of B_-, if $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot) \neq -1$ somewhere on the left of B_-, necessarily $\Theta^{(\zeta,\ell_-,\ell_+)}(m,\cdot) = 0$

somewhere to the left of B_-. Then by Theorem 5.6 there is ψ equal to m to the right of B_- (included) with $\Theta^{(\zeta,\ell_-,\ell_+)}(\psi,\cdot) = -1$ on the left of B_- and such that

$$F_L(m) \ge F_L(\psi) + c_1\zeta^2\ell_-^2.$$

The same argument used in the proof of Proposition 7.3 shows then that

$$F_L(m) \ge c_\beta L - c'' e^{-\omega(1/2-\theta_2)L} + c_1\zeta^2\ell_-^2$$

which leads to a contradiction because $L\epsilon = L^{-1} < c_1\zeta^2(\ell_-)^2 - c'' e^{-\omega(1/2-\theta_2)L}$ for L large enough. □

Lemma 7.1 and Theorem 7.5 below conclude the proof of Theorem 3.1.

Theorem 7.5. *Assume $m \in \Sigma_\alpha$, $|\vartheta_\alpha| \le \theta_0$ and that*

$$F_L(m) \le LF^{(1)}(\bar m) + \epsilon, \qquad \epsilon < L^{-600-[2+24(\beta-1)]}. \tag{7.12}$$

Then there exists ξ with $|\xi| \le \theta_0 L + 1$ such that

$$\|\bar m^e_\xi - m\|_2^2 \le L^{-100}.$$

Proof. To simplify notation we omit also in this proof the superscript "e" to denote extension to $Q_{\infty,L}$. We distinguish two cases, Case 1 is when (7.13) below is satisfied and Case 2 when it is not (we will see that the second case contradicts the assumptions of the theorem and thus it will not occur). Let θ_2 be as in Corollary 7.4.
Case 1. There exists $|\xi| \le \theta_1 L$ such that,

$$\|\bar m_\xi - m\|^2_{L^2(Q_{\theta_2 L,L})} \le L^{-300}. \tag{7.13}$$

We split the free energy as

$$F_L(m) = F_{Q_{\theta_2 L-1,L}}\left(m_{Q_{\theta_2 L-1,L}} \,|\, m_{Q^c_{\theta_2 L-1,L}}\right) + F_{Q^c_{\theta_2 L-1,L}}\left(m_{Q^c_{\theta_2 L-1,L}}\right), \tag{7.14}$$

where for $f, g \in L^\infty(Q_L, (-1,1))$ and $A \subseteq Q_L$,

$$F_A(f) := \int_A \phi_\beta(f)\,dr + \frac{1}{4}\int_{A\times A} J^{\text{neum}}(r,r')[f(r)-f(r')]^2 dr\,dr',$$

$$F_A(f|g) := F_A(f) + \frac{1}{2}\int_{A\times(Q_L\setminus A)} J^{\text{neum}}(r,r')[f(r)-g(r')]^2 dr\,dr'.$$

Since $|\phi_\beta(m) - \phi_\beta(\bar m)| \le c|m-\bar m|$, there is a new constant c so that

$$\begin{aligned}&\left|F_{Q_{\theta_2 L-1,L}}\left(m_{Q_{\theta_2 L-1,L}} \,|\, m_{Q^c_{\theta_2 L-1,L}}\right) - F_{Q_{\theta_2 L-1,L}}\left(\bar m_\xi \mathbf{1}_{Q_{\theta_2 L-1,L}} \,|\, \bar m_\xi \mathbf{1}_{Q^c_{\theta_2 L-1,L}}\right)\right| \\ &\quad \le cL\|\bar m_\xi - m\|_{L^2(Q_{\theta_2 L,L})}.\end{aligned}$$

By (7.13) and the exponential convergence of $\bar m_\xi$ to its asymptotes, for L large enough,

$$F_{Q_{\theta_2 L-1,L}}\left(m_{Q_{\theta_2 L-1,L}} \,|\, m_{Q^c_{\theta_2 L-1,L}}\right) \ge LF^{(1)}(\bar m) - L^{-148}.$$

By (7.14) and (7.12)

$$LF^{(1)}(\bar{m}) + \epsilon - LF^{(1)}(\bar{m}) + L^{-148} \geq F_{Q^c_{\theta_2 L-1,L}}\left(m_{Q^c_{\theta_2 L-1,L}}\right). \tag{7.15}$$

Using again the exponential convergence of $\bar{m}_\xi$, there are positive constants $\bar{c}$ and $\bar{c}'$ so that

$$\|m - \bar{m}_\xi\|_{L^2(Q^c_{\theta_2 L-1,L})} \leq \|m - \text{sign}(x)m_\beta\|_{L^2(Q^c_{\theta_2 L-1,L})} + \bar{c}e^{-\bar{c}'L}. \tag{7.16}$$

We postpone the proof that there is a constant $c > 0$ so that

$$F_{Q^c_{\theta_2 L-1,L}}\left(m_{Q^c_{\theta_2 L-1,L}}\right) \geq c \int_{|x|\geq L\theta_2 - 1} |m - \text{sign}(x)m_\beta|^2. \tag{7.17}$$

By (7.13) and(7.16)-(7.17)-(7.15), $\|\bar{m}_\xi - m\|_2^2 \leq L^{-300} + L^{-147}$, for L large enough. Since by assumption $m \in \Sigma_\alpha$ with $|\vartheta_\alpha| \leq \theta_0$, then, for L large enough, $|\xi| \leq \theta_0 + 1$ so that the analysis of Case 1 will be complete once we prove (7.17), which we do next.

By Corollary 7.4, for L sufficiently large, $\eta^{(\zeta,\ell_-)}(m,r) = \mp 1$ when $x < -\theta_2 L + 1$ and $x > \theta_2 L - 1$. Using this we are going to prove that for any $r \in Q^c_{\theta_2 L-1,L}$,

$$\phi_\beta(m(r)) + \frac{1}{2}\int_{Q^c_{\theta_2 L-1,L}} J^{\text{neum}}(r,r')[m(r) - m(r')]^2\, \mathrm{d}r' \geq c(m(r) - m_\beta)^2 \tag{7.18}$$

which yields (7.17). We consider only $x > 0$ as the case $x < 0$ is proved in exactly the same way. To prove (7.18), we first observe that $\phi_\beta(\pm m_\beta) = 0$, $\phi_\beta(m) > 0$ for $m \notin \{\pm m_\beta\}$ and $\phi_\beta(m)$ is strictly convex in $\pm m_\beta$. Therefore there exists a constant $c > 0$ such that

$$\phi_\beta(m) \geq c \min\{|m - m_\beta|^2,\ |m + m_\beta|^2\}.$$

Thus if $m(r) > 0$ the first term on the l.h.s. of (7.18) already yields the bound.

If $m(r) \leq 0$ we call $J^{(\ell_-)}(r,r') = \fint_{C^{(\ell_-)}_{r'}} J^{\text{neum}}(r,r'')dr''$, $r' \in Q^c_{\theta_2 L-1,L}$, and, by the Lipschitz continuity of J,

$$\begin{aligned}\int_{Q^c_{\theta_2 L-1,L}} & J^{\text{neum}}(r,r')\left(m(r) - m(r')\right)^2 \\ & \geq \int_{Q^c_{\theta_2 L-1,L}} J^{(\ell_-)}(r,r')\left(m(r) - m(r')\right)^2 - c\ell_-. \end{aligned} \tag{7.19}$$

By Cauchy-Schwartz,

$$\fint_{C^{(\ell_-)}_{r'}} \left(m(r) - m(r'')\right)^2 \geq \left(m(r) - \fint_{C^{(\ell_-)}_{r'}} m(r'')\right)^2 \geq \left(m_\beta - \zeta\right)^2,$$

which inserted in (7.19) gives

$$\int_{Q^c_{\theta_2 L-1,L}} J^{\mathrm{neum}}(r,r')\Big(m(r)-m(r')\Big)^2 \ge \frac{1}{2}(m_\beta-\zeta)^2 - c\ell_-$$

because $\int_{Q^c_{\theta_2 L-1,L}} J^{\mathrm{neum}}(r,r') > \frac{1}{2}$. Supposing ζ small enough, recall $\ell_- \le \zeta^2/2$, the r.h.s. $\ge m_\beta^2/4$ and (7.18) follows.

Case 2. The complementary case is when (7.13) does not hold, we will prove that such a case cannot actually happen. Indeed by Corollary 7.4 and Theorem 5.5 there is a function ψ equal to m on $|x| \le \theta_2 L$, such that $\psi = \pm m_\beta$ on $x > L/2-1$ and $x < -L/2+1$ and

$$F_L(m) \ge F_L(\psi) - ce^{-\omega(1/2-\theta_2)L}. \tag{7.20}$$

Calling ϕ the function on $Q_{\infty,L}$ equal to ψ on Q_L and to $\pm m_\beta$ on $x < -L/2$ and $x > L/2$, by Theorem 6.1 we have

$$\begin{aligned} F_L(\psi) = F_{Q_{\infty,L}}(\phi) &\ge c_\beta L + \inf_\xi cL^{-[2+24(\beta-1)]} \int_{|x|<\theta_2 L} |m-\bar m_\xi|^2 \\ &\ge c_\beta L + cL^{-[2+24(\beta-1)]-300}. \end{aligned} \tag{7.21}$$

Equations (7.20)–(7.21) contradict (7.12) for L large. □

8. Proof of Theorems 3.2 and 3.3

The proof follows the one dimensional analysis in [6], see also [7, 8, 14], and uses some spectral properties proved in an appendix, Sects. 9 and 10.

Recalling that $g_L(m) := -m + \tanh\{\beta J^{\mathrm{neum}} * m\}$, the first order term in f in the expansion of $g_L(\bar m_{\xi,L}+f)$, $|\xi| < L/2$, gives

$$\Omega_{\xi,L} f = -f + p_{\xi,L}\, J^{\mathrm{neum}} * f, \qquad p_{\xi,L} = \beta\, \cosh^{-2}\{\beta J^{\mathrm{neum}} \star \bar m^e_{\xi,L}\} \tag{8.1}$$

(the zero order term is however not missing because $\bar m_{\xi,L}$ is not a critical point unless $L=\infty$). We will regard here $\Omega_{\xi,L}$ as an operator on $L^\infty(Q_L)$, fix in the sequel $r \in (0,1)$ and restrict to ξ such that $|\xi| \le rL/2$. $\Omega_{\xi,L}$ is a shorthand for $\Omega_{\bar m_{\xi,L}}$ which is among the operators Ω_m considered in Sect. 9. Due to the planar symmetry some of its spectral properties just follow from the $d=1$ analysis and are valid for all L large enough, see Subsect. 9.1, here we just mention that the maximal eigenvalue is $\lambda_{\xi,L}$ with eigenvector a strictly positive, planar function $e_{\xi,L}(\cdot)$. In the notation of Subsect. 9.1 eigenvalue and eigenfunction are denoted by λ_m and e_m respectively, where $m = \bar m_{\xi,L}$.

