A Local Projection Stabilization/Continuous Galerkin–Petrov Method for Incompressible Flow Problems

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March 22, 2018

Abstract

A local projection stabilization (LPS) method in space is considered to approximate the evolutionary Oseen equations. Optimal error bounds with constants independent of the viscosity parameter are obtained in the continuous-in-time case for both the velocity and pressure approximation. In addition, the fully discrete case in combination with higher order continuous Galerkin–Petrov (cGP) methods is studied. Error estimates of order $k+1$ are proved, where k denotes the polynomial degree in time, assuming that the convective term is time-independent. Numerical results show that the predicted order is also achieved in the general case of time-dependent convective terms.

AMS subject classification: 65M12, 65M15, 65M60
Keywords: Evolutionary Oseen problem, inf-sup stable pairs of finite element spaces, local projection stabilization (LPS) methods, continuous Galerkin–Petrov (cGP) methods

1 Introduction

The behavior of incompressible fluid flows is modeled by the incompressible Navier–Stokes equations. Analyzing numerical schemes for these equations faces several difficulties. First, the unresolved problem of the uniqueness of the weak solution of the Navier–Stokes equations in three dimensions requires to assume uniqueness, which is usually done by assuming sufficient regularity of the weak solution. Moreover, the estimate of the nonlinear term often uses the Gronwall lemma, such that an exponential factor occurs in the error bounds, depending on some norm of the velocity, e.g., on $\|\nabla \boldsymbol{u}\|_\infty$ as in [20]. As result, the obtained estimates are by far too pessimistic in practice. For these reasons, this paper will deal, with respect to the numerical analysis, with a related but simpler problem, namely the evolutionary or transient Oseen equations. They read in dimensionless form as follows:

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Find $\boldsymbol{u}(t,\boldsymbol{x}) : (0,T] \times \Omega \to \mathbb{R}^d$, $d \in \{2,3\}$, and $p(t,\boldsymbol{x}) : (0,T] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \sigma\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } (0,T]\times\Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } (0,T]\times\Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } (0,T]\times\partial\Omega, \\ \boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_0 && \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with polyhedral Lipschitz boundary $\partial\Omega$, $\nu = \mathrm{Re}^{-1} > 0$ (viscosity) and $\sigma > 0$ are positive constants, $\boldsymbol{b}(t,\boldsymbol{x})$ is a given velocity field with $\operatorname{div}\boldsymbol{b} = 0$, $\boldsymbol{u}_0$ is the initial velocity field, and T is a given final time. Without loss of generality, one can assume $\sigma > 0$, since if it is not the case then a simple change of variable transforms the problem into (1) with $\sigma > 0$, see [19, Sect. 1].

The numerical solution of (1) requires discretizations in time and space. Concerning the temporal discretization, continuous Galerkin–Petrov methods of order $k \geq 1$, cGP(k), will be considered. With respect to space, finite element methods will be studied. Since the paper will study the convection-dominated regime, where ν is smaller than an appropriate norm of $\boldsymbol{b}$ by several orders of magnitude, a stabilization of the standard finite element discretization becomes necessary.

Considering the situation that the viscosity is much smaller than the convection, it is well known that stabilized discretizations have to be applied. The most popular stabilized finite element method is probably the streamline upwind Petrov–Galerkin (SUPG) method from [16, 22]. Often, the SUPG stabilization is used in combination with the pressure-stabilization Petrov–Galerkin (PSPG) method, [29]. However, the SUPG/PSPG method possesses some drawbacks, see [15]: it introduces a velocity-pressure coupling for which no physical explanation is known and the non-symmetry of the stabilization might be of disadvantage. In the time-dependent case, the consistent application of the method leads to a number of additional terms which have to be assembled, see [28, 31]. Because of the drawbacks of the SUPG/PSPG method, we think that it is worth to investigate different approaches in detail, in particular such approaches that are symmetric and that do not introduce an additional velocity-pressure coupling. Local projection stabilization (LPS) methods belong to this class of methods and will be the topic of this paper.

A different approach from this class was studied recently in [19], where a grad-div stabilized method is used to discretize the evolutionary Oseen equations. Optimal bounds for the divergence of the velocity and the $L^2(\Omega)$ norm of the pressure are proved for this method.

The LPS method was originally proposed for the Stokes problem in [12] and it was successfully extended to transport problems in [13]. Finite element analysis for the LPS method applied to the stationary Oseen equations can be found in [14, 34] and to convection-diffusion-reaction problems in [5, 7, 11, 35]. The stabilization term of the LPS method is based on a projection that is defined on the finite element space which approximates the solution and which maps into a discontinuous space. Compared with the standard Galerkin approach, the LPS method provides additional control over (parts of) the fluctuation of the gradient. The method is weakly consistent and the construction should lead to a consistency error that does not spoil the optimal rate of convergence. Originally, the LPS method was proposed as a two-level approach defining the projection spaces on coarser grids. This approach leads to additional couplings between neighboring mesh cell and consequently, the sparsity of the matrix decreases. This drawback does not appear in the one-level approach, where both spaces are defined on the same grid. In this approach, the approximation spaces have to be enriched in comparison with the standard finite element spaces. The additional degrees of freedoms that are introduced by the enrichment can be eliminated using static condensation. Altogether, the one-level approach is, in our opinion, more appealing from the point of view of implementation and this variant of the LPS method will be considered in this paper.

LPS stabilized finite element methods for the evolutionary Oseen problem were studied in [18]. In this paper, the streamline derivative is stabilized with the LPS methodology and an additional augmentation with a grad-div stabilization term is used. Note that this method is a different LPS method than studied here. In the general case, error bounds were derived in [18] under the condition that a certain measure of the mesh size is of the same order as the square root of the viscosity. To avoid the restriction on the mesh size for small viscosity, finite element spaces were then considered that satisfy a certain element-wise compatibility condition between the finite element velocity space and the projection space. Optimal error bounds for the pressure were not obtained in [18]. A similar LPS method, applied to the time-dependent Navier–Stokes equations, was analyzed in [8] and error estimates for the velocity in the continuous-in-time case were proved. An analysis of the fully discretized so-called high-order term-by-term LPS method was presented in [2].

As mentioned above, cGP(k) methods, which treat the temporal derivative in a finite element manner, will be considered as temporal discretization. For incompressible flow problems, usually θ-schemes are used. These schemes are simple to implement, however, they are at most of second order, like the Crank–Nicolson scheme or the fractional-step θ-scheme. In addition, they do not allow an efficient adaptive time step control. There are only few studies, e.g., [24, 27, 30], which consider higher order schemes, like diagonally implicit Runge–Kutta (DIRK) methods, Rosenbrock–Wanner (ROW) methods, or just cGP(2). To the best of our knowledge, there is no numerical analysis available for the first two classes of schemes applied to incompressible flow problems or even to convection-diffusion equations. The situation is different for cGP(k), which are a class of finite element methods in time using discrete solution spaces with continuous piecewise polynomials of degree less than or equal to k and test spaces consisting of discontinuous polynomials of degree up to order $k-1$. This setup enables the performance of a standard time marching algorithm and it does not require the solution of a global system in space and time as in space-time finite element methods.

The cGP method in time for the heat equation was investigated in [10]. Optimal error estimates and super-convergence results were derived at the end point of the discrete time intervals. For nonlinear systems of ordinary differential equations in d space dimensions, the methods cGP(k) were studied in [37] even in an abstract Hilbert space setting. A-stability and optimal error estimates were proved. It was also shown that cGP(k) methods have an energy decreasing property for the gradient flow equation of an energy functional. In [5], transient convection-diffusion-reaction equations were considered using cGP(k) in time combined with LPS in space. Optimal a-priori error estimates were derived for the fully discrete scheme. It was shown numerically that cGP(k) is super-convergent of order $(k+2)$ in the integrated norm and of order $2k$ at discrete time points. In addition, the obtained results were compared with discontinuous Galerkin (dG) time stepping schemes. Numerical studies for the time-dependent Stokes equations in [23], the evolutionary Oseen equations in [4], and transient convection-diffusion-reaction equations in [5] showed the expected orders of convergence for cGP(k), $k \in \{1, 2\}$. The dG(k) method was analyzed for the transient Stokes equations in [1]. In addition, the higher order convergence of cGP(2) compared with the discontinuous Galerkin discretization dG(1), both methods possessing the same complexity, was demonstrated. An efficient adaptive time step control can be performed with cGP(k) methods, e.g., as applied in [3] to transient convection-diffusion-reaction equations. The adaptive time step control is based on a post-processed discrete solution, which can be computed with affordable costs. It was shown that the adaptive time step control leads to lengths of the time steps that properly reflect the dynamics of the solution.

However, there is also a certain drawback of cGP(k) methods for $k \geq 2$: a coupled system of k equations has to be solved at each time instance. Utilizing a clever construction proposed in [37], the coupling is not strong, but it cannot be removed completely. Efficient solvers for this coupled problem in case of the Navier–Stokes equations were studied in [24], where a coupled multigrid method with Vanka-type smoothers was used.

Altogether, cGP(k) is in our opinion an attractive alternative to θ-schemes since a higher order in time can be achieved and an efficient and inexpensive time step control is possible.

The goal of this paper consists in studying the combination of the LPS method in space with the cGP(k) method in time. The numerical analysis will be performed for the transient Oseen equations (1). Thus, this paper presents the first numerical analysis of a higher order time stepping scheme for an incompressible flow problem with convection. In the continuous-in-time case, optimal error bounds for velocity and pressure with constants that do not depend on the viscosity parameter ν are derived with the assumption that the solution is sufficiently smooth. In addition, error estimates for the fully discrete problem of order $k+1$ are proved, assuming, as in other recently published papers, that the convective term does not depend on time. Numerical results show that the predicted order can be also observed in the case of time-dependent convective terms.

The remainder of the paper is organized as follows: Section 2 introduces the basic notation, it presents some preliminaries, and the semi-discretization (continuous-in-time) of the LPS method will be described. In Section 3, the error bounds for the semi-discrete problem are derived. Section 4 presents the error analysis of the fully discrete problem using a temporal discretization with a cGP(k) method. Numerical studies can be found in Section 5.

2 Preliminaries

Throughout this paper, standard notation and conventions will be used. For a measurable set $G \subset \mathbb{R}^d$, the inner product in $L^2(G)$, $L^2(G)^d$, and $L^2(G)^{d\times d}$ will be denoted by $(\cdot,\cdot)_G$. The norm and the semi-norm in $W^{m,p}(G)$ are given by $\|\cdot\|_{m,p,G}$ and $|\cdot|_{m,p,G}$, respectively. In the case $p=2$, $H^m(G)$, $\|\cdot\|_{m,G}$, and $|\cdot|_{m,G}$ are written instead of $W^{m,2}(G)$, $\|\cdot\|_{m,2,G}$, and $|\cdot|_{m,2,G}$. If $G=\Omega$, the index G in inner products, norms, and semi-norms will be omitted. The dual pairing between a space Z and its dual Z' will be denoted by $\langle\cdot,\cdot\rangle$. The temporal derivative of a function f is denoted by $\partial_t f$ and the i-th temporal derivative by $\partial_t^i f$. The subspace of functions from $H^1(\Omega)$ having a vanishing boundary trace is denoted by $H_0^1(\Omega)$ with $H^{-1}(\Omega)$ being its dual space with the associated norm $\|v\|_{-1} = \sup_{\varphi\in H_0^1(\Omega)\setminus\{0\}} \frac{\langle v,\varphi\rangle}{\|\nabla\varphi\|_0}$. Let Z be a Banach space with norm $\|\cdot\|_Z$, then the following spaces are defined

$$
\begin{aligned}
L^2(0,t;Z) &:= \left\{ v \,:\, (0,t)\to Z \,:\, \int_0^t \|v(s)\|_Z^2 \, ds < \infty \right\},\\
H^1(0,t;Z) &:= \left\{ v\in L^2(0,t;Z) \,:\, \partial_t v \in L^2(0,t;Z)\right\},\\
C(0,t;Z) &:= \{ v \,:\, (0,t)\to Z \,:\, v \text{ is continuous with respect to time}\},
\end{aligned}
$$

where $\partial_t v$ is the temporal derivative of v in the sense of distributions. If $t=T$, then the abbreviations $L^2(Z)$, $H^1(Z)$, and $C(Z)$ are used and it will not be indicated whether it is a scalar-valued or vector-valued space.

