Weierstrass Institute for
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# Brownian intersection local times

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[K.&M., *Ann. Probab.* 2002], [K.&M., *Trans. Amer. Math. Soc.* 2006]
[Chen&M., *JLMS* 2009], [K.&Mukherjee, *CPAM* 2013]

## Brownian intersection local times

- $p$ independent Brownian motions $W_1, \dots, W_p$ in $\mathbb{R}^d$, running up to time $T_i = \infty$ (in $d \geq 3$) or until the exit time $T_i$ from a given large ball, respectively.
- Intersection of the motion paths:

$$S = \bigcap_{i=1}^{p} W_i([0, T_i)).$$

- Classical: $S$ is nonempty $\iff p(d-2) < d$.
- Natural measure on $S$: intersection local time (ISLT)

$$\ell(A) = \int_A \mathrm{d}y \prod_{i=1}^{p} \int_0^{T_i} \mathrm{d}s\, \delta_y(W_i(s)), \qquad A \subset \mathbb{R}^d \text{ mb.}$$

- Three classical constructions:
  local time analysis, renormalization of measure on sausages, Hausdorff measure.
- Main goal today: Describe how the motions achieve a high amount of intersections in a given set $U \subset \mathbb{R}^d$.

## Some questions

Fix an open bounded set $U \subset \mathbb{R}^d$. We fix $d = 2$ and $p \in \mathbb{N}$ or $d = 3$ and $p = 2$.

- upper tails: $\mathbb{P}(\ell(U) > a) \approx ???$ as $a \to \infty$.
- law of large masses: $\ell/\ell(U) \approx ???$ on $\{\ell(U) > a\}$ as $a \to \infty$.
- Hausdorff dimension spectrum: Find a gauge function $\varphi$ such that

$$0 < \sup_{x \in S} \limsup_{r \downarrow 0} \frac{\ell(B_r(x))}{\varphi(r)} < \infty$$

  and determine the Hausdorff dimension of the set of thick points,

$$f(a) = \dim \Big\{ x \in S\colon \limsup_{r \downarrow 0} \frac{\ell(B_r(x))}{\varphi(r)} = a \Big\}.$$

- stretched-exponential moments: Find criteria for nonnegative bounded functions $\phi_1, \dots, \phi_n$ such that

$$\mathbb{E}\Big[ \exp\Big\{ \sum_{j=1}^n \langle \phi_j, \ell \rangle^{1/p} \Big\} \Big] < \infty.$$

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Via polynomial moments! For any positive variable $X$,

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How do we find $\mathbb{E}[\ell(U)^k]$? Here is a recipe:

- Explicitly write out the $k$-th moments of $\ell(U)$ by using the heuristic mentioned formula
$$\ell(U) = \int_U \mathrm{d}y \prod_{i=1}^{p} \int_0^{T_i} \mathrm{d}s\, \delta_y(W_i(s))$$
- summarize and transform the arising multi-integrals over space $\mathrm{d}y_1 \dots \mathrm{d}y_k$ and time $\mathrm{d}s_1^{(i)} \dots \mathrm{d}s_k^{(i)}$ as far as possible,
- bring all the times into chronological order and use Markov property to express it in terms of products of transition probabilities,
- integrate out over the times to write it in terms of Green functions,
- use integrability of the $p$-th power of the Green function around its singularity.

## Upper tails (2)

By $G(x,y) = c|x-y|^{d-2}$ we denote Green's function, and by $\mathfrak{A}h(x) = \int_U G(x,y)h(y)\,\mathrm{d}y$ the Green operator.