8.1. Fibers. Following [6], we introduce fibers in the space $L^\infty(Q_L; [-1,1])$, defined as

$$\mathcal{B}_{\xi,L} := \left\{ m \in L^\infty(Q_L; [-1,1]) : m = \bar{m}_{\xi,L} + \phi, \pi_{\xi,L}(\phi) = 0 \right\}, \tag{8.2}$$

where

$$\pi_{\xi,L}(\phi) = \frac{\langle e_{\xi,L}\phi\rangle_{\xi,L}}{\langle e_{\xi,L}e_{\xi,L}\rangle_{\xi,L}}, \quad \langle f, g\rangle_{\xi,L} := \int_{Q_L} fg\; p_{\xi,L}^{-1} \tag{8.3}$$

and call

$$\begin{aligned} \mathcal{B}_{\epsilon,\xi,L} &:= \left\{ \bar{m}_{\xi,L} + \phi \in \mathcal{B}_{\xi,L} : \|\phi\|_\infty \le \epsilon \right\}, \\ \mathcal{B}'_{\epsilon,\xi,L} &:= \left\{ m \in \mathcal{B}_{\epsilon,\xi,L} : m(x,y) = m(x,0) \right\}. \end{aligned} \tag{8.4}$$

8.2. Spectral analysis. The crucial property of $\Omega_{\xi,L}$ is invertibility: the inverse $\Omega_{\xi,L}^{-1}$ of $\Omega_{\xi,L}$ exists and it is a bounded operator on the space $\{\phi : \pi_{\xi,L}(\phi) = 0\}$, namely on the points of the fiber $\mathcal{B}_{\xi,L}$ parameterized by $\phi = m - \bar{m}_{\xi,L}$. More precisely there is a constant $\kappa > 0$ so that

$$\sup_{\phi:\pi_{\xi,L}(\phi)=0, \|\phi\|_\infty=1} \|\Omega_{\xi,L}^{-1}\phi\|_\infty \le \kappa L^2. \tag{8.5}$$

In $d = 1$ the bound on the r.h.s. is independent of L, the extension to $d = 2$ is proved in the appendix as a direct consequence of Theorem 9.4, see (9.26).

Theorem 8.1. *For any $r < 1$ and all L large enough, the only solution of $g_L(m) = 0$ with $m \in \mathcal{B}_{L^{-3},\xi,L}$, $|\xi| \le rL/2$ is $\hat{m}_L^e$.*

Proof. The analogous property in $d = 1$ has been proved in a stronger form in [6], thus the theorem will follow once we show that any solution of $g_L(m) = 0$ in $\{\mathcal{B}_{L^{-3},\xi,L}, |\xi| \le rL/2\}$ is necessarily in $\{\mathcal{B}'_{L^{-3},\xi,L}, |\xi| \le rL/2\}$.

Following Sect. 4 of [6], we consider the auxiliary equation

$$g_L(m) - \pi_{\xi,L}\big(g_L(m)\big)e_{\xi,L} = 0, \quad m \in \mathcal{B}_{L^{-3},\xi,L}. \tag{8.6}$$

We will prove that any solution of (8.6) is in $\mathcal{B}'_{L^{-3},\xi,L}$. The theorem will then follow because if $g_L(m) = 0$ then m satisfies (8.6).

For ϕ as above we define

$$R_{\xi,L}(\phi) = g_L(\bar{m}_{\xi,L} + \phi) - g_L(\bar{m}_{\xi,L}) - \Omega_{\xi,L}\phi.$$

By a Taylor expansion to second order, there is c so that

$$\|R_{\xi,L}(\phi)\|_\infty \le c\|\phi^2\|_\infty, \quad \|R_{\xi,L}(\phi_1) - R_{\xi,L}(\phi_2)\|_\infty \le c\{\|\phi_1\|_\infty + \|\phi_2\|_\infty\}\|\phi_1 - \phi_2\|_\infty. \tag{8.7}$$

For L large enough, let $\mathcal{A}_{\xi,L}$ be the following operator on $\mathcal{B}_{\xi,L}$:

$$\begin{aligned} \mathcal{A}_{\xi,L}(\phi) := -\Omega_{\xi,L}^{-1} \Big\{ &\big[g_L(\bar{m}_{\xi,L}) - \pi_{\xi,L}\left(g_L(\bar{m}_{\xi,L})\right) e_{\xi,L}\big] \\ &+ \big[R_{\xi,L}(\phi) - \pi_{\xi,L}(R_{\xi,L}(\phi))e_{\xi,L}\big]\Big\}. \end{aligned}$$

If ϕ is a fixed point of $\mathcal{A}_{\xi,L}(\cdot)$ and $\|\phi\| \le L^{-3}$, then $\bar{m}_{\xi,L} + \phi$ solves (8.6).

In [13] it has been proved that there is C so that, for α as in (6.3),

$$\|g_L(\bar{m}_{\xi,L})\|_\infty \le Ce^{-\alpha(L-2|\xi|)}$$

which implies that for L large enough

$$\|g_L(\bar{m}_{\xi,L}) - \pi_{\xi,L}\big(g_L(\bar{m}_{\xi,L})\big)e_{\xi,L}\|_\infty \le Ce^{-\bar{c}L}.$$

From (8.5) and (8.7) it then follows that

$$\|\mathcal{A}_{\xi,L}(\phi)\|_\infty \le c_0 L^2\left[Ce^{-\bar{c}L} + L^{-6}\right].$$

Thus, for all L large enough $\mathcal{A}_{\xi,L}$ maps the set $\mathcal{B}_{L^{-3},\xi,L}$ into itself. Moreover $\mathcal{A}_{\xi,L}$ maps $\mathcal{B}'_{L^{-3},\xi,L}$ into itself. By (8.7) and (8.5) we have

$$\|\mathcal{A}_{\xi,L}(\phi_1) - \mathcal{A}_{\xi,L}(\phi_2)\|_\infty \le \delta\|\phi_1 - \phi_2\|_\infty, \qquad \delta < L_1^{-1},$$

so that $\mathcal{A}_{\xi,L}$ is a contraction on $\mathcal{B}_{L^{-3},\xi,L}$ and since $\mathcal{B}'_{L^{-3},\xi,L}$ is invariant, the unique fixed point ϕ_ξ is in $\mathcal{B}'_{L^{-3},\xi,L}$, namely it has planar symmetry. As already remarked, solutions of (8.6) are fixed points of $\mathcal{A}_{\xi,L}$. We have thus shown that solutions of (8.6) have planar symmetry, which, as argued before, proves the theorem. □

8.3. Proof of Theorem 3.2. Let m, L and ϵ as in Theorem 3.2. Since the function $t \to \alpha(t)$, $\alpha(t) = \fint_{Q_L} T_t(m)$, $t \ge 0$, is continuous and since $|\vartheta_{\alpha(0)}| \le \theta_0$, either there is a time $t^* \ge 0$ when $|\vartheta_{\alpha(t^*)}| = \theta_0$ or else any limit point m^* (in L^∞) of $T_t(m)$ is in Σ_α with $|\vartheta_\alpha| \le \theta_0$.

Being a limit point, m^* is stationary and by lower semi-continuity, $F_L(m^*) \le F_L(m) < LF_L^{(1)}(\hat{m}_L) + \epsilon$. Since $\epsilon < \epsilon_0(L)$, by Theorem 3.1 there is ξ, $|\xi| \le \theta_0 L + 1$, so that $\|m^* - \bar{m}_{\xi,L}\|_2 < L^{-100}$. Since m^* is stationary its derivative is bounded, hence there is a constant c so that

$$\|m^* - \bar{m}_{\xi,L}\|_\infty \le c\left(\|m^* - \bar{m}_{\xi,L}\|_2\right)^{2/3} < c(L^{-100})^{2/3}. \tag{8.8}$$

We omit the proof that if $\|m - \bar{m}_{\xi,L}\|_\infty < \zeta$, ζ small enough, then m is in a fiber $\mathcal{B}_{\xi',L}$ with $|\xi' - \xi| \le c\zeta$, which is analogous to its $d = 1$ version proved in [6]. Using such a statement by (8.8) for L large enough $m \in \mathcal{B}_{L^{-3},\xi',L}$, $|\xi'| \le rL/2$, $r < 1$ and by Theorem 8.1 we then conclude that $m^* = \hat{m}_L^e$. Theorem 3.2 is proved.

8.4. Proof of Theorem 3.3. By symmetry we may restrict to $m \in \Sigma_\alpha$ with $\vartheta_\alpha = -\theta_0$. By assumption $\|m - \bar{m}_{\xi,L}\|_2 < L^{-100}$; we are going to show that for L large enough,

$$|-\theta_0 - \frac{\xi}{L}| \le L^{-100}. \tag{8.9}$$

Indeed, $\|m - \bar{m}_{\xi,L}\|_1 \le 4\|m - \bar{m}_{\xi,L}\|_2 < 4L^{-100}$, so that $|\fint_{Q_L} m - \fint_{Q_L} \bar{m}_{\xi,L}| \le \dfrac{4L^{-100}}{L^2}$

and (8.9) follows for L large enough because $\fint_{Q_L} m = \alpha$, $\vartheta_\alpha = -\theta_0$ and using (6.3).