In order to derive a variational form of (1), the spaces

$$
V := H_0^1(\Omega)^d, \quad Q := L_0^2(\Omega), \quad X := \left\{ \boldsymbol{v} \in L^2(V),\ \partial_t \boldsymbol{v} \in L^2(V') \right\}
$$

and the bilinear form

$$
a\big((\boldsymbol{u},p);(\boldsymbol{v},q)\big) := \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u},\boldsymbol{v}) + (\sigma\boldsymbol{u},\boldsymbol{v}) - (\operatorname{div}\boldsymbol{v},p) + (\operatorname{div}\boldsymbol{u},q)
$$

are introduced. Then, a variational form of (1) reads as follows:

Find $\boldsymbol{u} \in X$ and $p \in L^2(Q)$ such that

$$\langle \partial_t \boldsymbol{u}(t), \boldsymbol{v}(t)\rangle + a\big((\boldsymbol{u}(t), p(t)); (\boldsymbol{v}(t), q(t))\big) = (\boldsymbol{f}(t), \boldsymbol{v}(t)) \quad \forall\, \boldsymbol{v} \in L^2(V),\ q \in L^2(Q) \tag{2}$$

for almost all $t \in (0, T]$ and $\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0$. Note that this initial condition is well defined since functions belonging to X are continuous in time.

If the initial condition $\boldsymbol{u}_0$ is different from $\boldsymbol{0}$, the velocity $\boldsymbol{u}$ can be decomposed in the form

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \boldsymbol{\psi}(t), \quad \boldsymbol{\psi} \in X_0 := \{\boldsymbol{v} \in X : \ \boldsymbol{v}(0, \cdot) = \boldsymbol{0}\}.$$

Then for the given initial velocity field $\boldsymbol{u}_0$, one has to find $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{\psi}(t)$, with $\boldsymbol{\psi}(t) \in X_0$, and $p \in L^2(Q)$, where $(\boldsymbol{\psi}, p)$ is the solution of the problem

$$(\partial_t \boldsymbol{\psi}(t), \boldsymbol{v}(t)) + a\big((\boldsymbol{\psi}(t), p(t)); (\boldsymbol{v}(t), q(t))\big) = (\boldsymbol{g}(t), \boldsymbol{v}(t))$$

with

$$(\boldsymbol{g}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) - \nu(\nabla \boldsymbol{u}_0, \nabla \boldsymbol{v}) - ((\boldsymbol{b} \cdot \nabla)\boldsymbol{u}_0, \boldsymbol{v}) - (\sigma \boldsymbol{u}_0, \boldsymbol{v}).$$

For this reason, one can assume $\boldsymbol{u}_0 = \boldsymbol{0}$, which will be done in the sequel. Note that this choice of the initial condition will result in error bounds that do not contain contributions depending on $\boldsymbol{u}_0$.

Let $\Pi : L^2(\Omega)^d \to H^{\mathrm{div}}$ be the Leray projector that maps each function in $L^2(\Omega)^d$ onto its divergence-free part, where the Hilbert space H^{div} is defined by $H^{\mathrm{div}} = \{\boldsymbol{v} \in L^2(\Omega)^d : \nabla \cdot \boldsymbol{v} = 0,\ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\}$. The Stokes operator in Ω is defined by

$$A : \mathcal{D}(A) \subset H^{\mathrm{div}} \to H^{\mathrm{div}}, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\Omega)^d \cap V^{\mathrm{div}},$$

where the space $V^{\mathrm{div}} = \{\boldsymbol{v} \in H_0^1(\Omega)^d : \nabla \cdot \boldsymbol{v} = 0\}$ is equipped with the inner product of $H_0^1(\Omega)^d$.

Let $\{\mathcal{T}_h\}$ be a family of quasi-uniform triangulations of Ω into compact d-simplices, quadrilaterals, or hexahedra such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$. The diameter of $K \in \mathcal{T}_h$ will be denoted by h_K and the mesh size h is defined by $h := \max_{K\in\mathcal{T}_h} h_K$. Let $Y_h \subset H_0^1(\Omega)$ be a finite element space of scalar, continuous, piecewise mapped polynomial functions over $\mathcal{T}_h$. The finite element space V_h for approximating the velocity field is given by $V_h := Y_h^d \cap V$. The pressure is discretized using a finite element space $Q_h \subset Q$ of continuous or discontinuous functions with respect to $\mathcal{T}_h$. In this paper, inf-sup stable pairs (V_h, Q_h) will be considered, i.e., there is a positive constant β_0, independent of the triangulation, such that

$$\inf_{q_h\in Q_h\setminus\{0\}} \sup_{\boldsymbol{v}_h\in V_h\setminus\{\boldsymbol{0}\}} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_1 \|q_h\|_0} \geq \beta_0 > 0. \tag{3}$$

Since it is assumed that the family of meshes is quasi-uniform, the following inverse inequality holds

$$\|\boldsymbol{v}_h\|_{m,K} \leq C_{\mathrm{inv}} h_K^{l-m} \|\boldsymbol{v}_h\|_{l,K}, \tag{4}$$

for each $\boldsymbol{v}_h \in V_h$ and $0 \leq l \leq m \leq 1$, e.g., see [17, Thm. 3.2.6].

The space of discretely divergence-free functions is denoted by

$$V_h^{\mathrm{div}} = \{\boldsymbol{v}_h \in V_h : (\nabla \cdot \boldsymbol{v}_h, q_h) = 0 \quad \forall\, q_h \in Q_h\}.$$

The linear operator $A_h : V_h^{\mathrm{div}} \to V_h^{\mathrm{div}}$ is defined by

$$(A_h \boldsymbol{v}_h, \boldsymbol{w}_h) = (\nabla \boldsymbol{v}_h, \nabla \boldsymbol{w}_h) \quad \forall\, \boldsymbol{w}_h \in V_h^{\mathrm{div}}. \tag{5}$$

From (5), one concludes that

$$\|A_h^{1/2}\boldsymbol{v}_h\|_0 = \|\nabla \boldsymbol{v}_h\|_0, \quad \|\nabla A_h^{-1/2}\boldsymbol{v}_h\|_0 = \|\boldsymbol{v}_h\|_0 \quad \forall\, \boldsymbol{v}_h \in V_h^{\mathrm{div}}. \tag{6}$$

The so-called discrete Leray projection $\Pi_h^{\mathrm{div}} : L^2(\Omega)^d \to V_h^{\mathrm{div}}$, being the L^2-orthogonal projection of $L^2(\Omega)^d$ onto V_h^{div}, is given by

$$(\Pi_h^{\mathrm{div}}\boldsymbol{v}, \boldsymbol{w}_h) = (\boldsymbol{v}, \boldsymbol{w}_h) \quad \forall\, \boldsymbol{w}_h \in V_h^{\mathrm{div}}. \tag{7}$$

By definition, it follows that this projection is stable in the L^2 norm: $\|\Pi_h^{\mathrm{div}}\boldsymbol{v}\|_0 \le \|\boldsymbol{v}\|_0$ for all $\boldsymbol{v} \in L^2(\Omega)^d$.

The continuous-in-time standard Galerkin finite element method applied to (2) consists in finding $\boldsymbol{u}_h \in H^1(V_h)$ and $p_h \in L^2(Q_h)$ such that

$$(\partial_t \boldsymbol{u}_h(t), \boldsymbol{v}_h) + a\big((\boldsymbol{u}_h(t), p_h(t)); (\boldsymbol{v}_h, q_h)\big) = (\boldsymbol{f}(t), \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in V_h,\ q_h \in Q_h.$$

In the convection-dominated case, it is well-known that this method is unstable, unless h is sufficiently small. The use of a stabilized discretization is necessary.

This paper concentrates on the one-level variant of the LPS method in which approximation and projection spaces are defined on the same mesh. Let $D(K)$, $K \in \mathcal{T}_h$, be local finite-dimensional spaces and $\pi_K : L^2(K) \to D(K)$ the local L^2 projection into $D(K)$. The local fluctuation operator $\kappa_K : L^2(K) \to L^2(K)$ is given by $\kappa_K v := v - \pi_K v$. It is applied component-wise to vector-valued and tensor-valued arguments. The stabilization term S_h is defined by

$$S_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{K \in \mathcal{T}_h} \mu_K \big(\kappa_K \nabla \boldsymbol{u}_h, \kappa_K \nabla \boldsymbol{v}_h\big)_K,$$

where $\{\mu_K\}$, $K \in \mathcal{T}_h$, are non-negative constants. This kind of LPS method gives additional control on the fluctuation of the gradient. Also other variants of LPS methods are possible, e.g., by replacing in both arguments of $S_h(\cdot,\cdot)$ the gradient $\nabla \boldsymbol{w}_h$ by the derivative in the streamline direction $(\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_h$ or, even better [32, 33], by $(\boldsymbol{b}_K \cdot \nabla)\boldsymbol{w}_h$, where $\boldsymbol{b}_K$ is a piecewise constant approximation of $\boldsymbol{b}$. But in this method, one has to add the grad-div term $(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{v}_h)$ to S_h, see [36].

For performing the numerical analysis, the linear operator $C_h : V_h^{\mathrm{div}} \to V_h^{\mathrm{div}}$ with

$$(C_h \boldsymbol{v}_h, \boldsymbol{w}_h) = \sum_{K \in \mathcal{T}_h} \mu_K (\kappa_K \nabla \boldsymbol{v}_h, \kappa_K \nabla \boldsymbol{w}_h)_K \quad \forall\, \boldsymbol{v}_h, \boldsymbol{w}_h \in V_h^{\mathrm{div}}, \tag{8}$$

the linear operator $D_h : L^2(\Omega) \to V_h^{\mathrm{div}}$ with

$$(D_h q, \boldsymbol{w}_h) = (\operatorname{div} \boldsymbol{w}_h, q) \quad \forall\, \boldsymbol{w}_h \in V_h^{\mathrm{div}}, \tag{9}$$

the stabilized bilinear form

$$a_h\big((\boldsymbol{u}, p), (\boldsymbol{v}, q)\big) = a\big((\boldsymbol{u}, p); (\boldsymbol{v}, q)\big) + S_h(\boldsymbol{u}, \boldsymbol{v})$$

on the product space (V_h, Q_h), and the mesh-dependent norm

$$|||\boldsymbol{v}||| := \left\{ \nu |\boldsymbol{v}|_1^2 + \sigma \|\boldsymbol{v}\|_0^2 + \sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{v}\|_{0,K}^2 \right\}^{1/2}$$

are introduced.

It will be assumed that $\boldsymbol{b} \in L^\infty(L^\infty(\Omega) \cap H^{\mathrm{div}}(\Omega))$ and $\nabla \cdot \boldsymbol{b}(t) = 0$ for almost all $t \in [0,T]$. Then, a straightforward calculation shows that

$$a_h\left((\boldsymbol{v}_h, q_h), (\boldsymbol{v}_h, q_h)\right) = |||\boldsymbol{v}_h|||^2 \quad \forall\, \boldsymbol{v}_h \in V_h,\, q_h \in Q_h. \tag{10}$$

The semi-discrete LPS problem reads:

Find $\boldsymbol{u}_h \in H^1(V_h)$ and $p_h \in L^2(Q_h)$ such that for almost every $t \in (0,T]$

$$\big(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h\big) + a_h\left((\boldsymbol{u}_h, p_h); (\boldsymbol{v}_h, q_h)\right) = \big(\boldsymbol{f}, \boldsymbol{v}_h\big) \quad \forall\, \boldsymbol{v}_h \in V_h,\, q_h \in Q_h. \tag{11}$$

For performing the analysis of LPS schemes, certain compatibility conditions between the approximation space and local projection space have to be satisfied, see [34].

Assumption A1 There are interpolation operators $j_h : H^2(\Omega)^d \to V_h$ and $i_h : H^2(\Omega) \to Q_h$ with the approximation properties

$$\left.\begin{array}{rcll} \|\boldsymbol{w} - j_h \boldsymbol{w}\|_{0,K} + h_K |\boldsymbol{w} - j_h \boldsymbol{w}|_{1,K} & \le & C h_K^l \|\boldsymbol{w}\|_{l,K} & \forall\, \boldsymbol{w} \in H^l(K)^d,\ 2 \le l \le r+1, \\ \|q - i_h q\|_{0,K} + h_K |q - i_h q|_{1,K} & \le & C h_K^l \|q\|_{l,K} & \forall\, q \in H^l(K),\ 2 \le l \le r, \end{array}\right\} \tag{12}$$

for all $K \in \mathcal{T}_h$. The pressure interpolation operator i_h satisfies the orthogonality condition

$$(q - i_h q, r_h)_K = 0 \qquad \forall\, q \in Q \cap H^2(\Omega),\ r_h \in D(K). \tag{13}$$

The pairs $V_h/Q_h = \mathbb{Q}_r/\mathbb{P}_{r-1}^{\mathrm{disc}}$ together with $D(K) = \mathbb{P}_{r-1}(K)$ fulfill for $r \ge 2$ assumption A1 if j_h is the usual Lagrangian interpolation operator and i_h the L^2 projection. Further examples of inf-sup stable pairs V_h/Q_h, associated interpolation operators j_h and i_h, and projection spaces satisfying assumption A1 can be found in [36].