### Upper tails of $\ell(U)$, [K.&M. 2002]

$$\lim_{a\to\infty} \frac{1}{a^{1/p}} \log \mathbb{P}(\ell(U) > a) = -\frac{p}{\rho^*},$$

where

$$\rho^* = \sup\Big\{ \langle g^{2p-1}, \mathfrak{A} g^{2p-1} \rangle \colon g \in L^{2p}(U), \|g\|_{2p} = 1 \Big\}$$

- For $p = 1$, $\rho^*$ is just the principal eigenvalue of $\mathfrak{A}$.
- Maximiser(s) exist and satisfy $\frac{1}{2}\Delta g = -\frac{1}{\rho^*} g^{2p-1}$ in $U$.
- Uniqueness is unknown in general.
- Alternative formula [K.&M. 2006]:

$$\frac{1}{\rho^*} = \inf\Big\{ \frac{1}{2}\|\nabla\psi\|_2^2 \colon \psi \in H_0^1, \|\psi\|_{2p} = 1 \Big\}.$$

## Hausdorff dimension spectrum

**Thick points in $d=2$, $p\in\mathbb{N}$ [Dembo, Peres, Rosen, Zeitouni 2001-2]**

$$\sup_{x\in S}\limsup_{r\downarrow 0}\frac{\ell(B_r(x))}{r^2[\log r]^{2p}}=\Big(\frac{2}{p}\Big)^p,$$

and

$$\dim\Big\{x\in\mathbb{R}^d\colon \limsup_{r\downarrow 0}\frac{\ell(B_r(x))}{r^2[\log 1/r]^{2p}}=a\Big\}=\big[2-pa^{1/p}\big]_+,\qquad a\geq 0.$$

Here, many intersections come mainly from many returns.

**Thick points in $d=3$, $p=2$ [K.&M. 2002]**

$$\sup_{x\in S}\limsup_{r\downarrow 0}\frac{\ell(B_r(x))}{r[\log 1/r]^2}=\Big(\frac{\rho^*(B_1(0))}{2}\Big)^2,$$

and

$$\dim\Big\{x\in\mathbb{R}^d\colon \limsup_{r\downarrow 0}\frac{\ell(B_r(x))}{r[\log 1/r]^2}=a\Big\}=\Big[1-\sqrt{a}\frac{2}{\rho^*(B_1(0))}\Big]_+,\qquad a\geq 0.$$

Here, many intersections come from finitely many, very long stays.

## Law of large masses

Next goal: Large deviation principle for $\ell/\ell(U)$ on $\{\ell(U) > a\}$ as $a \to \infty$.

In particular: $\ell/\ell(U)$ should converge towards the set of maximisers.

Idea: More detailed test functions for describing mixed $k$-th moments.

## Law of large masses

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### Mixed stretched-exponential moments, [K.&M. 2006]

For any nonnegative bounded functions $\phi_1, \dots, \phi_n$,

$$\mathbb{E}\Big[\exp\Big\{\sum_{j=1}^n \langle \phi_j^{2p}, \ell\rangle^{1/p}\Big\}\Big] \begin{cases} < \infty & \text{if } \Theta(\phi) > 1, \\ = \infty & \text{if } \Theta(\phi) < 1, \end{cases}$$

where

$$\Theta(\phi) = \inf\Big\{\frac{p}{2}\|\nabla\psi\|_2^2 \colon \sum_{j=1}^n \|\phi_j \psi\|_{2p}^2 = 1\Big\}.$$

### Law of large masses [K.&M. 2006]

If $\mathfrak{M}$ denotes the set of minimisers $\psi^{2p}$, then

$$\lim_{a\to\infty} \mathbb{P}\big(\mathrm{d}(\tfrac{\ell}{\ell(U)}, \mathfrak{M}) > \varepsilon | \ell(U) > a\big) = 0, \qquad \varepsilon > 0.$$

## Comments

- Derivation of high $k$-th moments via the lemma

$$\lim_{k\to\infty}\frac{1}{k}\log\mathbb{E}\Big[\frac{1}{k!^p}X^{kp}\Big]=-p\log\frac{\Theta}{p}\qquad\Longrightarrow\qquad \mathbb{E}[e^X]\begin{cases}<\infty & \text{if } \Theta>1,\\ =\infty & \text{if } \Theta<1.\end{cases}$$

- However, approach with $k$-th moments not suitable for deriving large-deviations principle (via the Gärtner-Ellis lemma)!
- Reason: Limiting functional of $k$-moments not Gâteaux-differentiable in the test functions in any sense.
- Reason: True rate function not convex. $\Longrightarrow$ Different approaches necessary.

## Large-deviations principle

New setting: deterministic time $\to \infty$, motions restricted to staying in a compact set.