Theorem 8.2. *For any ϵ and $r \in (0,1)$ there is $L(\epsilon, r)$ so that for all $L > L(\epsilon, r)$ the following holds. Let $m \in L^\infty(Q_L)$ be such that there is $\xi_0 \in (-\frac{L}{2}, -\frac{rL}{2})$ so that $\|m - \bar{m}_{\xi_0,L}\|_\infty \le \epsilon$, then*

$$\lim_{t\to\infty} \|T_t(m) - m^+\|_\infty = 0. \tag{8.10}$$

Proof. By assumption

$$m(x,y) \ge \bar{m}_{\xi_0}(x) - 2\epsilon, \qquad \text{for all } (x,y) \in Q_L.$$

In Proposition 8.2, Theorem 8.3 and Proposition 8.4 of [4] it has been proved that for L large (how large depending on ϵ and r) $T_t\big(\bar{m}_{\xi_0} - 2\epsilon\big)$ converges to m^+. Thus (8.10) follows from the comparison theorem. □

By Lemmas 6.4 and 6.5,

$$\|T_t(m) - \bar{m}_{\xi,L}\|_\infty \le 2e^{-t} + c\left(\left[e^{-t} + e^{2(\beta-1)t}\right]\|m - \bar{m}_{\xi,L}\|_2\right)^{2/3}.$$

Choosing t suitably large (independently of L) the r.h.s. becomes $< \epsilon$ and by (8.9), $T_t(m)$ satisfies the assumption of Theorem 8.2 with $r < \theta_0$ and L large enough. Then Theorem 3.3 follows from Theorem 8.2, noticing that convergence in $L^\infty(Q_L)$ implies convergence in $L^2(Q_L)$, because Q_L is bounded.

Appendix

9. Spectral Estimates, sup Norms

The analysis in this appendix refers to functions on the square Q_L and on the channel $Q_{\infty,L}$; in the latter case we will consider only one function, the instanton $\bar{m}^e$. For brevity we call planar a function or a kernel where the dependence on the point r is only via its x coordinate $x = r \cdot e_1$.

Definition. *The set $\mathcal{M}_L$ consists of the instanton $\bar{m}^e \in L^\infty(Q_{\infty,L}, (-1,1))$ and of the family of planar functions $m \in L^\infty(Q_L, [-1,1])$ which are in either one of the following two classes (r below a fixed number in $(0,1)$):*

- $\bar{m}_{\xi,L}$, $|\xi| \le \dfrac{rL}{2}$
- $\|m - \hat{m}^e_L\|_\infty \le \epsilon(L)$, $\epsilon(L) > 0$ *a small number which will be fixed later.*

9.1. Maximal eigenvalue and eigenvector. We call A_m, $m \in \mathcal{M}_L$, the operator on $L^\infty(Q_L)$ or $L^\infty(Q_{\infty,L})$ if $m = \bar{m}^e$, whose kernel is

$$A_m(r,r') = p_m(r) J^{\text{neum}}(r,r'). \tag{9.1}$$

If $m = \bar{m}_{\xi,L}$, then $p_m = \cosh^{-2}\{\beta J^{\text{neum}} * m\}$, otherwise $p_m = \beta(1-m^2)$. If $m = \bar{m}^e$ or $m = \hat{m}^e_L$ the two expressions coincide. The different choices are due to different applications, e.g. if we linearize around the flow $T_t(m)$ or $S_t(m)$.

In [13] it is proved that given $r \in [0,1)$ there are L_r and $\epsilon(L)$ so that for all $L \ge L_r$ and any $m \in \mathcal{M}_L$, there are $\lambda_m > 0$ and e_m so that, with $s_m = p_m^{-1} e_m$,

$$\int A_m(r,r') e_m(r') dr' = \lambda_m e_m(r), \quad \int s_m(r) A_m(r,r') dr' = \lambda_m s_m(r'). \tag{9.2}$$

e_m is a strictly positive, smooth planar function in $L^\infty(Q_L)$ that we normalize so that

$$\int s_m e_m = \int e_m^2 p_m^{-1} = \langle e_m, e_m \rangle_m = 1. \tag{9.3}$$

λ_m is an eigenvalue of A_m with strictly positive right and left eigenvectors, e_m and s_m, in agreement with the Perron-Frobenius theorem which is indeed behind the proof of the above statements. The function $e_m^{(1)}(x)$ on $[-L/2, L/2]$ or $\mathbb{R}$ for the instanton, defined by $x \to e_m(r)$, $r \cdot e_1 = x$, is the eigenvector for the $d = 1$ problem with interaction j as in (2.2), however $\int e_m^{(1)}(x)^2 dx = L^{-1}$ due to (9.3). In the case $m = \bar{m}^e$, $\lambda_m = 1$ and $e_m(r) = c\bar{m}'(r \cdot e_1)/\sqrt{L}$, c a normalization independent of L.

The above statements are verified in a large class of functions m, those which follow are instead more restrictive. All bounds below are uniform in $\mathcal{M}_L$ but we keep reference to the specific $m \in \mathcal{M}_L$ for future applications.

- There are $c_\pm > 0$ and $\alpha'_m > 0$ so that

$$1 - c_+ e^{-2\alpha'_m L} \le \lambda_m \le 1 + c_+ e^{-2\alpha'_m L}. \tag{9.4}$$

- For each $m \in \mathcal{M}_L$ define x_m as $x_m = 0$ if $m = \bar{m}^e$, $x_m = \xi$ if $m = \bar{m}_{\xi,L}$ and $x_m = 0$ for the remaining m. Then there are $s > 0$ and $\delta < 1$ so that

$$p_m(r) \le \delta, \quad |r \cdot e_1 - x_m| \ge s, \tag{9.5}$$

  and there are $\alpha_m > 0$, $\alpha''_m > 0$ and c so that

$$e_m(r) \le \frac{c}{\sqrt{L}}\, e^{-\alpha_m |r\cdot e_1 - x_m|}, \quad e_m(r)^{-1} \le c\sqrt{L}\, e^{\alpha''_m |r\cdot e_1 - x_m|}. \tag{9.6}$$

- We will also use that there is a constant c so that

$$\|p_m^{-1}\|_\infty \le c. \tag{9.7}$$

- As mentioned, all the previous bounds are uniform in $\mathcal{M}_L$, by suitably resetting the coefficients.

9.2. Reduction to Markov chains. Let K_m be the Markov operator whose transition probability kernel is

$$K_m(r, r') = \frac{A_m(r, r')e_m(r')}{\lambda_m e_m(r)}. \tag{9.8}$$

Since $A_m^n(r, r') = e_m(r)\lambda_m^n K_m^n(r, r')e_m(r')^{-1}$ we will derive bounds on A_m^n and consequently on the spectrum of A_m and of $\Omega_m := A_m - 1$ from properties of K_m^n. The important point of the transformation (9.8) is that K_m is a Perron-Frobenius Markov kernel to which the high temperature Dobrushin techniques apply.

Calling $x = r \cdot e_1$ and $y = r \cdot e_2$ we can write $K_m(r, r')$ as

$$K_m(r, r') = P_m(x, x')q_{x,x'}(y, y'), \tag{9.9}$$

where, relative to the measure $K_m(r,r')dr'$, $P_m(x,x')$ is the marginal distribution of x' and $q_{x,x'}(y,y')$ is the conditional distribution of y' given x' (to simplify notation we drop sometimes the suffix m). The explicit expression of $P_m(x,x')$ is

$$P_m(x,x') = \frac{p_m(x) j^{\text{neum}}(x,x') e_m(x')}{\lambda_m e_m(x)}, \tag{9.10}$$

where $j(x,x')$ is defined in (2.2) and $e_m(x) \equiv e_m(x,y)$ (recall that $e_m(r)$ is planar). Equation (9.10) is (9.8) in the $d = 1$ case with interaction $j(x,x')$. Notice that due to the planar symmetry assumption the marginal $P_m(x,x')$ does not depend on the y coordinate of r. In the sequel we will consider the probability density

$$q_{x,x'}(z) = \frac{J((x,0),(x',z))}{j(x,x')} \tag{9.11}$$

on $\mathbb{R}$ noticing that the variable $y' := y + z$ modulo reflections at $\pm nL/2$ has the law $q_{x,x'}(y,y')$ and sometimes, by an abuse of notation, we will write $q_{x,x'}(z)$ for $q_{x,x'}(y,y')$.

9.3. One dimensional results. To study the dependence on the initial point r of the Markov chain with transition probability kernel $K_m(r,r')$ we will use couplings (for brevity we may shorthand $x = r \cdot e_1$ and $y = r \cdot e_2$). We first recall some one dimensional results proved in [13]. Call

$$W^{(1)}[x,x'] = \left[w_m(x) + w_m(x')\right] \mathbf{1}_{\{x \neq x'\}}, \quad w_m(x) := e_m(x)^{-1}; \tag{9.12}$$

e_m and $m \in \mathcal{M}_L$ below are regarded as functions of x.

Theorem 9.1. *There are c and $\omega^{(1)}$ positive and for any $(x_0,x_0') \in [-L/2,L/2]^2$ (or $\mathbb{R}^2$) a process on $([-L/2,L/2]^2)^{\mathbb{N}}$ (or $(\mathbb{R}^2)^{\mathbb{N}}$) whose expectation is denoted by $\mathcal{E}^{(1)}_{x_0,x_0'}$ so that its marginal distributions are the Markov chains with transition probability* (9.10) *and, for any L large enough and $n \geq 1$,*

$$\mathcal{E}^{(1)}_{x_0,x_0'}\left(W^{(1)}[x(n),x'(n)]\right) \leq c W^{(1)}[x(0),x'(0)] e^{-\omega^{(1)} n}. \tag{9.13}$$

Moreover if for some n, $x(n) = x'(n)$ then $x(n+k) = x'(n+k)$ for all $k \geq 0$.

9.4. Couplings and Wasserstein distance. For any $(r_0,r_0') \in Q_L \times Q_L$ (or $Q_{\infty,L} \times Q_{\infty,L}$ if $m = \bar{m}^e$) we define a process $\{r(n), r'(n), n \in \mathbb{N}\}$, $r(0) = r_0$, $r'(0) = r_0'$, with values on $Q_L \times Q_L$ (or $Q_{\infty,L} \times Q_{\infty,L}$) as follows: The marginal distribution of $\{x(n), x'(n), n \in \mathbb{N}\}$ is set equal to the law $\mathcal{P}^{(1)}_{x_0,x_0'}$ of the process defined in Theorem 9.1. To complete the definition we must give the law of $\{y(n), y'(n), n \in \mathbb{N}\}$ conditioned on the trajectory

$$(\underline{x}, \underline{x}') = \{(x(n), x'(n)), n \in \mathbb{N}\},$$

which we consider in the sequel as fixed. Define then $n_0,\ n_1 \in \mathbb{N} \cup \{+\infty\}$ as

$$n_0 := \inf\left\{n \in \mathbb{N} :\ x(n_0) = x'(n_0)\right\}, \quad n_1 := \inf\left\{n \in \mathbb{N} :\ n \geq n_0 \text{ and } |y(n) - y'(n)| \leq 1\right\},$$

where the infimum over the empty set is defined as $+\infty$. This means that n_0 is the first time when the x-coordinates couple, and n_1 is the first time at which the y-coordinates get close after the x-coordinates have coupled.