Assumption A2 The fluctuation operator satisfies the approximation property

$$\|\kappa_K q\|_{0,K} \le C h_K^l |q|_{l,K} \qquad \forall\, K \in \mathcal{T}_h,\ \forall\, q \in H^l(K),\ 0 \le l \le r. \tag{14}$$

For performing the numerical analysis, the steady-state Stokes problem

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{g} && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \end{aligned} \tag{15}$$

will be considered. The standard Galerkin approximation $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ is the solution of the mixed finite element approximation to (15), given by

$$\begin{aligned} \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (\operatorname{div} \boldsymbol{v}_h, p_h) &= (\boldsymbol{g}, \boldsymbol{v}_h) && \forall\, \boldsymbol{v}_h \in V_h, \\ (\nabla \cdot \boldsymbol{u}_h, q_h) &= 0 && \forall\, q_h \in Q_h. \end{aligned} \tag{16}$$

From [21, 25], it is known that the following estimates hold

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \le C \left(\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \nu^{-1} \inf_{q_h \in Q_h} \|p - q_h\|_0 \right), \tag{17}$$

$$\|p - p_h\|_0 \le C \left(\nu \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \inf_{q_h \in Q_h} \|p - q_h\|_0 \right), \tag{18}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \le C h \left(\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \nu^{-1} \inf_{q_h \in Q_h} \|p - q_h\|_0 \right). \tag{19}$$

It can be observed that the error bounds for the velocity depend on a negative power of ν.

As suggested in [19], a projection of $(\boldsymbol{u},p)$ into $V_h \times Q_h$ is used in the finite element analysis, where the bounds for the velocity are uniform in ν. Let $(\boldsymbol{u},p)$ be the solution of (1) with $\boldsymbol{u} \in H^1\left(V \cap H^{l+1}(\Omega)^d\right)$, $p \in L^2\left(Q \cap H^l(\Omega)\right)$, $l \geq 1$, and define the right-hand side of the Stokes problem (15) by

$$\boldsymbol{g} = \boldsymbol{f} - \partial_t \boldsymbol{u} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - \sigma\boldsymbol{u} - \nabla p, \tag{20}$$

such that $(\boldsymbol{u},0)$ is the solution of (15). Denoting the corresponding Galerkin approximation in $V_h \times Q_h$ by $(\boldsymbol{s}_h, l_h)$, one obtains from (17)–(19)

$$\|\boldsymbol{u} - \boldsymbol{s}_h\|_0 + h\|\boldsymbol{u} - \boldsymbol{s}_h\|_1 \leq Ch^{l+1}\|\boldsymbol{u}\|_{l+1}, \tag{21}$$

$$\|l_h\|_0 \leq C\nu h^l \|\boldsymbol{u}\|_{l+1}, \tag{22}$$

where the constant C does not depend on ν.

Remark 1. *Assuming the necessary smoothness in time and considering* (15) *with*

$$\boldsymbol{g} = \boldsymbol{g}^i = \partial_t^i\left(\boldsymbol{f} - \partial_t\boldsymbol{u} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - \sigma\boldsymbol{u} - \nabla p\right), \quad i \geq 1,$$

one can derive error bounds of form (21) *and* (22) *also for* $\partial_t^i\boldsymbol{u} - \boldsymbol{s}_h(\boldsymbol{g}^i)$ *and* $l_h(\boldsymbol{g}^i)$*, where* $\boldsymbol{s}_h(\boldsymbol{g}^i)$ *and* $l_h(\boldsymbol{g}^i)$ *denote the solution of* (16) *with right-hand side* $\boldsymbol{g} = \boldsymbol{g}^i$*. Hence, the estimates*

$$\begin{aligned} \|\partial_t^i\boldsymbol{u} - \boldsymbol{s}_h(\boldsymbol{g}^i)\|_0 + h\|\partial_t^i\boldsymbol{u} - \boldsymbol{s}_h(\boldsymbol{g}^i)\|_1 &\leq Ch^{l+1}\|\partial_t^i\boldsymbol{u}\|_{l+1}, \\ \|l_h(\boldsymbol{g}^i)\|_0 &\leq C\nu h^l\|\partial_t^i\boldsymbol{u}\|_{l+1}, \end{aligned}$$

can be derived.

3 Error analysis for the continuous-in-time case

In this section, error bounds for velocity and pressure will be derived with constants independent of ν for a sufficiently smooth solution. The analysis follows the lines of [19].

Theorem 2. *Let* $(\boldsymbol{u},p)$ *be the solution of* (2) *and let* $(\boldsymbol{u}_h,p_h)$ *be the solution of* (11)*. Assume* $\boldsymbol{b} \in L^\infty(L^\infty)$ *and the regularities*

$$(\boldsymbol{u},p) \in L^2(H^{r+1}) \times L^2(H^r), \quad \partial_t\boldsymbol{u} \in L^2(H^r). \tag{23}$$

Choosing the stabilization parameters of the LPS method such that $\mu_K \sim 1$ *with respect to the mesh width, then the following error estimate holds for all* $t \in (0,T]$

$$\begin{aligned} &\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|_0^2 + \nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(0,t;L^2)}^2 + \sigma\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(0,t;L^2)}^2 \\ &\quad + \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(0,t;L^2(K))}^2 \\ &\leq Ch^{2r}\left(\|\boldsymbol{u}\|_{L^2(0,t;H^{r+1})}^2 + \|\partial_t\boldsymbol{u}\|_{L^2(0,t;H^r)}^2 + \|p\|_{L^2(0,t;H^r)}^2\right), \end{aligned} \tag{24}$$

where $C = C\left(\sigma, \|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$ *is independent of* ν *and* h.

Proof. The proof of the error bound is based on the comparison of $(\boldsymbol{u}_h,p_h)$ with the approximation $(\boldsymbol{s}_h,l_h)$ of the Stokes equations with right-hand side (20). Let $\boldsymbol{e}_h = \boldsymbol{u}_h - \boldsymbol{s}_h$, then a direct calculation yields

$$\begin{aligned} &(\partial_t\boldsymbol{e}_h,\boldsymbol{v}_h) + a_h((\boldsymbol{e}_h,p_h-l_h),(\boldsymbol{v}_h,q_h)) \\ &= (\partial_t(\boldsymbol{u}-\boldsymbol{s}_h),\boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h),\boldsymbol{v}_h) \\ &\quad + (\nabla p,\boldsymbol{v}_h) - S_h(\boldsymbol{s}_h,\boldsymbol{v}_h) \quad \forall\ \boldsymbol{v}_h \in V_h,\ q_h \in Q_h. \end{aligned} \tag{25}$$

Taking $(\boldsymbol{v}_h, q_h) = (\boldsymbol{e}_h, p_h - l_h)$ in (25), one gets with integrating by parts, using that $\boldsymbol{e}_h$ has discrete divergence equal to zero, and (13)

$$(\nabla p, \boldsymbol{e}_h) = -(p, \nabla\cdot\boldsymbol{e}_h) = -(p - i_h p, \nabla\cdot\boldsymbol{e}_h) = (i_h p - p, \kappa_K \nabla\cdot\boldsymbol{e}_h).$$

With the Cauchy–Schwarz inequality and Hölder's inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + \nu\|\nabla\boldsymbol{e}_h\|_0^2 + \sigma\|\boldsymbol{e}_h\|_0^2 + \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2 \\
&\le \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_0\|\boldsymbol{e}_h\|_0 + \|\boldsymbol{b}\|_\infty\|\nabla(\boldsymbol{u}-\boldsymbol{s}_h)\|_0\|\boldsymbol{e}_h\|_0 + \sigma\|\boldsymbol{u}-\boldsymbol{s}_h\|_0\|\boldsymbol{e}_h\|_0 \\
&\quad + \left(\sum_{K\in\mathcal{T}_h}\mu_K^{-1}\|p - i_h p\|_{0,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2\right)^{1/2} + |S_h(\boldsymbol{s}_h,\boldsymbol{e}_h)|\,.
\end{aligned}$$

Now, the term with the stabilization has to be bounded. The Cauchy–Schwarz inequality gives

$$\begin{aligned}
S_h(\boldsymbol{s}_h,\boldsymbol{e}_h) &= S_h(\boldsymbol{s}_h-\boldsymbol{u},\boldsymbol{e}_h) + S_h(\boldsymbol{u},\boldsymbol{e}_h) \\
&\le S_h^{1/2}(\boldsymbol{s}_h-\boldsymbol{u},\boldsymbol{s}_h-\boldsymbol{u})S_h^{1/2}(\boldsymbol{e}_h,\boldsymbol{e}_h) + S_h^{1/2}(\boldsymbol{u},\boldsymbol{u})S_h^{1/2}(\boldsymbol{e}_h,\boldsymbol{e}_h).
\end{aligned} \tag{26}$$

Applying the stability of the fluctuation operator κ_K and the choice $\mu_K \sim 1$ of the stabilization parameters yields

$$S_h(\boldsymbol{s}_h,\boldsymbol{e}_h) \le C\left(\|\boldsymbol{s}_h-\boldsymbol{u}\|_1 + \|\kappa_K\nabla\boldsymbol{u}\|_0\right)\left(\sum_{K\in\mathcal{T}_h}\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2\right)^{1/2}, \tag{27}$$

such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + \nu\|\nabla\boldsymbol{e}_h\|_0^2 + \sigma\|\boldsymbol{e}_h\|_0^2 + \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2 \\
&\le \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_0\|\boldsymbol{e}_h\|_0 + \|\boldsymbol{b}\|_\infty\|\nabla(\boldsymbol{u}-\boldsymbol{s}_h)\|_0\|\boldsymbol{e}_h\|_0 + \sigma\|\boldsymbol{u}-\boldsymbol{s}_h\|_0\|\boldsymbol{e}_h\|_0 \\
&\quad + C\big(\|p-i_h p\|_0 + \|\boldsymbol{s}_h-\boldsymbol{u}\|_1 + \|\kappa_K\nabla\boldsymbol{u}\|_0\big)\left(\sum_{K\in\mathcal{T}_h}\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2\right)^{1/2}.
\end{aligned}$$

With Young's inequality and hiding terms on the left-hand side, one obtains

$$\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + 2\nu\|\nabla\boldsymbol{e}_h\|_0^2 + \sigma\|\boldsymbol{e}_h\|_0^2 + \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2 \\
&\le C\left(\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_0^2 + \|\boldsymbol{b}\|_\infty^2\|\nabla(\boldsymbol{u}-\boldsymbol{s}_h)\|_0^2 + \sigma^2\|\boldsymbol{u}-\boldsymbol{s}_h\|_0^2\right) \\
&\quad + C\left(\|p-i_h p\|_0^2 + \|\boldsymbol{s}_h-\boldsymbol{u}\|_1^2 + \|\kappa_K\nabla\boldsymbol{u}\|_0^2\right).
\end{aligned} \tag{28}$$

Assuming now for $t \le T$ the regularities (23), integrating (28) on $(0,t)$, taking into account that $\boldsymbol{e}_h(0) = \boldsymbol{0}$, since $\boldsymbol{u}_0 = \boldsymbol{0}$, and applying estimates (21), (12), and (14), one gets

$$\begin{aligned}
&\|\boldsymbol{e}_h(t)\|_0^2 + 2\nu\|\nabla\boldsymbol{e}_h\|_{L^2(0,t;L^2)}^2 + \sigma\|\boldsymbol{e}_h\|_{L^2(0,t;L^2)}^2 + \sum_{K\in\mathcal{T}_h}\mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|_{L^2(0,t;L^2(K))}^2 \\
&\le Ch^{2r}\left(\|\boldsymbol{u}\|_{L^2(0,t;H^{r+1})}^2 + \|\partial_t\boldsymbol{u}\|_{L^2(0,t;H^r)}^2 + \|p\|_{L^2(0,t;H^r)}^2\right),
\end{aligned} \tag{29}$$

where $C = C\left(\sigma, \|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$ is independent of ν and h.

The final result is obtained by applying the triangle inequality to the left-hand side of (24) and using (29) and (21). □

In the next step of the error analysis, a bound for the pressure error will be derived.

Theorem 3. *Let the assumptions of Theorem 2 hold and let* $\nu \leq 1$*, then it holds*

$$\|p - p_h\|_{L^2(0,t;L^2)} \leq Ch^r \quad \forall\, t \in (0,T], \tag{30}$$

where $C = C\left(\beta_0^{-1}, \|\boldsymbol{u}\|_{L^2(0,t;H^{r+1})}, \|\partial_t \boldsymbol{u}\|_{L^2(0,t;H^r)}, \|p\|_{L^2(0,t;H^r)}, \sigma, \|\boldsymbol{b}\|_{L^\infty(0,t;L^\infty)}\right)$ *is independent of* ν *and* h.