Introduce the occupation measures $\ell_t^{(i)} = \int_0^t \mathrm{d}s\, \delta_{W_i(s)}$ and the ISLT

$$\ell_{tb}(A) = \int_A \mathrm{d}y \prod_{i=1}^p \ell_{tb_i}(\mathrm{d}y), \qquad b = (b_1, \dots, b_p) \in (0,\infty)^p.$$

### LDP, [K.&Mukherjee 2013]

Under $\mathbb{P}(\cdot \cap \bigcap_{i=1}^p \{\tau_i > tb_i\})$, as $t \to \infty$, the tuple

$$\Big( \frac{\ell_{tb}}{t^p \prod_{i=1}^p b_i}; \frac{\ell^{(1)}}{tb_1}, \dots, \frac{\ell^{(p)}}{tb_p} \Big)$$

satisfies an LDP on the set $\mathscr{M} \times \mathscr{M}_1^p$ with speed $t$ and rate function

$$I_b\Big( \prod_{i=1}^p \psi_i^2; \psi_1^2, \dots, \psi_p^2 \Big) = \frac{1}{2} \sum_{i=1}^p b_i \|\nabla \psi_i\|_2^2.$$

## Comments

- Contains the famous Donsker-Varadhan-Gärtner LDP as a special case.
- Gives a rigorous meaning to the above formula for the ISLT in the high-density limit.
- Also $\ell/\ell(U)$ satisfies an LDP under $\{\ell(U)>a\}$ with speed $a^{1/p}$ and rate function

$$\psi^{2p}\mapsto \inf\Big\{\frac{1}{2}\sum_{i=1}^{p}\|\nabla\psi\|_2^2\colon \psi^{2p}=\prod_{i=1}^{p}\psi_i^2\Big\}.$$

- Proof steps:
  (1) LDP for an $\varepsilon$-smoothed version (easy)
  (2) $\Gamma$-convergence of the rate function (more work)
  (3) tightness of exponential approximation (very heavy).
  In (3) we use $k$-th moments for difference of integrals against continuous bounded test functions $f$:

$$\mathbb{E}^{(tb)}\Big[\big|\langle \ell_{tb}-\ell_{tb,\varepsilon},f\rangle\big|^k\Big]\leq k!^p C_\varepsilon^k,\qquad k\in\mathbb{N},t\in(0,\infty),$$

with $\lim_{\varepsilon\downarrow 0}C_\varepsilon=0$.

## Analogous results for random walks

The method of high polynomial moments was also fruitful in the spatially discrete setting:

(1) for self-intersections of a single random walk and

(2) for mutual intersection local time of several walks in the super-critical dimensions:

**Rescaled Self-ISLT, [van der Hofstad/K./M. 2006]**

Let $(S_t)_{t\in[0,\infty)}$ denote a continuous-time simple random walk in $\mathbb{Z}^d$ with local times $\ell_t(z) = \int_0^t \mathrm{d}r\, \delta_z(S_r)$ with $p > 0$ small enough, then for any $1 \ll \alpha_t \ll t^{1/(d+1)}$,

$$\mathbb{E}\big(\|\ell_t\|_p^{pk} \mathbb{1}\{S_{[0,t]} \subset B_{L\alpha_t}\}\big) \le k^{kp} C^k \alpha_t^{k[d+(2-d)p]}, \qquad k \ge \frac{t}{\alpha_t^2}.$$

**ISLT in super-critical dimensions, [Chen/M. 2009]**

The (mutual) intersection local time $I$ of $p$ random walks in $\mathbb{Z}^d$ with $d > \frac{2p}{p-1}$ satisfies

$$\lim_{a\to\infty} a^{-1/p} \log \mathbb{P}(I > a) = -p\chi_{d,p},$$

where

$$\chi_{d,p} = \inf\Big\{\frac{1}{2}\|\nabla^{(\mathrm{d})} g\|_2^2 \colon g \in \ell^{2p}(\mathbb{Z}^d), \|g^2\|_p = 1\Big\}.$$

- Elegant compactification in terms of periodisation of the Green function.
- I suppressed many other people's works ...