For $n \le n_1$, $y(n)$ and $y'(n)$ are independent of each other and distributed with the law of the Markov chain with transition probability (9.11) which starts respectively from y_0 and y'_0. If $n_1 < \infty$ the conditional law of $\{y(n), y'(n), n \in [n_1, n_1 + k_0]\}$, k_0 as in Lemma 9.2 below, given $y(n_1), y'(n_1)$ is Π, Π the probability in Lemma 9.2 below. If $y(n_1 + k_0) = y'(n_1 + k_0)$, $y'(n) = y(n)$ for $n \ge n_1 + k_0$ with $y(n)$ having the law of the Markov chain with transition probability (9.11). If instead $y(n_1 + k_0) \neq y'(n_1 + k_0)$ we repeat the previous procedure with n_0 replaced by $n_1 + k_0$ and so forth.

Lemma 9.2. *There are π_0 and k_0 positive and for any (y_0, y'_0, X), $|y_0 - y'_0| \le 1$, $X = (x_0, .., x_{k_0})$, a probability $\Pi = \Pi_{(y_0,y'_0,X)}$ on $[-L/2, L/2]^{k_0+1} \times [-L/2, L/2]^{k_0+1}$ such that the marginal distributions of $y(\cdot)$ and $y'(\cdot)$ are the Markov chains with transition probability* (9.11) *starting from y_0 and y'_0 and ($\mathcal{E}^{(1)}_{x_0,x'_0}$ below as in Theorem 9.1),*

$$\mathcal{E}^{(1)}_{x_0,x'_0}\Big(\Pi_{(y_0,y'_0,X)}\big(\{y(k_0) = y'(k_0)\}\big)\Big) \ge \pi_0. \tag{9.14}$$

The lemma follows easily from the smoothness properties of the transition kernel, its proof is just as in its one dimensional version in [13] and it is omitted.

We call $\mathcal{P}_{r_0,r'_0}$ the joint law of $\{r(n), r'(n), n \in \mathbb{N}\}$ as defined above and denote by $\mathcal{E}_{r_0,r'_0}$ expectation w.r.t. $\mathcal{P}_{r_0,r'_0}$. $\mathcal{P}_{r_0,r'_0}$ is a coupling of the Markov chains starting from r_0 and r'_0 and with transition probability K_m. Indeed, for any $f \in L^\infty(Q_L)$ or $f \in L^\infty(Q_{\infty,L})$, and any $n \ge 1$,

$$\mathcal{E}_{r_0,r'_0}\big(f(r(n)\big) = \int_{Q_L} K^n_m(r_0, r) f(r), \quad \mathcal{E}_{r_0,r'_0}\big(f(r'(n))\big) = \int_{Q_L} K^n_m(r'_0, r) f(r). \tag{9.15}$$

Recalling (9.12) we define a distance $W[r, r']$, on $Q_L \times Q_L$ or on $Q_{\infty,L} \times Q_{\infty,L}$ as

$$W[r, r'] = \big[w_m(r) + w_m(r')\big]\, \mathbf{1}_{\{r \neq r'\}} = W^{(1)}[x, x'], \quad w_m(r) := e_m(r)^{-1} \tag{9.16}$$

($x = r \cdot e_1$ above) and call

$$R_{n,r_0,r'_0} = \mathcal{E}_{r_0,r'_0}\left(W[(r(n), r'(n)]\right). \tag{9.17}$$

R_{n,r_0,r'_0} is an upper bound for the Wasserstein distance between $K^n_m(r_0, \cdot)$ and $K^n_m(r'_0, \cdot)$ relative to the distance (9.16).

Theorem 9.3. *There are positive constants L^*, c and ω so that for any of the above chains and any $L > L^*$, $n \ge 1$:*

$$R_{n,r_0,r'_0} \le c e^{-(\omega/L^2)n} W[r_0, r'_0]. \tag{9.18}$$

The proof of Theorem 9.3, which is postponed, uses Theorem 9.1 to reduce to the case when $x(\cdot) = x'(\cdot)$. Then the y coordinates (regarded on the whole axis and then reduced to $[-L/2, L/2]$ by reflections) perform independent random walks with increments having the same law (which depends on $\underline{x}$) till when they get to distance ≤ 1. By Lemma 9.2 after a time they couple k_0 with probability $\pi_0 > 0$ and the proof of Theorem 9.3 will then be concluded with an estimate of the probability of the time when two independent walks get closer than 1. We will see that such a probability is positive independent of L and of the starting points provided the time is proportional to L^2 (recall that the y coordinates are defined modulo reflections at $\pm nL/2$).

9.5. L^∞ bounds. The Markov chain K_m has an invariant probability measure $\mu(r)dr$ (recall the normalization of e_m and s_m in Subsect. 9.1)

$$\mu(r) := s_m(r)e_m(r) = e_m(r)^2 p_m(r)^{-1}, \quad \int \mu(r)K_m(r,r')dr = \mu(r'). \tag{9.19}$$

Let $\psi \in L^\infty(Q_L)$ and $u = \psi w_m$. By the invariance of μ,

$$\int K_m^n(r_0,r')[u(r') - \mu(u)]dr' = \int \mu(r_0')\mathcal{E}_{r_0,r_0'}\left(u(r(n)) - u(r'(n))\right) dr_0'. \tag{9.20}$$

We write $u(r) - u(r') = \dfrac{\tilde u(r)}{w_m(r)} w_m(r) - \dfrac{\tilde u(r')}{w_m(r')} w_m(r')$, $\tilde u = u - \mu(u)$ where, by an abuse of notation, $\mu(u) = \int \mu(r)u(r)dr$. Thus

$$\left|u(r) - u(r')\right| \le \|\frac{\tilde u}{w_m}\|_\infty W[r,r']. \tag{9.21}$$

Hence by (9.18)

$$\left|\int K_m^n(r_0,r')\left[u(r') - \mu(u)\right]\right| \le \|\frac{\tilde u}{w_m}\|_\infty c e^{-(\omega/L^2)n}\left(w_m(r_0) + C'\right). \tag{9.22}$$

The term with C' is obtained by writing $\int w_m(r)\mu(r) = \int e_m p_m^{-1}$ which, by (9.6) and (9.7), is bounded. Moreover, recalling that $u = \psi w_m$,

$$\frac{\tilde u(r)}{w_m(r)} = \tilde\psi(r), \quad \tilde\psi := \psi - e_m\langle\psi, e_m\rangle_m. \tag{9.23}$$

By (9.22) and (9.8),

$$\left|\int A_m^n(r_0,r')\tilde\psi(r')\right| \le e_m(r_0)\left\{\|\tilde\psi\|_\infty c\left[\lambda_m e^{-(\omega/L^2)}\right]^n \left(w_m(r_0) + C'\right)\right\} \tag{9.24}$$

which using (9.6) proves:

Theorem 9.4. *There are positive constants L^*, c and ω so that for any of the above chains, any $L > L^*$, $n \ge 1$ and any ψ such that $\langle\psi, e_m\rangle_m = 0$,*

$$\|A_m^n\psi\|_\infty \le c'[\lambda_m e^{-(\omega/L^2)}]^n \|\psi\|_\infty, \tag{9.25}$$

where $c' = c\left[1 + C'\|e_m\|_\infty\right]$ and for any $t > 0$, (recalling that $\Omega_m = A_m - 1$)

$$\|e^{\Omega_m t}\psi\|_\infty \le e^{-t}c'\|\psi\|_\infty \sum_{n=0}^\infty \frac{(\lambda_m e^{-(\omega/L^2)}t)^n}{n!} \le c' e^{-(\omega/2L^2)t}\|\psi\|_\infty. \tag{9.26}$$

The last bound follows for L large enough by bounding $-1+\lambda_m e^{-x} < |\lambda_m - 1| + e^{-x} - 1$, $e^{-x} - 1 \le -3x/4$ ($x > 0$ small enough), $|\lambda_m - 1| \le \omega/(4L^2)$, by (9.4) for L large enough.

9.6. A preliminary lemma. In the proof of Theorem 9.3 and in Sect. 10 as well we will use Lemma 9.5 below. With reference to (9.6), define for $m \in \mathcal{M}_L$,

$$w_{m;a}(r) = w_m(r)e^{a|r\cdot e_1 - x_m|}, \quad a \geq 0, \tag{9.27}$$

$$k_m(n, r) := \min\left\{n, (|r \cdot e_1 - x_m| - (s+1))\mathbf{1}_{|r\cdot e_1 - x_m| - (s+1) > 0}\right\}, \tag{9.28}$$

where s is as in (9.5).

Lemma 9.5. *Let δ be as in (9.5). Then there exist positive constants L^*, a_0, c and $\delta_1 \in (\delta, 1)$ such that for any $0 < a < a_0$ and $L^* > L$ the following holds. If $m \in \mathcal{M}_L$ then for any $n \geq 1$,*

$$\int K_m^n(r, r')w_{m;a}(r')dr' \leq c\delta_1^{k_m(n,r)} w_{m;a}(r). \tag{9.29}$$

All the above coefficients can be taken uniformly in $m \in \mathcal{M}_L$.