Proof. As usual, the bound is derived on the basis of the discrete inf-sup condition (3). It turns out that the derivation requires in particular a bound for $\|\partial_t \boldsymbol{e}_h\|_{-1}$, which will be proved first. By definition, it is

$$\|\partial_t \boldsymbol{e}_h\|_{-1} = \sup_{\boldsymbol{\varphi} \in H_0^1(\Omega)^d \setminus \{\boldsymbol{0}\}} \frac{|\langle \partial_t \boldsymbol{e}_h, \boldsymbol{\varphi} \rangle|}{\|\nabla \boldsymbol{\varphi}\|_0}.$$

The first step consists in reducing the bound of $\|\partial_t \boldsymbol{e}_h\|_{-1}$ to a bound of $\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0$. From [9, Lemma 3.11], it is known that

$$\|\partial_t \boldsymbol{e}_h\|_{-1} \leq Ch\|\partial_t \boldsymbol{e}_h\|_0 + C\|A^{-1/2}\Pi \partial_t \boldsymbol{e}_h\|_0, \tag{31}$$

where Π is the Leray projector introduced in Section 2. Applying [9, (2.15)] gives

$$\|A^{-1/2}\Pi \partial_t \boldsymbol{e}_h\|_0 \leq Ch\|\partial_t \boldsymbol{e}_h\|_0 + \|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0, \tag{32}$$

with A_h defined in (5). From (31), (32), the symmetry of A_h, (6), and the inverse inequality (4), it follows that

$$\begin{aligned}
\|\partial_t \boldsymbol{e}_h\|_{-1} &\leq Ch\|\partial_t \boldsymbol{e}_h\|_0 + C\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0 \\
&= Ch\|A_h^{1/2} A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0 + C\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0 \\
&= Ch\|\nabla(A_h^{-1/2} \partial_t \boldsymbol{e}_h)\|_0 + C\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0 \\
&\leq C\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0.
\end{aligned} \tag{33}$$

Next, a bound for $\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0$ will be derived. Projecting the error equation (25) onto the discretely divergence-free space V_h^{div} and using integration by parts yields

$$\begin{aligned}
&(\partial_t \boldsymbol{e}_h, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{e}_h, \nabla \boldsymbol{v}_h) + ((\boldsymbol{b} \cdot \nabla)\boldsymbol{e}_h + \sigma \boldsymbol{e}_h, \boldsymbol{v}_h) + S_h(\boldsymbol{e}_h, \boldsymbol{v}_h) \\
&\quad = \ (\partial_t(\boldsymbol{u} - \boldsymbol{s}_h), \boldsymbol{v}_h) + ((\boldsymbol{b} \cdot \nabla)(\boldsymbol{u} - \boldsymbol{s}_h) + \sigma(\boldsymbol{u} - \boldsymbol{s}_h), \boldsymbol{v}_h) \\
&\qquad\ -S_h(\boldsymbol{s}_h, \boldsymbol{v}_h) - (p - i_h p, \nabla \cdot \boldsymbol{v}_h).
\end{aligned}$$

Utilizing (9), one finds $(p - i_h p, \nabla \cdot \boldsymbol{v}_h) = (D_h(p - i_h p), \boldsymbol{v}_h)$, such that

$$\begin{aligned}
\partial_t \boldsymbol{e}_h &= -\nu A_h \boldsymbol{e}_h - \Pi_h^{\text{div}}\big((\boldsymbol{b} \cdot \nabla)\boldsymbol{e}_h + \sigma \boldsymbol{e}_h\big) - C_h \boldsymbol{e}_h + \Pi_h^{\text{div}}\big(\partial_t(\boldsymbol{u} - \boldsymbol{s}_h)\big) \\
&\quad +\Pi_h^{\text{div}}\big((\boldsymbol{b} \cdot \nabla)(\boldsymbol{u} - \boldsymbol{s}_h) + \sigma(\boldsymbol{u} - \boldsymbol{s}_h)\big) - C_h(\boldsymbol{s}_h) \\
&\quad -D_h(p - i_h p).
\end{aligned} \tag{34}$$

With (8), the Cauchy–Schwarz inequality, (6), the L^2 stability of the fluctuation operator κ_K, and $\mu_K \sim 1$, one obtains for all $\boldsymbol{v}_h \in V_h^{\text{div}}$

$$\begin{aligned}
\|A_h^{-1/2} C_h \boldsymbol{v}_h\|_0 &= \sup_{\boldsymbol{w}_h \in V_h^{\text{div}} \setminus \{\boldsymbol{0}\}} \frac{|\langle C_h \boldsymbol{v}_h, A_h^{-1/2} \boldsymbol{w}_h \rangle|}{\|\boldsymbol{w}_h\|_0} \\
&= \sup_{\boldsymbol{w}_h \in V_h^{\text{div}} \setminus \{\boldsymbol{0}\}} \frac{|\sum_{K \in \mathcal{T}_h} \mu_K (\kappa_K \nabla \boldsymbol{v}_h, \kappa_K \nabla (A_h^{-1/2} \boldsymbol{w}_h))_{0,K}|}{\|\boldsymbol{w}_h\|_0} \\
&\leq \sup_{\boldsymbol{w}_h \in V_h^{\text{div}} \setminus \{\boldsymbol{0}\}} \frac{\left(\sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{v}_h\|_{0,K}^2\right)^{1/2} C \|\nabla (A_h^{-1/2} \boldsymbol{w}_h)\|_0}{\|\boldsymbol{w}_h\|_0} \\
&\leq C \sup_{\boldsymbol{w}_h \in V_h^{\text{div}} \setminus \{\boldsymbol{0}\}} \frac{\left(\sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{v}_h\|_{0,K}^2\right)^{1/2} \|\boldsymbol{w}_h\|_0}{\|\boldsymbol{w}_h\|_0} \\
&= C \left(\sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{v}_h\|_{0,K}^2 \right)^{1/2}. \qquad (35)
\end{aligned}$$

Applying above argument to $\|A_h^{-1/2} D_h (p - i_h p)\|_0$ yields

$$\|A_h^{-1/2} D_h (p - i_h p)\|_0 \leq C \|p - i_h p\|_0. \qquad (36)$$

Definition (7) and the symmetry of A_h gives for any $\boldsymbol{g} \in L^2(\Omega)^d$ the equality $(A_h^{-1/2} \Pi_h^{\text{div}} \boldsymbol{g}, \boldsymbol{v}_h) = (\boldsymbol{g}, A_h^{-1/2} \boldsymbol{v}_h)$ for all $\boldsymbol{v}_h \in V_h^{\text{div}}$. It follows with $\boldsymbol{v}_h = A_h^{-1/2} \Pi_h^{\text{div}} \boldsymbol{g} \in V_h^{\text{div}}$ and (6) that

$$\|A_h^{-1/2} \Pi_h^{\text{div}} \boldsymbol{g}\|_0^2 \leq \|\boldsymbol{g}\|_{-1} \|\nabla (A_h^{-1/2} A_h^{-1/2} \Pi_h^{\text{div}} \boldsymbol{g})\|_0 = \|\boldsymbol{g}\|_{-1} \|A_h^{-1/2} \Pi_h^{\text{div}} \boldsymbol{g}\|_0$$

and hence

$$\|A_h^{-1/2} \Pi_h^{\text{div}} \boldsymbol{g}\|_0 \leq \|\boldsymbol{g}\|_{-1} \quad \forall\, \boldsymbol{g} \in L^2(\Omega)^d. \qquad (37)$$

In the next step, $A_h^{-1/2}$ is applied to (34). Using (35), (36), and (37) leads to

$$\begin{aligned}
\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0 \leq\ & \nu \|A_h^{1/2} \boldsymbol{e}_h\|_0 + \|(\boldsymbol{b} \cdot \nabla) \boldsymbol{e}_h + \sigma \boldsymbol{e}_h\|_{-1} + \left(\sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{e}_h\|_{0,K}^2 \right)^{1/2} \\
& + \|\partial_t (\boldsymbol{u} - \boldsymbol{s}_h)\|_{-1} + \|(\boldsymbol{b} \cdot \nabla)(\boldsymbol{u} - \boldsymbol{s}_h) + \sigma (\boldsymbol{u} - \boldsymbol{s}_h)\|_{-1} \\
& + \left(\sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{s}_h\|_{0,K}^2 \right)^{1/2} + \|p - i_h p\|_0. \qquad (38)
\end{aligned}$$

Taking the square of (38) and integrating on $(0,t)$ yields

$$\begin{aligned}
&\int_0^t \|A_h^{-1/2}\partial_s \boldsymbol{e}_h(s)\|_0^2\, ds\\
&\le\ C\Bigg(\int_0^t \nu^2\|A_h^{1/2}\boldsymbol{e}_h(s)\|_0^2\, ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h)(s)\|_{-1}^2\, ds\\
&\quad + \int_0^t \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{e}_h(s)\|_{0,K}^2\, ds + \int_0^t \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_{-1}^2\, ds\\
&\quad + \int_0^t \|(p-i_h p)(s)\|_0^2\, ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h))(s)\|_{-1}^2\, ds\\
&\quad + \int_0^t \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{s}_h(s)\|_{0,K}^2\, ds\Bigg). \qquad (39)
\end{aligned}$$

It will be shown that all the terms on the right-hand side of (39) are of sufficiently high asymptotic order, concretely that they are $\mathcal{O}(h^{2r})$. This asymptotic behavior is obtained for the first and third term directly from (29). Using the definition of the $H^{-1}(\Omega)^d$ norm for the second term in (39), applying integration by parts, and utilizing Poincaré's inequality leads to

$$\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h\|_{-1} \le C\left(\|\boldsymbol{b}\|_\infty + \sigma\right)\|\boldsymbol{e}_h\|_0.$$

Hence, it is

$$\int_0^t \|((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h)(s)\|_{-1}^2\, ds \le C\int_0^t \|\boldsymbol{e}_h(s)\|_0^2\, ds,$$

such that the order of convergence $\mathcal{O}(h^{2r})$ can be again deduced from (29). For estimating $\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1}$, the definition of the $H^{-1}(\Omega)^d$ norm and Poincaré's inequality are applied, which gives a bound of this term by $C\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_0$. Now, (21) is applied, see Remark 1, and using the regularity assumptions (23), the asymptotic bound for $\|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1}$ is $\mathcal{O}(h^r)$. Consequently, the integral of its square is also bounded, giving

$$\int_0^t \|\partial_s(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_{-1}^2\, ds \le Ch^{2r}\|\partial_t\boldsymbol{u}\|_{L^2(0,t;H^r)}^2.$$

The term involving the pressure is estimated with (12). Arguing as in (26)–(27) yields

$$\begin{aligned}
\int_0^t \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{s}_h(s)\|_{0,K}^2\, ds &\le C\int_0^t \left(\|(\boldsymbol{s}_h-\boldsymbol{u})(s)\|_1^2 + \|\kappa_K\nabla\boldsymbol{u}(s)\|_0^2\right) ds\\
&\le Ch^{2r}\|\boldsymbol{u}\|_{L^2(0,t;H^{r+1})}^2,
\end{aligned}$$

where (21) and (14) were applied in the last inequality. Finally, arguing in the same way as for the second term gives

$$\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} \le C\left(\|\boldsymbol{b}\|_\infty + \sigma\right)\|\boldsymbol{u}-\boldsymbol{s}_h\|_0,$$

from what follows that

$$\int_0^t \|((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h))(s)\|_{-1}^2\, ds \le C\int_0^t \|(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_0^2\, ds.$$

The bound for this term is finished by applying (21). Combining the estimates for (39) with (33), it is shown that

$$\int_0^t \|\partial_s(\boldsymbol{e}_h)(s)\|_{-1}^2 \, ds = \mathcal{O}(h^{2r}). \tag{40}$$

Using now the discrete inf-sup condition (3) and (25), one obtains

$$\begin{aligned}
&\beta_0\|p_h - i_h p\|_0 \\
&\leq \nu\|\nabla \boldsymbol{e}_h\|_0 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h\|_{-1} + \|\partial_t \boldsymbol{e}_h\|_{-1} + C\left(\sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{e}_h\|_{0,K}^2\right)^{1/2} \\
&\quad + \|\partial_t(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} + \|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h)\|_{-1} \\
&\quad + C\left(\sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{s}_h\|_{0,K}^2\right)^{1/2} + \|p - i_h p\|_0 + \|l_h\|_0.
\end{aligned}$$

Taking the square and integrating on $(0,t)$ leads to

$$\begin{aligned}
&\beta_0^2\int_0^t \|(p_h - i_h p)(s)\|_0^2 \, ds \\
&\leq C\left(\int_0^t \nu^2\|\nabla\boldsymbol{e}_h(s)\|_0^2 \, ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)\boldsymbol{e}_h + \sigma\boldsymbol{e}_h)(s)\|_{-1}^2 \, ds\right. \\
&\quad + \int_0^t \|\partial_s(\boldsymbol{e}_h)(s)\|_{-1}^2 \, ds + \int_0^t \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{e}_h(s)\|_{0,K}^2 \, ds \\
&\quad + \int_0^t \|\partial_s(\boldsymbol{u}-\boldsymbol{s}_h)(s)\|_{-1}^2 \, ds + \int_0^t \|((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{s}_h)(s) + \sigma(\boldsymbol{u}-\boldsymbol{s}_h))(s)\|_{-1}^2 \, ds \\
&\quad \left. + \int_0^t \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\boldsymbol{s}_h(s)\|_{0,K}^2 \, ds + \int_0^t \|(p - i_h p)(s)\|_0^2 \, ds + \int_0^t \|l_h(s)\|_0^2 \, ds\right).
\end{aligned}$$