Proof. Call $x = r \cdot e_1$ and $P_s(r, r') = K_m(r, r')\mathbf{1}_{|x - x_m| \geq s}$ and $= 0$ otherwise; let E_r be the expectation of the Markov process with transition probability K_m starting from r so that

$$\int K_m^n(r, r')w_{m;a}(r')dr' = E_r\left(w_{m;a}(r(n))\right).$$

We decompose the expectation on the r.h.s. by using the sets $A_0 = \{r(\cdot) : |x(0) - x_m| \leq s\}$,

$$A_k := \Big\{r(\cdot) : |x(t) - x_m| > s, \quad t = 0, .., k-1; \quad |x(k) - x_m| \leq s\Big\}, \quad k \geq 1,$$

$$B_h := \Big\{r(\cdot) : |x(t) - x_m| > s, \quad t = h, .., n; \quad |x(h-1) - x_m| \leq s\Big\}, \quad h \geq 1,$$

$$C_n = \Big\{r(\cdot) : |x(t) - x_m| > s, t = 0, .., n\Big\}, \quad D_n = \Big\{r(\cdot) : |x(n) - x_m| \leq s\Big\}.$$

Then,

$$\begin{aligned}\int K_m^n(r, r')w_{m;a}(r')dr' = &\int P_s^n(r, r')w_{m;a}(r')dr' \\ &+ \sum_{n \geq h > k} \int P_s^k(r, r_0)\mathbf{1}_{|x_0 - x_m| \leq s} E_{r_0} \\ &\quad \times \left(\mathbf{1}_{|r(h-k-1)\cdot e_1 - x_m| \leq s}\phi_{n-h}(r(h-k))\right) dr_0,\end{aligned} \tag{9.30}$$

where $\phi_l(r) := \int P_s^l(r, r')w_{m;a}(r')dr'$ for $l \in \mathbb{N}$. By (9.8), (9.5) and (9.7) there is c so that

$$\int P_s^l(r, r')w_{m;a}(r')dr' \leq c\left[\lambda_m^{-1}e^a\delta\right]^l w_{m;a}(r) \tag{9.31}$$

because $|x' - x| \leq l$. By (9.4) for L large and a small enough $\lambda_m^{-1}e^a\delta =: \delta_1 < 1$. Note that only for $k \geq k_m(r, n)$ the corresponding terms in (9.30) are nonzero, hence

$$\int K_m^n(r, r')w_{m;a}(r')dr' \leq c \sum_{n \geq h > k_m(r,n)} \delta_1^{k+n-h} w_{m;a}(r),$$

and (9.29) then follows. By the last item in Subsect. 9.1 all coefficients in the above bounds can be chosen uniformly in $m \in \mathcal{M}_L$ so that the proof of the lemma is complete. □

9.7. Proof of Theorem 9.3. Given n call n_0 the integer part of $n/2$ and shorthand $\xi_n = (r(n), r'(n))$. Then

$$\mathcal{E}_{\xi_0}\big(W[\xi_n]\big) = \mathcal{E}_{\xi_0}\Big(\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\Big\{\mathbf{1}_{x_{n_0}\neq x'_{n_0}} + \mathbf{1}_{x_{n_0}=x'_{n_0}}\Big\}\Big). \tag{9.32}$$

When $x_{n_0} \neq x'_{n_0}$ we bound $W[\xi_n] \le w_m\big(x(n)\big) + w_m\big(x'(n)\big)$, namely we drop the characteristic function that $r(n) \neq r'(n)$ so that the expectations relative to $r(\cdot)$ and $r'(\cdot)$ uncouple. Then by Lemma 9.5 with $a = 0$, $\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\mathbf{1}_{x_{n_0}\neq x'_{n_0}} \le cW^{(1)}[x_{n_0}, x'_{n_0}]$, hence by Theorem 9.1,

$$\mathcal{E}_{\xi_0}\Big(\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\mathbf{1}_{x_{n_0}\neq x'_{n_0}}\Big) \le c'W[\xi_0]e^{-\omega^{(1)}n_0}. \tag{9.33}$$

To bound $\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)$ with $x_{n_0} = x'_{n_0}$ we recall from Theorem 9.1 and the definition of $\mathcal{P}_{\xi_0}$ that $x(i) = x'(i)$ for all $i \ge n_0$, so that

$$W[\xi_n] = 2w_m(x(n))\mathbf{1}_{y_{n_0}\neq y'_{n_0}}. \tag{9.34}$$

We distinguish two cases: *First case,* $|x_0 - x_m| > n$. We bound $W[\xi_n] \le 2w_m(x(n))$ and get

$$\mathcal{E}_{\xi_0}\Big(\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\mathbf{1}_{x_{n_0}\neq x'_{n_0}}\Big) \le 2E_{r_0}\Big(w_m(r(n))\Big) \le c\delta_1^n w_m(r_0) \tag{9.35}$$

having used Lemma 9.5 with $a = 0$ and with c above a suitable constant.
Second case, $|x_0 - x_m| \le n$. To decouple x from (y, y') we use Hölder. Let $p^{-1}+q^{-1} = 1$ then, supposing (for instance) $w_m(x_0) \le w_m(x'_0)$,

$$\begin{aligned}\mathcal{E}_{\xi_0}\Big(\mathcal{E}_{\xi_{n_0}}\big(W[\xi_n]\big)\mathbf{1}_{x_{n_0}=x'_{n_0}}\Big) &\le 2E_{r_0}\Big(w_m(r(n))^p\Big)^{1/p}\\ &\quad\times\mathcal{P}_{\xi_0}\Big(\big\{x(n_0) = x'(n_0);\, y(n) \neq y'(n)\big\}\Big)^{1/q}.\end{aligned} \tag{9.36}$$

We use the second inequality in (9.6) to write

$$w_m(r)^p \le \Big[(c\sqrt{L})e^{\alpha''_m|x-x_m|}\Big]^{p-1}\; e_m^{-1}(r) = (c\sqrt{L})^{p-1}w_{m;a}(r), \quad a = \alpha''_m(p-1). \tag{9.37}$$

Taking $p-1 > 0$ small enough we can apply Lemma 9.5 and recalling that $|x_0-x_m| \le n$ we get

$$E_{r_0}\Big(w_m(x(n))^p\Big) \le c'(\sqrt{L})^{p-1}w_{m;a}(r_0)\delta_1^{|x_0-x_m|} \le c''(\sqrt{L})^{p-1}w_m(r_0). \tag{9.38}$$

The last inequality is valid for $p - 1 > 0$ small enough. Then

$$\Big\{E_{r_0}\Big(w_m(x(n))^p\Big)\Big\}^{1/p} \le C(\sqrt{L})^{1-1/p}w_m(r_0)e_m(r_0)^{1-1/p} \le C'w_m(r_0), \tag{9.39}$$

having used the first inequality in (9.6).
Conclusions. In the first case, $|x_0 - x_m| > n$, the bound (9.35) concludes the proof, while in the second case we need to prove that $\mathcal{P}_{\xi_0}\Big(\big\{x(n_0) = x'(n_0);\, y(n) \neq y'(n)\big\}\Big)$ is exponentially small, which is done in the next subsection.

9.8. Coupling the y coordinates. In this subsection we suppose $r_0 = (x_0, y_0)$ and $r_0' = (x_0, y_0')$, namely that the initial x coordinates are the same. This is indeed what happens at time n_0 in the case we have to study and, to simplify notation, we have just reset time n_0 equal to 0. We will prove that at a time cL^2 the y coordinates are the same with probability not smaller than a number $\pi > 0$, uniformly in ξ_0 and L. By iteration this will prove that (shorthanding $\xi_0 = (r_0, r_0')$)

$$\mathcal{P}_{\xi_0}\Big(\{y(n) \neq y'(n)\}\Big) \le (1-\pi)e^{-n/(cL^2)} \tag{9.40}$$

which inserted in (9.36) will conclude the proof of (9.18). Let

$$\tau = \inf\big\{n \in \mathbb{N} : |y(n) - y'(n)| \le 1\big\}. \tag{9.41}$$

We will prove that

Proposition 9.6. *There are $k_1 > 0$ and $\pi_1 > 0$ so that*

$$\inf_{x_0, y_0, y_0'} \mathcal{P}_{\xi_0}\Big(\{\tau \le k_1 L^2\}\Big) \ge \pi_1. \tag{9.42}$$

Proposition 9.6 and Lemma 9.2 prove (9.40) with $\pi = \pi_0\pi_1$ and $cL^2 > k_0 + k_1L^2$. In the sequel we will prove Proposition 9.6. Since $y(n)$ and $y'(n)$ are independent of each other till τ, we may as well and will in the sequel consider $\mathcal{P}_{\xi_0}$ defined so that $y(n)$ and $y'(n)$ are independent of each other at all times. Shorthand,

$$Z_n = [y'(n) - y'(0)] - [y(n) - y(0)],$$

and call

$$\sigma := \inf_x \int P(x, x')\, q_{x,x'}(z) z^2\, dx' dz > 0.$$

Positivity follows because there is c so that $\dfrac{e_m(r')}{e_m(r)} \le c$ for any $|(r' - r)\cdot e_1| \le 2$, see [13].

Lemma 9.7. *There is c so that for any $n \ge 1$ for any ξ_0 with $x_0 = x_0'$,*

$$\mathcal{E}_{\xi_0}\big(Z_n\big) = 0, \quad \mathcal{E}_{\xi_0}\big(Z_n^2\big) \ge 2\sigma\, n, \quad \mathcal{E}_{\xi_0}\big(Z_n^4\big) \le c\, n^2. \tag{9.43}$$

Proof. We write $z_n = Z_n - Z_{n-1}$, $n \ge 1$, so that $Z_n = z_1 + \cdots + z_n$. For any k, n with $k < n$ and any measurable function f on $\mathbb{R}$, using that $J(0, r)$ depends on $|r|$ and $q_{x,x'}(z) = q_{x,x'}(-z)$,

$$\begin{aligned}\mathcal{E}_{\xi_0}\big(f(z_k)z_n\big) &= \mathcal{E}_{\xi_0}\Big(f(z_k)\int (u'-u) q_{x_{n-1},x}(u) q_{x_{n-1},x}(u') P(x_{n-1}, x) du du' dx\Big),\\ &= 0\end{aligned}$$

hence the first equality in (9.43) after setting $f = 1$. Analogously, recalling also the definition of σ,

$$\begin{aligned}\mathcal{E}_{\xi_0}\big(z_n^2\big) &= \mathcal{E}_{r_0, r_0'}\Big(\int (u'-u)^2 q_{x_{n-1},x}(u) q_{x_{n-1},x}(u') P(x_{n-1}, x) du du' dx\Big)\\ &= \mathcal{E}_{\xi_0}\Big(\int (u'^2 + u^2) q_{x_{n-1},x}(u) q_{x_{n-1},x}(u') P(x_{n-1}, x) du du' dx\Big) \ge 2\sigma,\end{aligned}$$

hence the lower bound in (9.43). The upper bound in (9.43) is derived by noticing that by symmetry

$$\mathcal{E}_{\xi_0}\big(Z_n^4\big) = \mathcal{E}_{\xi_0}\Big(\sum_{j\le n} z_j^4 + 12\sum_{i<j\le n} z_i^2 z_j^2\Big) \le cn^2.$$

□

Proof of Proposition 9.6. We have $\{\tau \le n\} \supseteq \{|Z_n| > L, \mathrm{sign}(Z_n) \ne \mathrm{sign}(y'(0) - y(0))\}$ because $y'(k) - y(k)$ jumps at most by 2. By symmetry,

$$\mathcal{P}_{\xi_0}\big(|Z_n| > L, \mathrm{sign}(Z_n) \ne \mathrm{sign}(y_0' - y_0)\big) = \frac{1}{2}\mathcal{P}_{\xi_0}\big(|Z_n| > L\big)$$

so that

$$\mathcal{P}_{\xi_0}\big(\tau \le n\big) \ge \frac{1}{2}\mathcal{P}_{\xi_0}\big(|Z_n| > L\big).$$

We have

$$\begin{aligned}\mathcal{E}_{\xi_0}(Z_n^2) &= \mathcal{E}_{\xi_0}(Z_n^2 1_{|Z_n|\le L}) + \mathcal{E}_{\xi_0}(Z_n^2 \mathbf{1}_{|Z_n|>L})\\ &\le L^2 + \mathcal{E}_{\xi_0}(Z_n^4)^{1/2}\mathcal{P}_{\xi_0}(|Z_n| > L)^{1/2}.\end{aligned}$$

Moreover, using (9.43) and the choice of σn, we obtain that for $n > L^2\sigma^{-1}$,

$$\mathcal{P}_{\xi_0}(|Z_n| > L)^{1/2} \ge \frac{2\sigma n - L^2}{(cn^2)^{1/2}} \ge \frac{\sigma}{\sqrt{c}},$$

hence (9.42). □

10. Spectral Gap

We regard here $\Omega_m = A_m - 1$, $m \in \mathcal{M}_L$, as an operator on the weighted L^2-spaces $L^2(Q_L, p_m^{-1}dr)$ or on $L^2(Q_{\infty,L}, p^{-1}dr)$ if $m = \bar{m}^e$ and denote by $\langle\cdot,\cdot\rangle_m$ the scalar product. On such spaces Ω_m is self-adjoint, it has eigenvalue $\lambda_m - 1$ with eigenvector the planar function e_m. We will prove here that:

Theorem 10.1. *There is $a > 0$ so that for all L large enough,*

$$\sup_{f:\langle f,e_m\rangle_m=0} \frac{\langle f, \Omega_m f\rangle_m}{\langle f, f\rangle_m} \le -\frac{a}{L^2}. \tag{10.1}$$

A crucial point in the proof of Theorem 10.1, which is given in the remaining of this section, is that the operator Ω_m is self-adjoint. The mere existence of a spectral gap then follows from Weyl's theorem by the same argument used in [16] for the $d = 1$ case. The argument is however abstract and does not allow to determine the dependence on L of the spectral gap. Notice on the other hand that for the Allen-Cahn equation $m_t = \Delta m - V'(m)$ the question trivializes because the linearized operator is a sum of two commuting operators, $\{\frac{d^2}{dx^2} - V''(\bar{m}(x))\} + \frac{d^2}{dy^2}$, so that it is the non-local nature of the interaction which is behind all difficulties we find here.

Notation. To simplify notation we fix $m \in \mathcal{M}_L$ and shorthand $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_m$. We call M the self adjoint operator equal to A_m on $\{f : \langle f, e_m \rangle = 0\}$, while $M e_m = 0$. We denote by $\|M\|$ its norm:

$$\|M\| = \sup_{f \neq 0} \frac{|\langle f, Mf \rangle|}{\langle f, f \rangle} = \sup_{f : \langle f, e_m \rangle = 0} \frac{|\langle f, A_m f \rangle|}{\langle f, f \rangle}. \tag{10.2}$$

Lemma 10.2. *If there is* $a > 0$ *so that for all L large enough,*

$$\log \|M\| \leq -\frac{2a}{L^2} \tag{10.3}$$

then (10.1) *holds.*

Proof. If (10.3) holds, then

$$\sup_{f : \langle f, e_m \rangle = 0} \frac{\langle f, (A_m - 1) f \rangle}{\langle f, f \rangle} \leq -1 + \|M\| \leq -1 + e^{-2a/L^2} \leq -\frac{a}{L^2}$$

for L large enough. □

To bound $\log \|M\|$ we use the spectral theorem:

Proposition 10.3.

$$\log \|M\| = \sup_{f \neq 0, \|f\|_\infty < \infty} \liminf_{n \to \infty} \frac{1}{2n} \log \left\{ \frac{\langle f, M^{2n} f \rangle}{\langle f, f \rangle} \right\}. \tag{10.4}$$

Equality holds with limsup as well.

Proof. Equation (10.4) is a direct consequence of the spectral theorem for self-adjoint operators, as we are going to see. Let $\langle f, f \rangle = 1$ and n be even. Since $\langle f, M^n f \rangle \leq \|M\|^n$,

$$\limsup_{n \to \infty} \frac{1}{2n} \log \langle f, M^{2n} f \rangle \leq \log \|M\|. \tag{10.5}$$

For the reverse inequality we use the spectral theorem to say that for any $0 \leq \lambda < \|M\|$ there is a non-zero orthogonal projection P_λ which commutes with M and such that for any $n \geq 1$, $M^{2n} P_\lambda \geq \lambda^{2n} P_\lambda$. Since L^∞ is dense in L^2, given any $0 \leq \lambda < \|M\|$ there are f and R such that $\|f\|_\infty < R$ and $P_\lambda f \neq 0$. Then writing f in $\langle f, M^{2n} f \rangle$ as $f = P_\lambda f + (1 - P_\lambda) f$ and expanding,

$$\langle f, M^{2n} f \rangle \geq \langle P_\lambda f, M^{2n} P_\lambda f \rangle \geq \lambda^{2n} \langle P_\lambda f, P_\lambda f \rangle, \quad \langle P_\lambda f, P_\lambda f \rangle > 0,$$

the first inequality using that M and P_λ commute, so that the mixed terms vanish. Hence

$$\sup_{f : \langle f, f \rangle = 1, \|f\|_\infty < \infty} \liminf_{n \to \infty} \frac{1}{2n} \log \langle f, M^{2n} f \rangle \geq \log \lambda,$$

thus

$$\sup_{f : \langle f, f \rangle = 1, \|f\|_\infty < \infty} \liminf_{n \to \infty} \frac{1}{2n} \log \langle f, M^{2n} f \rangle_m \geq \log \|M\|$$

which, together with (10.5), yields (10.4). □

Proof of (10.3). We consider f such that $\langle f, e_m \rangle = 0$, $\langle f, f \rangle = 1$ and $\|f\|_\infty \le R$ and we look for upper bounds on $\langle f, M^n f \rangle$, n even. We define $g = f/e_m$. Recalling (9.19), $\int g\mu = \langle f, e_m \rangle = 0$. By (9.8) and shorthanding K for K_m,

$$\lambda_m^{-n} \langle f, M^n f \rangle = \int g[K^n g]\mu \, dr = \int g(r)g(r')[K^n(r,r') - \mu(r')]\, dr' \mu(r) dr$$
$$= \int g(r)g(r')[K^n(r,r') - K^n(r'',r')]\, \mu(r'')\, dr'' dr' \mu(r) dr,$$

where we have used the invariance of μ with respect to K, see (9.19). Calling $Q^n_{r_0,r'_0}(dr, dr')$ the distribution of $(r(n), r'(n))$ of the Markov chain with law $\mathcal{P}_{r_0,r'_0}$ defined in Subsect. 9.4 we have

$$\int g(r')[K^n(r,r') - K^n(r'',r')]dr' = \int [g(r_1) - g(r_2)]\, Q^n_{r,r''}(dr_1 dr_2). \quad (10.6)$$

With such notation,

$$\lambda_m^{-n} \langle f, M^n f \rangle = \int g(r)\, [g(r_1) - g(r_2)]\;\, Q^n_{r,r''}(dr_1 dr_2) \mu(r'')\, dr'' \mu(r) dr.$$

With reference to (9.5)-(9.6), we split the domain of integration into the two sets $\{|x - x_m| \le n, |x'' - x_m| \le n\}$ and its complement, denoting by x and x'' the x coordinates of r and r''. We call

$$I := \int_{\{|x-x_m|\le n, |x''-x_m|\le n\}} g(r)\, [g(r_1) - g(r_2)]\, Q^n_{r,r''}(dr_1 dr_2) \mu(r'')\, dr'' \mu(r) dr. \quad (10.7)$$

Recalling that $g = f/e_m$, $\|f\|_\infty \le R$ and with W defined in (9.16), proceeding as in (9.21),

$$I \le R^2 \int_{\{|x-x_m|\le n, |x''-x_m|\le n\}} \frac{W[r_1, r_2]}{e_m(r)}\, Q^n_{r,r''}(dr_1 dr_2) \mu(r'')\, dr'' \mu(r) dr$$
$$\le c R^2 e^{-(\omega/L^2)n} \int_{\{|x-x_m|\le n, |x''-x_m|\le n\}} \frac{W[r, r'']}{e_m(r)}\, \mu(r'')\, dr'' \mu(r) dr,$$

where we have used (9.18). By the definition of $W[r, r'']$ we have for $r \ne r''$,

$$\frac{W[r, r'']}{e_m(r)} = \frac{1}{e_m^2(r)} + \frac{1}{e_m(r'') e_m(r)},$$

hence

$$\int_{\{|x-x_m|\le n, |x''-x_m|\le n\}} \frac{W[r, r'']}{e_m(r)}\, \mu(r'')\, dr'' \mu(r) dr$$
$$\le \int_{\{|x-x_m|\le n\}} p_m^{-1} dr + \left[\int e_m p_m^{-1} dr \right]^2.$$

By (9.6) and (9.7), $\int e_m \le c\sqrt{L}$, so that for a suitable constant c,

$$I \le cR^2 e^{-(\omega/L^2)n} nL. \tag{10.8}$$

Note that in the case Q_L the better bound

$$\langle f, M^n f\rangle \le cR^2 e^{-(\omega/L^2)n} L^2$$

follows directly from the spectral gap in L^∞, see (9.25). For the channel Q_L, however, the entire analysis presented in this section is necessary.