Arguing exactly as in the estimates of the terms which are on the right-hand side of (39), using (40) for bounding $\int_0^t \|\partial_s(\boldsymbol{e}_h)(s)\|_{-1}^2 \, ds$, (22) to bound the last term, and applying finally the triangle inequality, proves (30). □

4 Error analysis for the fully discrete method with cGP(k)

The continuous Galerkin–Petrov method is studied as temporal discretization. Consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $I := [0,T]$ and set $I_n = (t_{n-1}, t_n]$, $\tau_n = t_n - t_{n-1}$, $n = 1,\ldots N$, and $\tau := \max_{1\leq n\leq N}\tau_n$. For a given non-negative integer k, the time-continuous and time-discontinuous velocity spaces are defined as follows

$$X_k^{\mathrm{c}} := \{\boldsymbol{u}\in C(V_h) : \boldsymbol{u}|_{I_n}\in\mathbb{P}_k(I_n,V_h)\}, \ X_k^{\mathrm{dc}} := \{\boldsymbol{u}\in L^2(V_h) : \boldsymbol{u}|_{I_n}\in\mathbb{P}_k(I_n,V_h)\},$$

and time-continuous and time-discontinuous pressure spaces are given by

$$Y_k^{\mathrm{c}} := \{q\in C(Q_h) : q|_{I_n}\in\mathbb{P}_k(I_n,Q_h)\}, \ Y_k^{\mathrm{dc}} := \{q\in L^2(Q_h) : q|_{I_n}\in\mathbb{P}_k(I_n,Q_h)\},$$

for $n = 1,\ldots,N$. Here,

$$\mathbb{P}_k(I_n,W_h) := \left\{u : I_n\to W_h : u(t) = \sum_{i=0}^k U_i t^i, \ \forall\, t\in I_n, U_i\in W_h, \forall\, i\right\}$$

denotes the space of W_h-valued polynomials of order k in time. The functions in the spaces X_k^{dc} and Y_k^{dc} are allowed to be discontinuous at the nodes t_n. Below, the combination of the LPS method as spatial discretization and the cGP(k) time stepping scheme is denoted by LPS/cGP.

Denote by $X_{k,0}^{\mathrm{c}} := X_k^{\mathrm{c}} \cap X_0$ the subspace of X_k^{c} with zero initial condition and introduce a bilinear form b_h given by

$$b_h\left((\boldsymbol{u},p);(\boldsymbol{v},q)\right) := \int_0^T \left[\big(\partial_t \boldsymbol{u}, \boldsymbol{v}\big) + a_h\big((\boldsymbol{u},p);(\boldsymbol{v},q)\big)\right]\, dt.$$

The LPS/cGP method reads as follows:

Find $\boldsymbol{u}_{h,\tau} \in X_{k,0}^{\mathrm{c}}$ and $p_{h,\tau} \in Y_k^{\mathrm{c}}$ such that

$$b_h\left((\boldsymbol{u}_{h,\tau}, p_{h,\tau});(\boldsymbol{v}_{h,\tau}, q_{h,\tau})\right) = \int_0^T \big(\boldsymbol{f}, \boldsymbol{v}_{h,\tau}\big)\, dt \quad \forall\, \boldsymbol{v}_{h,\tau} \in X_{k-1}^{\mathrm{dc}},\ q_{h,\tau} \in Y_{k-1}^{\mathrm{dc}}, \tag{41}$$

where the index h,τ refers to the discretization in space and time. The associated continuous problem is given by:

Find $\boldsymbol{u} \in X$ and $p \in L^2(Q)$ such that

$$\int_0^T \left[\big(\partial_t \boldsymbol{u}(t), \boldsymbol{v}(t)\big) + a\big((\boldsymbol{u}(t),p(t));(\boldsymbol{v}(t),q(t))\big)\right]\, dt = \int_0^T \big(\boldsymbol{f}(t), \boldsymbol{v}(t)\big)\, dt \tag{42}$$

for all $\boldsymbol{v} \in L^2(V)$, $q \in L^2(Q)$.

For a function w, which is smooth on each time interval I_n, the operator π_{k-1} is defined by

$$(\pi_{k-1} w)|_{I_n}(t) = \sum_{i=1}^{k} w(\tilde{t}_{n,i})\tilde{L}_{n,i}(t), \tag{43}$$

where $\tilde{t}_{n,i}$ denote the Gaussian quadrature points on I_n and $\tilde{L}_{n,i} \in \mathbb{P}_{k-1}(I_n)$ are the associated Lagrange basis functions. From (43), it can be concluded that $\pi_{k-1}\boldsymbol{w}_{h,\tau} \in X_{k-1}^{\mathrm{dc}}$ for all $\boldsymbol{w}_{h,\tau} \in X_k^{\mathrm{c}}$ and $\pi_{k-1} q_{h,\tau} \in Y_{k-1}^{\mathrm{dc}}$ for all $q_{h,\tau} \in Y_k^{\mathrm{c}}$. Furthermore, one has for all $\boldsymbol{w}_{h,\tau} \in X_k^{\mathrm{c}}$ that

$$\int_{I_n} (\boldsymbol{w}_{h,\tau}(t) - \pi_{k-1}\boldsymbol{w}_{h,\tau}(t))t^j\, dt = \boldsymbol{0}, \quad j = 0,\ldots,k-1,\ n = 1,\ldots,N, \tag{44}$$

where $\boldsymbol{0}$ denotes the zero element in V_h.

The finite element analysis considers the mesh-dependent norm

$$\|\boldsymbol{v}\|_{\mathrm{cGP}} := \left(\int_0^T |||\pi_{k-1}\boldsymbol{v}|||^2 dt + \frac{1}{2}\|\boldsymbol{v}(T)\|_0^2\right)^{1/2}.$$

It was already observed in [5] that $||\cdot||_{\mathrm{cGP}}$ is on $X_k^{\mathrm{c}} \subset X_k^{\mathrm{dc}}$ not only a semi-norm but a norm. For completeness of presentation, the corresponding arguments are repeated here. The first term inside the definition of $||\boldsymbol{v}||_{\mathrm{cGP}}$ guarantees that $||\boldsymbol{v}||_{\mathrm{cGP}} = 0$ results in a function $\boldsymbol{v}$ which is on each time interval I_n given by $L_k^{(n)}(t)\boldsymbol{\varphi}_h(x)$, where $L_k^{(n)}$ is the transformed k-th Legendre polynomial on I_n and $\boldsymbol{\varphi}_h \in V_h$. Due to $\boldsymbol{v}(T) = 0$ and $L_k^{(N)}(T) = 1$ the function $\boldsymbol{v}$ vanishes on the last time interval I_N. The continuity of $\boldsymbol{v}$ on I gives then $\boldsymbol{v}(t_{N-1}) = \boldsymbol{0}$. By recursion, one obtains $\boldsymbol{v} = \boldsymbol{0}$ on I and hence $||\cdot||_{\mathrm{cGP}}$ is a norm.

The following lemma proves a property of the bilinear form b_h that will be used in the derivation of the error bounds for the approximation to the velocity.

Lemma 4. *Assume that $\boldsymbol{b}$ and σ are constant with respect to time. Then, there exists a constant $C > 0$ independent of ν, h, and τ such that*

$$b_h\left((\boldsymbol{v}_{h,\tau}, q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right) = \|\boldsymbol{v}_{h,\tau}\|_{\mathrm{cGP}}^2 \quad \forall\ (\boldsymbol{v}_{h,\tau}, q_{h,\tau}) \in X_k^{\mathrm{dc}} \times Y_k^{\mathrm{dc}}$$

holds true.

Proof. It is

$$\begin{aligned} & b_h\left((\boldsymbol{v}_{h,\tau}, q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right) \\ = & \int_0^T \left[\left(\partial_t \boldsymbol{v}_{h,\tau}, \pi_{k-1}\boldsymbol{v}_{h,\tau}\right) + a_h\left((\boldsymbol{v}_{h,\tau}, q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right)\right]\ dt. \end{aligned}$$

Using the fact that the convection and reaction are time-independent functions, taking into account that

$$\begin{aligned} & \int_0^T \left[-\left(q_{h,\tau}, \operatorname{div} \pi_{k-1}\boldsymbol{v}_{h,\tau}\right) + \left(\pi_{k-1}q_{h,\tau}, \operatorname{div} \boldsymbol{v}_{h,\tau}\right)\right]\ dt \\ = & \int_0^T \left[-\left(\pi_{k-1}q_{h,\tau}, \operatorname{div} \pi_{k-1}\boldsymbol{v}_{h,\tau}\right) + \left(\pi_{k-1}q_{h,\tau}, \operatorname{div} \pi_{k-1}\boldsymbol{v}_{h,\tau}\right)\right]\ dt = 0, \end{aligned}$$

and (10), one obtains

$$\begin{aligned} & \int_0^T a_h\left((\boldsymbol{v}_{h,\tau}, q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right)\ dt \\ = & \int_0^T a_h\left((\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau}); (\pi_{k-1}\boldsymbol{v}_{h,\tau}, \pi_{k-1}q_{h,\tau})\right)\ dt = \int_0^T |||\pi_{k-1}\boldsymbol{v}_{h,\tau}|||^2\ dt. \end{aligned}$$

Concerning the first term, it is noted that $\partial_t \boldsymbol{v}_{h,\tau}$ is a discontinuous function in time of degree $k-1$. Using $\boldsymbol{v}_{h,\tau}(0) = \boldsymbol{0}$ yields

$$\begin{aligned} \int_0^T \left(\partial_t \boldsymbol{v}_{h,\tau}, \pi_{k-1}\boldsymbol{v}_{h,\tau}\right)\ dt &= \int_0^T \left(\partial_t \boldsymbol{v}_{h,\tau}, \boldsymbol{v}_{h,\tau}\right)\ dt = \frac{1}{2}\int_0^T \frac{d}{dt}\left\|\boldsymbol{v}_{h,\tau}\right\|_0^2\ dt \\ &= \frac{1}{2}\left\|\boldsymbol{v}_{h,\tau}(T)\right\|_0^2. \end{aligned}$$

□

The derivation of error bounds makes use of a time interpolation $\widetilde{w} \in C(H)$ of a sufficiently smooth function w, where H can be either a velocity space V or a pressure space Q, and $\widetilde{w}|_{I_n} \in \mathbb{P}_k(I_n, H)$, defined by

$$\widetilde{w}(t_{n-1}) = w(t_{n-1}), \quad \widetilde{w}(t_n) = w(t_n), \quad \int_{I_n} (w(t) - \widetilde{w}(t), z(t))\ dt = 0, \tag{45}$$

for all $z \in \mathbb{P}_{k-2}(I_n, H)$. The standard interpolation error estimate

$$\left(\int_{I_n} \|w - \widetilde{w}\|_m^2\ dt\right)^{1/2} \le C\tau_n^{k+1}\left(\int_{I_n} \|w^{(k+1)}\|_m^2\ dt\right)^{1/2} \tag{46}$$

holds true for $m \in \{0, 1\}$ and all time intervals I_n, $n = 1, \ldots, N$.

Theorem 5. *Assume that the spaces V_h, Q_h satisfy Assumptions A1 and A2, $\mu_K \sim 1$ for all $K \in \mathcal{T}_h$, and $\nu \leq 1$. Let $(\boldsymbol{u}, p)$ be the solution of (42) and $(\boldsymbol{u}_{h,\tau}, p_{h,\tau})$ the solution of (41). Further, assume that the solution $(\boldsymbol{u}, p)$ is smooth enough such that all the norms on the right-hand side of (47) are bounded. Then, there exists a positive constant C independent of ν, h, and τ such that the error estimate*

$$\begin{aligned} \left\|\boldsymbol{u}-\boldsymbol{u}_{h,\tau}\right\|_{\mathrm{cGP}} \quad &\leq \quad Ch^r \left(\|\boldsymbol{u}\|_{L^2(H^{r+1})} + \|\boldsymbol{u}\|_{H^1(H^r)} + \|p\|_{L^2(H^r)} + h\|\boldsymbol{u}(T)\|_{r+1}\right) \\ &\quad +C\tau^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(H^1)} \end{aligned} \tag{47}$$

is valid.