We will see below that all other contributions to $\lambda_m^{-n}\langle f, M^n f\rangle$ are smaller (for L large enough). In the complement of $\{|x - x_m| \le n, |x'' - x_m| \le n\}$ we use (10.6) backwards to rewrite the integrals in terms of $g(r')[K^n(r, r') - K^n(r'', r')]$. We will not exploit the minus sign and bound separately the two terms in the difference. We start with the term

$$\begin{aligned} Z := &\int_{\{|x-x_m|\ge n;|x''-x_m|\le n\}} g(r)g(r')K^n(r, r')\,\mu(r'')\,dr''dr'\mu(r)dr \\ \le &\int_{\{|x-x_m|\ge n\}} |g(r)g(r')|K^n(r, r')\,dr'\mu(r)dr =: \hat{Z}. \end{aligned} \tag{10.9}$$

Call $K_s(r, r') = K(r, r')$ if $|x - x_m| \ge s$ and 0 otherwise. Then $\hat{Z} = \sum_{h=0}^{n} Z_h$, where

$$\begin{aligned} Z_0 &= \int_{\{|x-x_m|\ge n\}} |g(r)g(r')|K_s^n(r, r')\,dr'\mu(r)dr, \\ Z_h &= \int_{\{|x-x_m|\ge n,|x''-x_m|\le s\}} |g(r)g(r')|K_s^{n-h}(r, r'')K^h(r'', r')\,dr''dr'\mu(r)dr. \end{aligned} \tag{10.10}$$

To bound Z_0 we use (9.5) to write

$$K_s(r, r') \le \delta J^{\text{neum}}(r, r')\frac{e_m(r')}{\lambda_m e_m(r)}$$

and get, with $\|f\|_2^2 = \int f^2$,

$$\begin{aligned} Z_0 &\le \lambda_m^{-n}\delta^{n-1}\int |f(r)f(r')|(J^{\text{neum}})^n(r, r')\,dr'dr \\ &\le \|f\|_2^2\lambda_m^{-n}\delta^{n-1}, \end{aligned} \tag{10.11}$$

$$\begin{aligned} Z_h &\le \gamma_h R \int_{\{|x-x_m|\ge n, |x''-x_m|\le s\}} |g(r)| K_s^{n-h}(r, r'')\, dr'' \mu(r) dr \\ &\le \gamma_h R^2 \int_{\{|x-x_m|\ge n\}} e_m(r), \\ \gamma_h &:= \sup_{|x''-x_m|\le s} \int e_m(x')^{-1} K^h(r'', r') dr' \le c\sqrt{L}. \end{aligned} \tag{10.12}$$

The last inequality follows from Lemma 9.5 and (9.6), with $c = c(s)$ a constant independent of h. By (9.6),

$$\sup_{|x-x_m|\ge n} \int e_m(r) \le c\sqrt{L} e^{-(\alpha_m/2)n} \tag{10.13}$$

so that $Z_h \le cR^2 L e^{-(\alpha_m/2)n}$. In conclusion, there is c so that

$$Z \le c\left(\|f\|_2^2 \lambda_m^{-n} \delta^n + nR^2 L e^{-(\alpha_m/2)n} \right). \tag{10.14}$$

The next term we examine is

$$\begin{aligned} B &:= \int_{\{|x-x_m|\ge n; |x''-x_m|\le n\}} |g(r)g(r')| K^n(r'', r')\, \mu(r'')\, dr'' dr' \mu(r) dr \\ &\le cR^2\sqrt{L} e^{-(\alpha_m/2)n} \int_{\{|x''-x_m|\le n\}} \frac{K^n(r'', r')}{e_m(x')}\, \mu(r'')\, dr'' dr' \\ &\le cR^2\sqrt{L} e^{-(\alpha_m/2)n} \int \frac{\mu(r')}{e_m(x')} dr' \le c' R^2 L e^{-(\alpha_m/2)n}, \end{aligned} \tag{10.15}$$

where we have used (10.13).

The next term is

$$C := \int_{\{|x-x_m|\le n; |x''-x_m|\ge n\}} |g(r)g(r')| K^n(r'', r')\, \mu(r'')\, dr'' dr' \mu(r) dr$$

which is equal to Z, see (10.9). The next one is

$$D := \int_{\{|x-x_m|\le n; |x''-x_m|\ge n\}} |g(r)g(r')| K^n(r, r')\, \mu(r'')\, dr'' dr' \mu(r) dr$$

which is equal to B, see (10.15). The last two terms are G and H :

$$\begin{aligned} G &:= \int_{\{|x-x_m|\ge n; |x''-x_m|\ge n\}} |g(r)g(r')| K^n(r, r')\, \mu(r'')\, dr'' dr' \mu(r) dr \\ &\le ce^{-2\alpha_m n} \int_{\{|x-x_m|\ge n\}} |g(r)g(r')| K^n(r, r')\, dr' \mu(r) dr = ce^{-2\alpha_m n} \hat{Z}, \end{aligned}$$

where $\hat{Z}$ is defined in (10.9). By (10.13) and (9.19),

$$H := \int_{\{|x-x_m|\geq n;|x''-x_m|\geq n\}} |g(r)g(r')|K^n(r'',r')\,\mu(r'')\,dr''dr'\mu(r)dr$$
$$\leq cR^2\sqrt{L}e^{-(\alpha_m/2)n}\int \frac{K^n(r'',r')}{e_m(r')}\,\mu(r'')\,dr''dr' \leq c'R^2Le^{-(\alpha_m/2)n}.$$

In conclusion we have proved that there is a constant c so that

$$\langle f, M^n f\rangle \leq c\lambda_m^n\left(R^2e^{-(\omega/L^2)n}nL + \|f\|^2\lambda_m^{-n}\delta^n + (n+1)R^2Le^{-(\alpha_m/2)n}\right).$$

Hence for L large enough,

$$\limsup_{n\to\infty}\frac{1}{2n}\log\langle f, M^{2n}f\rangle \leq \log\lambda_m - \frac{\omega}{L^2}$$

which by (9.4) yields and Proposition 10.3 yields (10.3). □

11. Extension to $d > 2$

While the paper is written for a strictly two dimensional system most of the arguments are not really two dimensional. We will sketch here the proof of the extension to $d = 3$ and see what is missing in $d > 3$. Thus we call $Q_L = [-L/2, L/2]^d$ with the cost of tunneling still denoted by P_L.

Theorem 11.1. *In $d = 3$ for L large enough $P_L = L^2F_L^{(1)}(\hat{m}_L)$ and* (2.21)–(2.22) *are also valid.*

Sketch of proof. The dimensional restriction to $d = 3$ comes uniquely from the limit Wulff problem as the isoperimetric inequality below is proved only in $d = 3$. The remaining parts of the proof work instead also in $d > 3$.

The obvious analogy of (5.1) is,

$$W_{\alpha,L} := m_\beta 1_{\{x:\, x_1\geq L\vartheta_\alpha\}} - m_\beta 1_{\{x:\, x_1<L\vartheta_\alpha\}},$$

and $\mathcal{N}_{\delta,L}$ is defined as in (7.2).

As in the two-dimensional case, the first step consists in showing that any minimizing path has to pass through a small neighborhood of $W_{\alpha,L}$ for α sufficiently small. This is a direct consequence of the 3-d version of the limit Wulff-problem (2.14). Indeed, Proposition 2.1 holds in three dimensions as well:

11.1. Isoperimetric problem in $d = 3$. For $\theta = 0$ the solution to the isoperimetric problem in the unit three dimensional cube Q_1 has the boundary which is parallel to one of the planes parallel to the faces of the cube, see Corollary 2 of [29]. On the other hand, for any $\theta \in (0, 1)$ the solutions to the isoperimetric problem in Q_1 belongs to one of the five types described in Fig. 9 of [29].

We claim that there exists an $\epsilon > 0$ such that the solution E_θ with $\theta \in (-\epsilon, \epsilon)$ having volume $1/2 - \theta$ is such that $Q_1 \cap \partial E_\theta$ is contained in plane parallel to a coordinate plane.

Assume now by contradiction that there exists a sequence $(\theta_n) \subset (0, 1)$ converging to $1/2$ such that $Q_1 \cap \partial E_{\theta_n}$ is not contained in a plane parallel to a coordinate plane. Then E_{θ_n} belongs to one of the four remaining types listed in [29]. Passing to a (not relabeled) subsequence if necessary, by the compactness theorem of bounded variation functions with uniformly bounded BV norm, it follows that the sequence (E_{θ_n}) converges in $L^1(Q_1)$ to a finite perimeter set E as $n \to +\infty$, the boundary of which cannot be contained in a plane parallel to a coordinate plane.

11.2. Couplings and Wasserstein distance. We refer here to Subsect. 9.4 (with the same title) where we have defined the coupling used in the proof of the spectral estimates and which must be modified in $d > 2$. Call $x_i, i = 1, .., d$, the coordinates, x_1 the one in the direction of the channel, $x_i, i > 1$ the transversal ones. The coupling is then defined iteratively so that once $x'_j = x''_j$, $j \le k-1$, they stay together while the other coordinates move independently until $x'_k = x''_k$ and so on. Thus at each step the requirement is that two one dimensional walks meet with each other and the estimate is then the same as in Sect. 9. The remaining analysis of the spectral estimates is essentially independent of the dimensions, and details are omitted.

11.3. The Bodineau-Ioffe argument. As a direct consequence of the Wulff estimate above we obtain Proposition 7.1. The next step consists in passing to the problem in the channel by finding vertical connections (which are now hyperplanes instead of stripes) close to the sides of the cube. In other words, we have to show that the interface is "flat" even on the mesoscopic scale, or, again phrased differently, that fingers do not grow too far. We sketch the proof for any dimension $d \ge 2$.

Define

$$S(n) := [n\ell_+, (n+1)\ell_+) \times [-L/2, L/2)^{d-1},$$

then we immediately obtain from the proof of Proposition 7.2 that for any $m \in \mathcal{N}_{\delta,L}$ there are $n_\pm \in Z_L^\pm$ such that $\Theta^{(\zeta,\ell_-,\ell_+)}(m, \cdot) \neq \pm 1$ in at most

$$N_\delta := c\delta L^{d-1}\left(\frac{\ell_+}{(\theta_1 - \theta_0)\zeta\,\ell_-^d}\right) \tag{11.1}$$

hypercubes of $\mathcal{D}^{(\ell_+)}$ inside $S(n_\pm)$. The constant c depends only on the dimension.

Let us now explain what we mean by a vertical connection (Definition 5.2) in higher dimensions.

Definition 11.2. *A* **vertical connection** *B is a $\mathcal{D}^{(\ell_+)}$-measurable connected set such that for any $(y_2, \dots, y_d) \in [-L/2, L/2]$ there exists $x \in [-L/2, L/2]$ such that $(x, y_2, \dots, y_n) \in B$.*

In particular the union of all cubes with sidelength ℓ_+ touching a given hyperplane normal to the x_1-axis is a vertical connection.