Proof. The error analysis starts by decomposing the error $\boldsymbol{e}_{h,\tau} = \boldsymbol{u}_{h,\tau} - \boldsymbol{u}$ into $\boldsymbol{\theta}_h := \tilde{\boldsymbol{s}}_h - \boldsymbol{u}$ and $\boldsymbol{\xi}_{h,\tau} := \boldsymbol{u}_{h,\tau} - \tilde{\boldsymbol{s}}_h$, with the velocity solution $\tilde{\boldsymbol{s}}_h$ of (16), where $\boldsymbol{g}$ in (20) is defined as $-\nu\Delta\tilde{\boldsymbol{u}}$. Then, it is

$$\boldsymbol{u}_{h,\tau} - \boldsymbol{u} = \boldsymbol{e}_{h,\tau} = \boldsymbol{\theta}_h + \boldsymbol{\xi}_{h,\tau}.$$

For the discrete error $\boldsymbol{\xi}_{h,\tau}$ Lemma 4 gives

$$\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}^2 = b_h\big((\boldsymbol{\xi}_{h,\tau}, p_{h,\tau}); (\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}p_{h,\tau})\big). \tag{48}$$

Applying a straightforward calculation yields

$$\begin{aligned} & b_h\big((\boldsymbol{\xi}_{h,\tau}, p_{h,\tau}); (\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}p_{h,\tau})\big) \\ &= \int_0^T \big(\partial_t \boldsymbol{u} - \partial_t \tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\big)\ dt + \int_0^T \nu\big(\nabla(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h), \nabla(\pi_{k-1}\boldsymbol{\xi}_{h,\tau})\big)\ dt \\ &\quad + \int_0^T \big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\big)\ dt + \int_0^T \big(\sigma(\boldsymbol{u}-\tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\big)\ dt \\ &\quad + \int_0^T (\nabla p, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt - \int_0^T S_h(\tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt. \end{aligned} \tag{49}$$

The six terms on the right-hand side have to be bounded.

For the first one, the error is split in two terms

$$\begin{aligned} \int_0^T \big(\partial_t \boldsymbol{u} - \partial_t \tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\big)\ dt \quad &= \quad \int_0^T \big(\partial_t(\boldsymbol{u}-\tilde{\boldsymbol{u}}), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\big)\ dt \\ &\quad + \int_0^T \big(\partial_t(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\big)\ dt. \end{aligned} \tag{50}$$

Integration by parts and using (45) yield for the first term on the right-hand side of (50)

$$\begin{aligned} & \int_0^T \big(\partial_t(\boldsymbol{u}-\tilde{\boldsymbol{u}}), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\big)\ dt \\ &= \sum_{n=1}^N \left(-\int_{I_n} (\boldsymbol{u}-\tilde{\boldsymbol{u}}, \partial_t(\pi_{k-1}\boldsymbol{\xi}))\ dt + (\boldsymbol{u}-\tilde{\boldsymbol{u}}, \pi_{k-1}\boldsymbol{\xi}_{h\tau})\Big|_{t_{n-1}}^{t_n} \right) = 0. \end{aligned} \tag{51}$$

For the second term on the right-hand side of (50), the application of the Cauchy–Schwarz inequality and (21) gives

$$\begin{aligned}
&\int_0^T \left(\partial_t(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\
&\leq \sum_{n=1}^N \int_{I_n} \|\partial_t \tilde{\boldsymbol{u}} - \partial_t \tilde{\boldsymbol{s}}_h\|_0 \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_0\ dt \\
&\leq \left(\sum_{n=1}^N \int_{I_n} \|\partial_t \tilde{\boldsymbol{u}} - \partial_t \tilde{\boldsymbol{s}}_h\|_0^2\ dt\right)^{1/2} \left(\sum_{n=1}^N \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_0^2\ dt\right)^{1/2} \\
&\leq Ch^r \left(\sum_{n=1}^N \int_{I_n} \|\partial_t \tilde{\boldsymbol{u}}\|_r^2\ dt\right)^{1/2} \left(\sum_{n=1}^N \int_{I_n} \sigma \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_0^2\ dt\right)^{1/2} \\
&\leq Ch^r \|\boldsymbol{u}\|_{H^1(H^r)} \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}},
\end{aligned} \tag{52}$$

where in the last estimate the inequality $\|\tilde{\boldsymbol{u}}\|_{H^1(H^r)} \leq C\|\boldsymbol{u}\|_{H^1(H^r)}$ was applied. Thus, from (50), (51), and (52) one derives the bound for the first term on the right-hand side of (49)

$$\int_0^T \left(\partial_t \boldsymbol{u} - \partial_t \tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \leq Ch^r \|\boldsymbol{u}\|_{H^1(H^r)} \|\boldsymbol{\xi}\|_{\mathrm{cGP}}. \tag{53}$$

To bound the third term on the right-hand side of (49), the error splitting

$$\begin{aligned}
\int_0^T \left((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt &= \int_0^T \left((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u} - \tilde{\boldsymbol{u}}), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\
&\quad + \int_0^T \left((\boldsymbol{b}\cdot\nabla)(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt
\end{aligned}$$

is used. Then, applying (46) and (21) yields

$$\begin{aligned}
&\int_0^T \left((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\
&\leq \int_0^T \|\boldsymbol{b}\|_\infty \left(\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_1 + \|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}_h\|_1\right) \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_0\ dt \\
&\leq C\left[\left(\sum_{n=1}^N \tau_n^{2k+2} \int_{I_n} \|\boldsymbol{u}^{(k+1)}\|_1^2\ dt\right)^{1/2} + \left(h^{2r} \sum_{n=1}^N \int_{I_n} \|\tilde{\boldsymbol{u}}\|_{r+1}^2\ dt\right)^{1/2}\right] \\
&\qquad \times \left(\sum_{n=1}^N \int_{I_n} |||\pi_{k-1}\boldsymbol{\xi}|||_0^2\ dt\right)^{1/2} \\
&\leq \left(C\tau^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(H^1)} + Ch^r \|\boldsymbol{u}\|_{L^2(H^{r+1})}\right) \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}.
\end{aligned} \tag{54}$$

Arguing exactly as before gives for the second and the fourth term on the right-hand side of (49)

$$\begin{aligned}
&\int_0^T \nu \left(\nabla(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \nabla(\pi_{k-1}\boldsymbol{\xi}_{h,\tau})\right)\ dt + \int_0^T \left(\sigma(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \pi_{k-1}\boldsymbol{\xi}_{h,\tau}\right)\ dt \\
&\quad \leq \left(C(\nu^{1/2} + \sigma^{1/2}h)h^r \|\boldsymbol{u}\|_{L^2(H^{r+1})} + C(\nu^{1/2} + \sigma^{1/2})\tau^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(H^1)}\right) \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}.
\end{aligned} \tag{55}$$

To bound the fifth term on the right-hand side of (49) observe that

$$\int_0^T (\nabla p, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt = \int_0^T -(p, \nabla\cdot\pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt = \int_0^T -(p, \pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt,$$

since the time projection π_{k-1} and the divergence commute. In addition, it is

$$\begin{aligned}\int_{I_n} (i_h p, \pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt &= \int_{I_n} (\pi_{k-1}(i_h p), \pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt \\ &= \int_{I_n} (\pi_{k-1}(i_h p), \nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt = 0, \qquad (56)\end{aligned}$$

since $\tilde{\boldsymbol{s}}_h$ has discrete divergence equal to zero and the relation $\int_{I_n}(\nabla\cdot\boldsymbol{u}_{h,\tau}, q_{h,\tau})\ dt = 0$ holds by definition for all $q_{h,\tau}\in Y_{k-1}^{\mathrm{dc}}$. Thus, for the fifth term on the right-hand side of (49), integration by parts with respect to space, applying the orthogonality condition (13), using (56), $\mu_K\sim 1$, and (12) lead to

$$\begin{aligned}&\int_0^T (\nabla p, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\ &= \int_0^T (i_h p - p, \pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})\ dt = \int_0^T \sum_{k\in\mathcal{T}_h} (i_h p - p, \kappa_K\pi_{k-1}\nabla\cdot\boldsymbol{\xi}_{h,\tau})_K\ dt \\ &\le \int_0^T \left(\sum_{K\in\mathcal{T}_h} \mu_K^{-1}\|i_h p - p\|_{0,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla\pi_{k-1}\boldsymbol{\xi}_{h,\tau}\|_{0,K}^2\right)^{1/2} dt \\ &\le C\left(\int_0^T \|i_h p - p\|_0^2\ dt\right)^{1/2} \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}} \le Ch^r\|p\|_{L^2(H^r)}\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}. \qquad (57)\end{aligned}$$

Finally, to bound the last term on the right-hand side of (49), the following decomposition is considered

$$\begin{aligned}\int_0^T S_h(\tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt &= \int_0^T S_h(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\ &\quad + \int_0^T S_h(\tilde{\boldsymbol{u}} - \boldsymbol{u}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt + \int_0^T S_h(\boldsymbol{u}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt. \qquad (58)\end{aligned}$$

For the first term on the right-hand side of (58), the L^2 stability of the fluctuation operator κ_K, $\mu_K\sim 1$, and (21) are applied to obtain

$$\begin{aligned}&\int_0^T S_h(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\ &\le \int_0^T \left(\sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_{0,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\pi_{k-1}\nabla\boldsymbol{\xi}_{h,\tau}\|_{0,K}^2\right)^{1/2} dt \\ &\le \left(\int_0^T \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\nabla(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_{0,K}^2\ dt\right)^{1/2} \left(\int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}|||^2\ dt\right)^{1/2} \\ &\le Ch^r\|\boldsymbol{u}\|_{L^2(H^{r+1})}\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}. \qquad (59)\end{aligned}$$

Applying the stability of the fluctuation operator κ_K, $\mu_K \sim 1$, and (46) gives for the second term on the right-hand side of (58)

$$\begin{aligned}
&\int_0^T S_h(\tilde{\boldsymbol{u}} - \boldsymbol{u}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\
&\leq \int_0^T \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla(\tilde{\boldsymbol{u}} - \boldsymbol{u})\|_{0,K}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \pi_{k-1} \nabla \boldsymbol{\xi}_{h,\tau}\|_{0,K}^2 \right)^{1/2} dt \\
&\leq \left(\int_0^T \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla(\tilde{\boldsymbol{u}} - \boldsymbol{u})\|_{0,K}^2\ dt \right)^{1/2} \left(\int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}|||^2\ dt \right)^{1/2} \\
&\leq C\tau^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(H^1)} \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}. \qquad (60)
\end{aligned}$$

To finish the estimate of the last term on the right-hand side of (49), the Cauchy–Schwarz inequality, the approximation properties (14) of the fluctuation operator κ_K, and $\mu_K \sim 1$ are used to get

$$\begin{aligned}
&\int_0^T S_h(\boldsymbol{u}, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \\
&\leq \int_0^T \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \nabla \boldsymbol{u}\|_{0,K}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K \pi_{k-1} \nabla \boldsymbol{\xi}_{h,\tau}\|_{0,K}^2 \right)^{1/2} dt \\
&\leq \left(\int_0^T \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_K (\nabla \boldsymbol{u})\|_{0,K}^2\ dt \right)^{1/2} \left(\int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}|||^2\ dt \right)^{1/2} \\
&\leq Ch^r \|\boldsymbol{u}\|_{L^2(H^{r+1})} \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}. \qquad (61)
\end{aligned}$$

Inserting (59), (60), and (61) in (58) gives

$$\int_0^T S_h(\tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{\xi}_{h,\tau})\ dt \leq \left(Ch^r \|\boldsymbol{u}\|_{L^2(H^{r+1})} + C\tau^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(H^1)} \right) \|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}}. \qquad (62)$$

Inserting (49) in (48) and utilizing (53), (54), (55), (57), and (62) lead to

$$\|\boldsymbol{\xi}_{h,\tau}\|_{\mathrm{cGP}} \leq Ch^r \Big[\|\boldsymbol{u}\|_{L^2(H^{r+1})} + \|\boldsymbol{u}\|_{H^1(H^r)} + \|p\|_{L^2(H^r)} \Big] + C\tau^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(H^1)}. \qquad (63)$$

Applying the triangle inequality, the bound (21), and the interpolation error estimates in time gives the statement of the theorem. □

Arguing similarly as in [5, Thm. 4.4], one can prove the following theorem.

Theorem 6. *Under the assumptions of Theorem 5, the following error estimate is valid*

$$\begin{aligned}
&\left(\int_0^T \|\boldsymbol{u}(t) - \boldsymbol{u}_{h,\tau}(t)\|_0^2\ dt \right)^{1/2} \\
&\leq C(1 + T^{1/2}) h^r \left[\|\boldsymbol{u}\|_{L^2(H^{r+1})} + \|\boldsymbol{u}\|_{H^1(H^r)} + \|p\|_{L^2(H^r)} \right] \\
&\quad + C(1 + T^{1/2}) \tau^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(H^1)}, \qquad (64)
\end{aligned}$$

with C independent of ν, h, and τ.