Now we show that Proposition 7.2 can be extended to higher dimensions. Let us prove the existence of the positive connection B_+, the existence of B_- being similar. To proceed we need the following definitions, see also Fig. 3:

$$\begin{aligned} A(i) &:= \{x : x_1 \in [i\ell_+, (i+1)\ell_+)\} \cap Q_L, \\ H(i) &:= A(i) \cap \{x : \Theta^{(\zeta,\ell_-,\ell_+)}(m, x) \in \{0, -1\}\}, \\ f(i) &:= |H(i)|. \end{aligned}$$

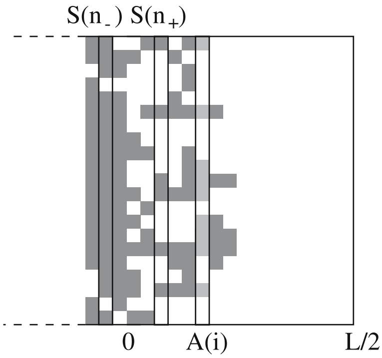

Fig. 3. We depict -for simplicity in a two-dimensional setting- the plus-squares (white), the minus- and zero-squares (dark), the slices $A(i)$, $S(n_\pm)$ and, in gray color, the set $H(i)$

Proof. Case 1. Let $M_L := \ell_+^{-1} L/2$ be the number of slices in the right half of the cube. Assume that there exists $i_0 \in \{n_+, \ldots, M_L\}$ such that

$$f(i_0) < C_0 \zeta^2 \ell_-^d \sum_{j=i_0}^{M_L} (f(j))^{\frac{d-2}{d-1}}, \tag{11.2}$$

where C_0 is a constant to be specified later, which depends only on J, β and the dimension. We show the existence of a connection B_+ by contradiction, i.e. if there is no connection, then we can construct a function m_2 such that a connection exists for m_2 and $F(m) \geq F(m_2) + \epsilon'$ for some $\epsilon' > 0$, thus deriving the desired contradiction as in the 2d-case.

The function m_2 is constructed in two steps. First, we obtain a function m_1 by "cutting" the contour in the most naive way at level i_0 : Define $m_1(x) := m_\beta$ for $x \in H(i_0)$ and $m_1(x) = m(x)$ elsewhere. Then there is a c_1 depending only on β, J and the dimension such that

$$F_L(m_1) \leq F_L(m) + c_1 f(i_0) \ell_+^{d-1}.$$

m_1 has the property that the sets $\{x : x_1 > \ell_+ i_0$ and $\Theta^{(\zeta,\ell_-,\ell_+)}(m,x) \in \{0,-1\}\}$ and $A(n_+)$ are not connected.

We can then apply Theorem 5.6 and conclude that there exists m_2 such that $m_2 = m$ on $\{x_1 < \ell_+ n_+\}$ and

$$F_L(m_2) \leq F_L(m_1) - c_2 \zeta^2 (\ell_-)^d N_0 \leq F_L(m) + f(i_0) c_1 \ell_+^{d-1} - c_2 \zeta^2 (\ell_-)^d N_0,$$

where N_0 denotes the number of zero-cubes in $\{x : (i_0+1)\ell_+ \leq x_1 \leq \ell_+ M_L\}$.

It remains to estimate N_0. Note that by definition of contours the boundary of $\bigcup_{j \geq i_0} H(j)$ is a union of 0-cubes, hence

$$N_0 \geq (2d\ell_+^{d-1})^{-1} \sum_{j \geq i_0} \mathcal{H}^{d-1}(\partial P[H(j)]),$$

where $\mathcal{H}^{d-1}$ is the (d-1)-dimensional Hausdorff measure, and $P[(x_1,\ldots,x_n)] = (x_2,\ldots,x_n)$ is the projection on the plane $\{x_1 = 0\}$. By the isoperimetric inequality,

$$\mathcal{H}^{d-1}(\partial P[H(j)]) \geq c_{d-1} P[H(j)]^{\frac{d-2}{d-1}} = c_{d-1} f(j)^{\frac{d-2}{d-1}},$$

where c_{d-1} is the isoperimetric constant in the interior of $[0, L]^{d-1}$. (Note that the isoperimetric inequality holds because we may assume that $|f(j)| \leq (1/2)L^{d-1}$ for all $i_0 \leq j \leq M_L$, or $N_0 \sim M_L/4$ follows immediately for δ sufficiently small.)

Therefore, if (11.2) holds, then $N_0 \geq \frac{2d\ell_+^{d-1}}{C_0 c_{d-1}\zeta^2 \ell_-^d} f(i_0)$, and

$$F(m) - F(m_2) \geq f(i_0)\ell_+^{d-1}\,(c_1 - 2dc_2/(c_{d-1}C_0)) > \alpha f(i_0),$$

where we can require $\alpha > 0$ for an appropriate choice of C_0. Note that we may assume $|f(i_0)| \geq 1$, or $H(i_0)$ contains no cube and therefore $A(i_0)$ is a connection. Hence this contradicts (7.4) for L sufficiently large and C_0 chosen appropriately.

Case 2. On the other hand, if (11.2) is false for all $i \in \{n_+, \ldots, \ell_+^{-1}(L/2)\}$ then, by solving the resulting difference inequality for the function $g(i) := \sum_{j=n_+}^{i+1} (f(j))^{\frac{d-2}{d-1}}$ we obtain that $f(n_+) \geq cL^{d-1}$, where c does not depend on δ. (See also Sect. 4.12 (proof of Lemma 4.4) in [5].) This contradicts the fact that $f(n_+) \leq \delta L^{d-1}$ for δ sufficiently small. $\square$

Acknowledgements. We are grateful to Dmitry Ioffe and Laszlo Zsido for helpful discussions.

References

1. Alberti, G., Bellettini, G., Cassandro, M., Presutti, E.: Surface tension in Ising systems with Kac potentials. J. Stat. Phys. **82**, 743–796 (1996)
2. Bellettini, G., De Masi, A., Dirr, N., Presutti, E.: Optimal orbits, invariant manifolds and finite volume instantons Preprint http://cvgmt.sns.it/people/belletti/, 2005
3. Bellettini, G., De Masi, A., Presutti, E.: Tunnelling for nonlocal evolution equations J. Nonlin. Math. Phys. **12**(Suppl.) 1, 50–63 (2005)
4. Bellettini, G., De Masi, A., Presutti, E.: Energy levels of a non local evolution equation. J. Math. Phys. **46**, 1–31 (2005)
5. Bodineau, T., Ioffe, D.: Stability of interfaces and stochastic dynamics in the regime of partial wetting. Ann. Henri Poincaré **5**, 871–914 (2004)
6. Buttà, P., De Masi, A., Rosatelli, E.: Slow motion and metastability for a nonlocal evolution equation. J. Stat. Phys. **112**, 709–764 (2003)
7. Chen, X.: Existence, uniqueness and asymptotic stability of traveling waves in non local evolution equations. Adv. Diff. Eqs. **2**, 125–160 (1997)
8. Chmaj, A., Ren, X.: Homoclinic solutions of an integral equation: existence and stability. J. Diff. Eq. **155**, 17–43 (1999)
9. Comets, F.: Nucleation for a long range magnetic model. Ann. Inst. H. Poincaré Probab. Statist. **23** (2), 135–178 (1987)
10. Cassandro, M., Olivieri, E., Picco, P.: Small random perturbations of infinite dimensional dynamical systems and the nucleation theory. Ann. Inst. Henri Poincaré **44**, 343–396 (1986)
11. De Masi, A., Dirr, N., Presutti, E.: Interface Instability under Forced Displacements. Annales Henri Poincaré **7**(3), 471–511 (2006)
12. De Masi, A., Gobron, T., Presutti, E.: Traveling fronts in non-local evolution equations. Arch. Rat. Mech. Anal. **132**, 143–205 (1995)
13. De Masi, A., Olivieri, E., Presutti, E.: Spectral properties of integral operators in problems of interface dynamics and metastability. Markov Process. Related Fields **4**, 27–112 (1998)
14. De Masi, A., Olivieri, E., Presutti, E.: Critical droplet for a non local mean field equation. Markov Process. Related Fields **6**, 439–471 (2000)

15. De Masi, A., Orlandi, E., Presutti, E., Triolo, L.: Glauber evolution with Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics. Nonlinearity **7**, 1–67 (1994)
16. De Masi, A., Orlandi, E., Presutti, E., Triolo, L.: Stability of the interface in a model of phase separation. Proc. Roy. Soc. Edinburgh Sect. A **124**, 1013–1022 (1994)
17. De Masi, A., Orlandi, E., Presutti, E., Triolo, L.: Uniqueness and global stability of the instanton in non local evolution equations. Rend. Math. Appl. (7) **14**, 693–723 (1994)
18. Faris, W.G., Jona Lasinio, G.: Large fluctuations for nonlinear heat equation with noise. J. Phys. A.: Math. Gen. **15**, 3025–3055 (1982)
19. Freidlin, M.I., Wentzell, A.D.: *Random Perturbations of Dynamical Systems*. Grund. der. Math. Wiss. **260**, Berlin-New York: Springer-Verlag, 1984
20. Fife, P.C.: *Mathematical aspects of reacting and diffusing systems*. Lectures Notes in Biomathematics **28**, Berlin-New York: Springer-Verlag, 1979
21. Fusco, G., Hale, J.K.: Slow motion manifolds, dormant instability and singular perturbations. J. Dyn. Diff. Eqs. **1**, 75–94 (1989)
22. Jona-Lasinio G., Mitter, P.K.: Large deviation estimates in the stochastic quantization of ϕ_2^4. Commun. Math. Phys. **130**, 111–121 (1990)
23. Kohn, R.V., Reznikoff, M.G., Tonegawa, Y.: The sharp interface limit of the action functional for Allen Cahn in one space dimension. Calc. Var. Partial Diff. Eqs. **25**(4), 503–534 (2006)
24. Kohn, R.V., Otto, F., Reznikoff, M.G., Vanden-Eijnden, E.: Action minimization and sharp interface limits for the stochastic Allen-Cahn Equation. Preprint (2005)
25. Massari, U., Miranda, M.: *Minimal Surfaces of Codimension One*. Notas de Matemática, Amsterdam: North-Holland, 1984
26. Martinelli, F.: On the two-dimensional dynamical Ising model in the phase coexistence region. J. Stat. Phys. **76**(5–6),1179–1246 (1994)
27. Olivieri, E., Vares, M.E.: *Large Deviations and Metastability*. Cambridge: Cambridge University Press, 2005
28. Presutti, E.: *From Statistical Mechanics towards Continuum Mechanics*. Course given at Max-Plank Institute, Leipzig, 1999
29. Ros, A.: The isoperimetric problem. In: *Global theory of minimal surfaces*. Clay Math. Proc., **2**, Providence, RI: Amer. Math. Soc., 2005, pp. 175–209

Communicated by J.L. Lebowitz