Proof. Denoting as before $\boldsymbol{\xi}_{h,\tau} = \boldsymbol{u}_{h,\tau} - \tilde{\boldsymbol{s}}_h$. Applying the ideas leading to (63) on $[0, t_n]$, $n = 1, \ldots, N$, instead of on $[0, T]$ gives

$$\int_0^{t_n} |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}(t)|||^2 \, dt + \frac{1}{2}\|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2 \\ \leq \; Ch^{2r}\Big[\|\boldsymbol{u}\|^2_{L^2(H^{r+1})} + \|\boldsymbol{u}\|^2_{H^1(H^r)} + \|p\|^2_{L^2(H^r)}\Big] + C\tau^{2k+2}\|\boldsymbol{u}\|^2_{H^{k+1}(H^1)},$$

where the norms on the right-hand side were extended from $[0, t_n]$ to $[0, T]$ by using that these norms do not decrease by extending the time interval. After having neglected the non-negative integral on the left-hand side and having multiplied by τ_n, the summation over $n = 1, \ldots, N$ yields

$$\sum_{n=1}^{N} \tau_n \|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2 \;\leq\; \left(\sum_{n=1}^{N} \tau_n\right) Ch^{2r}\Big[\|\boldsymbol{u}\|^2_{L^2(H^{r+1})} + \|\boldsymbol{u}\|^2_{H^1(H^r)} + \|p\|^2_{L^2(H^r)}\Big] \\ + \left(\sum_{n=1}^{N} \tau_n\right) C\tau^{2k+2}\|\boldsymbol{u}\|^2_{H^{k+1}(H^1)}. \tag{65}$$

Since $\boldsymbol{\xi}_{h,\tau}$ is a piecewise polynomial of degree less than or equal to k in time, the equivalence of all norms in finite-dimensional spaces gives

$$\int_{t_{n-1}}^{t_n} \|\boldsymbol{\xi}_{h,\tau}\|_0^2 \, dt \leq C_k \left(\int_{t_{n-1}}^{t_n} \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}(t)\|_0^2 \, dt + \tau_n \|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2 \right),$$

where C_k depends on the polynomial degree k but it is independent of τ_n and h. Hence, applying (65) and (63) leads to

$$\begin{aligned} \int_0^T \|\boldsymbol{u}_{h,\tau}(t) - \tilde{\boldsymbol{s}}_h(t)\|_0^2 \, dt \;&\leq\; C_k \sum_{n=1}^{N} \left(\int_{t_{n-1}}^{t_n} \|\pi_{k-1}\boldsymbol{\xi}_{h,\tau}(t)\|_0^2 \, dt + \tau_n \|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2 \right) \\ &\leq\; C_k \left(\int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}(t)|||_0^2 \, dt + \sum_{n=1}^{N} \tau_n \|\boldsymbol{\xi}_{h,\tau}(t_n)\|_0^2 \right) \\ &\leq\; C(1+T)h^{2r}\Big[\|\boldsymbol{u}\|^2_{L^2(H^{r+1})} + \|\boldsymbol{u}\|^2_{H^1(H^r)} + \|p\|^2_{L^2(H^r)}\Big] \\ &\quad +C(1+T)\tau^{2k+2}\|\boldsymbol{u}\|^2_{H^{k+1}(H^1)}. \end{aligned} \tag{66}$$

Now, the statement of the theorem follows by applying the triangle inequality and the time interpolation error estimates (46) together with (21). □

Theorem 7. *Let the assumptions of Theorem 5 hold and let in addition $(\boldsymbol{u}, p)$ be smooth enough such that the norms on the right-hand side of (67) are bounded. Then, there exists a positive constant C independent of ν, h, and τ such that the error estimate*

$$\left(\int_0^T \|\pi_{k-1}\left(p_{h,\tau}(t) - p(t)\right)\|_0^2 \, dt \right)^{1/2} \\ \leq \; C(1+T)h^r\left[\|\boldsymbol{u}\|_{H^1(H^{r+1})} + \|\boldsymbol{u}\|_{H^2(H^r)} + \|p\|_{H^1(H^r)}\right] \\ +C(1+T)\tau^k(1+\tau)\|\boldsymbol{u}\|_{H^{k+2}(H^1)} + C\tau^{k+1}\|p\|_{H^{k+1}(L^2)} \\ +Ch^r\left[\|\boldsymbol{u}\|_{L^2(H^{r+1})} + \|\boldsymbol{u}\|_{H^1(H^r)} + \|p\|_{H^1(H^r)}\right] \tag{67}$$

holds.

Proof. A straightforward calculation shows that for all $\boldsymbol{v}_{h,\tau} \in X_{k-1}^{\mathrm{dc}}$ and $q_{h,\tau} \in Y_{k-1}^{\mathrm{dc}}$, it holds

$$
\begin{aligned}
&b_h((\boldsymbol{u}_{h,\tau} - \tilde{\boldsymbol{s}}_h, p_{h,\tau}); (\boldsymbol{v}_{h,\tau}, q_{h,\tau})) \\
&= \int_0^T (\partial_t \boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\, dt + \int_0^T \nu(\nabla \boldsymbol{\xi}_{h,\tau}, \nabla \boldsymbol{v}_{h,\tau})\, dt + \int_0^T \big((\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau}\big)\, dt \\
&\quad + \int_0^T \sigma(\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\, dt - \int_0^T (\nabla\cdot \boldsymbol{v}_{h,\tau}, p_{h,\tau})\, dt + \int_0^T S_h(\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\, dt \\
&= \int_0^T (\partial_t(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \boldsymbol{v}_{h,\tau})\, dt + \int_0^T \nu(\nabla(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \nabla \boldsymbol{v}_{h,\tau})\, dt \\
&\quad + \int_0^T ((\boldsymbol{b}\cdot\nabla)(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h), \boldsymbol{v}_{h,\tau})\, dt + \int_0^T \sigma(\boldsymbol{u} - \tilde{\boldsymbol{s}}_h, \boldsymbol{v}_{h,\tau})\, dt \\
&\quad - \int_0^T S_h(\tilde{\boldsymbol{s}}_h, \boldsymbol{v}_{h,\tau})\, dt + \int_0^T (\nabla p, \boldsymbol{v}_{h,\tau})\, dt.
\end{aligned}
$$

From this equation, one obtains

$$
\begin{aligned}
&\int_0^T (p_{h,\tau} - i_h\tilde{p}, \nabla\cdot \boldsymbol{v}_{h,\tau})\, dt \\
&= \int_0^T (p - i_h\tilde{p}, \nabla\cdot \boldsymbol{v}_{h,\tau})\, dt + \int_0^T (\partial_t \boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\, dt + \int_0^T \nu(\nabla \boldsymbol{\xi}_{h,\tau}, \nabla \boldsymbol{v}_{h,\tau})\, dt \\
&\quad + \int_0^T ((\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\, dt + \int_0^T \sigma(\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\, dt + \int_0^T S_h(\boldsymbol{\xi}_{h,\tau}, \boldsymbol{v}_{h,\tau})\, dt \\
&\quad + \int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \boldsymbol{v}_{h,\tau})\, dt + \int_0^T \nu(\nabla(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \nabla \boldsymbol{v}_{h,\tau})\, dt \\
&\quad + \int_0^T ((\boldsymbol{b}\cdot\nabla)(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \boldsymbol{v}_{h,\tau})\, dt + \int_0^T \sigma(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}, \boldsymbol{v}_{h,\tau})\, dt + \int_0^T S_h(\tilde{\boldsymbol{s}}_h, \boldsymbol{v}_{h,\tau})\, dt. \qquad (68)
\end{aligned}
$$

To derive the error estimates, the Gaussian quadrature rule with k points will be used for the numerical integration of the time integral. Hence, one has

$$
\int_0^T q_{2k-1}(t)\, dt = \sum_{n=1}^{N} \frac{\tau_n}{2} \sum_{i=1}^{k} \hat{\omega}_i q_{2k-1}(\tilde{t}_{n,i}) \qquad (69)
$$

for all $q_{2k-1} \in \mathbb{P}_{2k-1}(I_n)$, where $\tilde{t}_{n,i}$ denote the corresponding quadrature points on I_n and $\hat{\omega}_i$ are the weights of the Gaussian formula on $(-1,1)$ which satisfy $\hat{\omega}_i > 0$. Let $\tilde{t}_{n,0} = t_{n-1}$ be an additional point.

Using the discrete inf-sup condition (3), one can construct $\boldsymbol{w}_{h,\tau} \in \mathbb{P}_k(I_n, V_h)$ such that

$$
\begin{aligned}
\beta_0 \|\pi_{k-1}\left(p_{h,\tau}(\tilde{t}_{n,i}) - i_h\tilde{p}(\tilde{t}_{n,i})\right)\|_0^2 &\le \left(\pi_{k-1}\left(p_{h,\tau}(\tilde{t}_{n,i}) - i_h\tilde{p}(\tilde{t}_{n,i})\right), \nabla\cdot \boldsymbol{w}_{h,\tau}(\tilde{t}_{n,i})\right), \qquad (70)\\
\|\boldsymbol{w}_{h,\tau}(\tilde{t}_{n,i})\|_1 &= \|\pi_{k-1}\left(p_{h,\tau}(\tilde{t}_{n,i}) - i_h\tilde{p}(\tilde{t}_{n,i})\right)\|_0. \qquad (71)
\end{aligned}
$$

Since $\boldsymbol{w}_{h,\tau} \in \mathbb{P}_k(I_n, V_h)$, it follows that $\pi_{k-1}\boldsymbol{w}_{h,\tau} \in \mathbb{P}_{k-1}(I_n, V_h)$. Setting $\boldsymbol{v}_{h,\tau} = \pi_{k-1}\boldsymbol{w}_{h,\tau}$ and using (43), (44), one obtains

$$\begin{aligned}\int_0^T (p_{h,\tau} - i_h\tilde{p}, \nabla \cdot \boldsymbol{v}_{h,\tau})\ dt &= \sum_{n=1}^{N} \int_{I_n} ((p_{h,\tau} - i_h\tilde{p}), \pi_{k-1}(\nabla \cdot \boldsymbol{w}_{h,\tau}))\ dt \\ &= \sum_{n=1}^{N} \int_{I_n} (\pi_{k-1}(p_{h,\tau} - i_h\tilde{p}), \nabla \cdot \boldsymbol{w}_{h,\tau})\ dt \\ &\geq \beta_0 \int_0^T \|\pi_{k-1}(p_{h,\tau} - i_h\tilde{p})\|_0^2\ dt, \end{aligned} \tag{72}$$

where the exactness of the quadrature rule for polynomials of degree $(2k-1)$, the positivity of the quadrature weights, (69), and (70) were used.

Setting $\boldsymbol{v}_{h,\tau} = \pi_{k-1}\boldsymbol{w}_{h,\tau}$ in (68), using (72), the assumption that $\boldsymbol{b}$ and σ are constants with respect to time, and (43), it follows that

$$\begin{aligned}&\beta_0 \int_0^T \|\pi_{k-1}(p_{h,\tau} - i_h\tilde{p})\|_0^2\ dt \\ &\leq \int_0^T (p_{h,\tau} - i_h\tilde{p}, \pi_{k-1}(\nabla \cdot \boldsymbol{w}_{h,\tau}))\, dt \\ &= \int_0^T (\pi_{k-1}(p - i_h\tilde{p}), \pi_{k-1}(\nabla \cdot \boldsymbol{w}_{h,\tau}))\ dt + \int_0^T (\partial_t \boldsymbol{\xi}_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\ &\quad + \int_0^T \nu(\pi_{k-1}(\nabla\boldsymbol{\xi}_{h,\tau}), \pi_{k-1}(\nabla\boldsymbol{w}_{h,\tau}))\ dt + \int_0^T ((\boldsymbol{b}\cdot\nabla)\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\ &\quad + \int_0^T \sigma(\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt + \int_0^T S_h(\pi_{k-1}\boldsymbol{\xi}_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\ &\quad + \int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt + \int_0^T \nu(\pi_{k-1}\nabla(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}(\nabla\boldsymbol{w}_{h,\tau}))\ dt \\ &\quad + \int_0^T (\pi_{k-1}(\boldsymbol{b}\cdot\nabla)(\tilde{\boldsymbol{s}}_h - \boldsymbol{u})_{h,\tau}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\ &\quad + \int_0^T \sigma(\pi_{k-1}(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt + \int_0^T S_h(\pi_{k-1}\tilde{\boldsymbol{s}}_h, \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt. \end{aligned} \tag{73}$$

The seventh term on the right-hand side of (73) is decomposed in the form

$$\begin{aligned}\int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt &= \int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\ &\quad + \int_0^T (\partial_t(\tilde{\boldsymbol{u}} - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt.\end{aligned}$$

For the second term on the right-hand side, integrating by parts with respect to time and using (45) yield

$$\begin{aligned}&\int_0^T (\partial_t(\tilde{\boldsymbol{u}} - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\ &= -\sum_{n=1}^{N} \left(\int_{I_n} (\tilde{\boldsymbol{u}} - \boldsymbol{u}, \partial_t(\pi_{k-1}\boldsymbol{w}_{h,\tau}))\ dt + (\boldsymbol{u} - \tilde{\boldsymbol{u}}, \pi_{k-1}\boldsymbol{w}_{h,\tau})\Big|_{t_{n-1}}^{t_n} \right) = 0.\end{aligned}$$

It follows that

$$
\begin{aligned}
\int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt &\le \int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0 \|\pi_{k-1}\boldsymbol{w}_{h,\tau}\|_0\ dt \\
&\le C\int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0 \|\nabla\pi_{k-1}\boldsymbol{w}_{h,\tau}\|_0\ dt,
\end{aligned}
$$

where Poincaré's inequality was applied in the last line.

Using (69), (43), and (71) gives

$$
\begin{aligned}
\int_0^T \|\nabla\pi_{k-1}\boldsymbol{w}_{h,\tau}\|_0^2\ dt &= \sum_{n=1}^N \frac{\tau_n}{2}\sum_{i=1}^k \hat{\omega}_i \|\pi_{k-1}\nabla\boldsymbol{w}_h(\tilde{t}_{n,i})\|_0^2 \\
&\le C\sum_{n=1}^N \frac{\tau_n}{2}\sum_{i=1}^k \hat{\omega}_i \|\nabla\boldsymbol{w}_h(\tilde{t}_{n,i})\|_0^2 \\
&= C\sum_{n=1}^N \frac{\tau_n}{2}\sum_{i=1}^k \hat{\omega}_i \|\pi_{k-1}\left(p_{h,\tau}(\tilde{t}_{n,i}) - i_h\tilde{p}(\tilde{t}_{n,i})\right)\|_0^2 \\
&= C\int_0^T \|\pi_{k-1}\left(p_{h,\tau} - i_h\tilde{p}\right)\|_0^2\ dt, \qquad (74)
\end{aligned}
$$

where $\tilde{t}_{n,i}$, $i = 1, \ldots, k$, denote the node of Gaussian quadrature on I_n and $\hat{\omega}_i$, $i = 1, \ldots, k$, are the corresponding weights on $[-1, 1]$.

Applying (74) yields

$$
\begin{aligned}
&\int_0^T (\partial_t(\tilde{\boldsymbol{s}}_h - \boldsymbol{u}), \pi_{k-1}\boldsymbol{w}_{h,\tau})\ dt \\
&\quad\le C\int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0^2\ dt + \frac{\beta_0}{12}\int_0^T \|\pi_{k-1}\left(p_{h,\tau} - i_h\tilde{p}\right)\|_0^2\ dt.
\end{aligned}
$$

Arguing in the same way for the rest of the terms on the right-hand side of (73) leads to

$$
\begin{aligned}
&\int_0^T \|\pi_{k-1}(p_{h,\tau} - i_h\tilde{p})\|_0^2\ dt \\
&\le C\bigg[\int_0^T \|\pi_{k-1}(p - i_h\tilde{p})\|_0^2\ dt + \int_0^T \|\partial_t\boldsymbol{\xi}_{h,\tau}\|_{-1}^2\ dt + \int_0^T |||\pi_{k-1}\boldsymbol{\xi}_{h,\tau}|||^2\ dt \\
&\quad + \int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0^2\ dt + \int_0^T \nu\|\pi_{k-1}\nabla(\tilde{\boldsymbol{s}}_h - \boldsymbol{u})\|_0^2\ dt \\
&\quad + \int_0^T (\|\boldsymbol{b}\|_\infty + \sigma)\|\pi_{k-1}(\tilde{\boldsymbol{s}}_h - \boldsymbol{u})\|_0^2\ dt + \int_0^T \sum_{K\in\mathcal{T}_h} \mu_K\|\kappa_K\pi_{k-1}\nabla\tilde{\boldsymbol{s}}_h\|_{0,K}^2\ dt\bigg]. \qquad (75)
\end{aligned}
$$

Now, the terms on the right-hand side of (75) need to be bounded. The estimates for the third term follows from Theorem 5. In the following, the L^2 stability of the projection π_{k-1} and the interpolation operator with respect to time, i.e., $\|\pi_{k-1}\boldsymbol{v}\|_0 \le C\|\boldsymbol{v}\|_0$ and $\|\tilde{\boldsymbol{v}}\|_0 \le C\|\boldsymbol{v}\|_0$ will be often used. For the first term on the right-hand side of (75), applying (46) and (13) gives

$$\begin{aligned}\int_0^T \|\pi_{k-1}(p - i_h\tilde{p})\|_0^2 \, dt &\leq C \left(\int_0^T \|p - \tilde{p}\|_0^2 \, dt + \int_0^T \|\tilde{p} - i_h\tilde{p}\|_0^2 \, dt \right) \\ &\leq C\tau^{2k+2} \int_0^T \|p^{(k+1)}\|_0^2 \, dt + Ch^{2r} \int_0^T \|\tilde{p}\|_r^2 \, dt \\ &\leq C \left(\tau^{2k+2} \|p\|_{H^{k+1}(L^2)}^2 + h^{2r} \|p\|_{L^2(H^r)}^2 \right).\end{aligned}$$

For bounding the second term on the right-hand side of (75), one first observes that $\int_0^T \|\partial_t \boldsymbol{\xi}_{h,\tau}\|_{-1} \, dt \leq \int_0^T \|\partial_t \boldsymbol{\xi}_{h,\tau}\|_0 \, dt$. Now, since it is assumed that $\boldsymbol{b}$ and σ are independent of t, the error bounds for $\|\boldsymbol{\xi}_{h,\tau}\|_0$ can also be applied to its time derivative so that applying (66) to $\partial_t \boldsymbol{\xi}_{h,\tau}$ leads to

$$\begin{aligned}\int_0^T \|\partial_t \boldsymbol{\xi}_{h,\tau}\|_0^2 \, dt &\leq C(1+T)h^{2r} \left[\|\partial_t \boldsymbol{u}\|_{L^2(H^{r+1})}^2 + \|\partial_t \boldsymbol{u}\|_{H^1(H^r)}^2 + \|\partial_t p\|_{L^2(H^r)}^2 \right] \\ &\quad + C(1+T)\tau^{2k} \|\partial_t \boldsymbol{u}\|_{H^{k+1}(H^1)}^2.\end{aligned}$$

For the truncation errors involving $\tilde{\boldsymbol{s}}_h - \boldsymbol{u}$ (the last four terms), one argues as in Theorem 5 to get

$$\begin{aligned}\int_0^T \|\partial_t(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_0^2 \, dt &\leq Ch^{2r} \|\boldsymbol{u}\|_{H^1(H^r)}^2, \\ \int_0^T \nu \|\pi_{k-1} \nabla(\tilde{\boldsymbol{s}}_h - \boldsymbol{u})\|_0^2 \, dt &\leq C\nu \left(h^{2r} \|\boldsymbol{u}\|_{L^2(H^{r+1})}^2 + \tau^{2k+2} \|\boldsymbol{u}\|_{H^{k+1}(H^1)}^2 \right), \\ \int_0^T (\|\boldsymbol{b}\|_\infty + \sigma) \|\tilde{\boldsymbol{s}}_h - \boldsymbol{u}\|_0^2 \, dt &\leq C \left(h^{2r} \|\boldsymbol{u}\|_{L^2(H^r)}^2 + \tau^{2k+2} \|\boldsymbol{u}\|_{H^{k+1}(H^1)}^2 \right).\end{aligned}$$

The bound for the last term (similarly as in the estimates (58)–(61)) uses the error splitting with respect to space and time, the L^2 stability of the fluctuation operator κ_K, $\mu_K \sim 1$, and the approximation properties of κ_K. One obtains

$$\begin{aligned}&\int_0^T \sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_K \pi_{k-1} \nabla \tilde{\boldsymbol{s}}_h\|_{0,K}^2 \, dt \\ &\leq 3 \int_0^T \sum_{K \in \mathcal{T}_h} \|\kappa_K \nabla(\tilde{\boldsymbol{s}}_h - \tilde{\boldsymbol{u}})\|_{0,K}^2 \, dt + 3 \int_0^T \sum_{K \in \mathcal{T}_h} \|\kappa_K \nabla(\tilde{\boldsymbol{u}} - \boldsymbol{u})\|_{0,K}^2 \, dt \\ &\quad + 3 \int_0^T \sum_{K \in \mathcal{T}_h} \|\kappa_K \nabla \boldsymbol{u}\|_{0,K}^2 \, dt \\ &\leq C \left(h^{2r} \|\boldsymbol{u}\|_{L^2(H^{r+1})}^2 + \tau^{2k+2} \|\boldsymbol{u}\|_{H^{k+1}(H^{r+1})}^2 \right).\end{aligned}$$

The statement of the theorem follows by collecting the bounds for all terms on the right-hand side of (75), and by applying the triangle inequality and the bounds (13) and (46) for the interpolation errors in space and time. □

Remark 8. *Instead of using* $\int_0^T \|\partial_t \boldsymbol{\xi}_{h,\tau}\|_{-1}^2 \, dt \leq \int_0^T \|\partial_t \boldsymbol{\xi}_{h,\tau}\|_0^2 \, dt$, *one could use*

$$\int_0^T \|\partial_t \boldsymbol{\xi}_{h,\tau}\|_{-1}^2 \, dt \leq C \int_0^T \|A_h^{-1/2} \partial_t \boldsymbol{\xi}_{h,\tau}\|_0^2 \, dt$$

and then argue as in the proof of Theorem 3. However, since it is assumed that $\boldsymbol{b}$ *is time-independent, the proof presented above is shorter although it requires a higher regularity of the solution.*

5 Numerical studies

Two examples will be presented that support the theoretical results obtained in the previous sections. In the first example, an analytical solution is considered and very small time steps are applied to support the error analysis of Section 3. In the second example the solution is polynomial in the space such that the approximation will be exact in the spatial part and the discretization error in time dominates. This example will support the analytical results from Section 4. A third example presents a brief comparison of the method studied in this paper with a different stabilized method for the evolutionary Oseen equations.

All simulations were performed on uniform quadrilateral grids where the coarsest grid (level 1) is obtained by dividing the unit square into four squares. Mapped finite element spaces [17] were used, where the enriched spaces on the reference cell $\hat{K} = [-1, 1]^2$ are given by

$$\mathbb{Q}_r^{\text{bubble}}(\hat{K}) := \mathbb{Q}_r(\hat{K}) + \operatorname{span}\left\{(1-\hat{x}_1^2)(1-\hat{x}_2^2)\hat{x}_i^{r-1},\ i = 1, 2\right\}.$$

The combination $\mathbb{Q}_r^{\text{bubble}}(\hat{K})$ with $D(K) = \mathbb{P}_{r-1}(K)$ provides for $r \geq 2$ suitable spaces for LPS methods, see [36]. The simulations were performed with the code MooNMD [26].

Example 9. *An example with negligible temporal error.* Consider the Oseen problem (1) with $\Omega = (0, 1)^2$, $\nu = 10^{-10}$, $\boldsymbol{b} = \boldsymbol{u}$, $\sigma = 1$, and $T = 1$. The right-hand side $\boldsymbol{f}$ and the initial condition $\boldsymbol{u}_0$ were chosen such that

$$\boldsymbol{u}(t, x, y) = \sin(t)\begin{pmatrix}\sin(\pi x)\sin(\pi y)\\ \cos(\pi x)\cos(\pi y)\end{pmatrix},$$
$$p(t, x, y) = \sin(t)\left(\sin(\pi x) + \cos(\pi y) - \frac{2}{\pi}\right)$$

is the solution of (1) equipped with non-homogeneous Dirichlet boundary conditions.

This example studies the convergence order with respect to space. To this end, the time discretization scheme cGP(2) with the small time step length $\tau = 1/1280$ was used. Numerical studies concerning the choice of stabilization parameters for convection-dominated problems suggest that a good choice is $\mu_K \in (0, 1)$, e.g., see [6]. Based on these studies and our own experience, the stabilization parameters were set to be $\mu_K = 0.1$. The convergence plots for simulations with the finite element spaces $V_h/Q_h = \mathbb{Q}_3^{\text{bubble}}/\mathbb{P}_2^{\text{disc}}$ and the projection space $D(K) = \mathbb{P}_2(K)$ are presented in Figure 1 and Table 1. One can see fourth order convergence for the $L^2(L^2)$ norm and the L^2 norm at the final time. For all other norms on the left-hand side of (24) and the $L^2(L^2)$ norm of pressure, third order of convergence can be observed. It can be seen in Figure 1 and Table 1 that $\|\kappa_K \nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(L^2)}$ is the dominant term among the velocity errors on the left-hand side of (24). Altogether, the order of convergence is exactly as predicted in (24) and (30).

Example 10. *An example with dominant temporal error.* Let $\Omega = (0, 1)^2$, $\nu = 10^{-10}$, $\boldsymbol{b} = \boldsymbol{u}$, $\sigma = 1$, $T = 1$ and consider the Oseen equations (1) with the prescribed solution

$$\boldsymbol{u} = \begin{pmatrix}\sin(40t)y\\ \cos(t)x\end{pmatrix}, \quad p(t, x, y) = \cos(40t)(x - 0.5) + \sin(40t)(2y - 1).$$

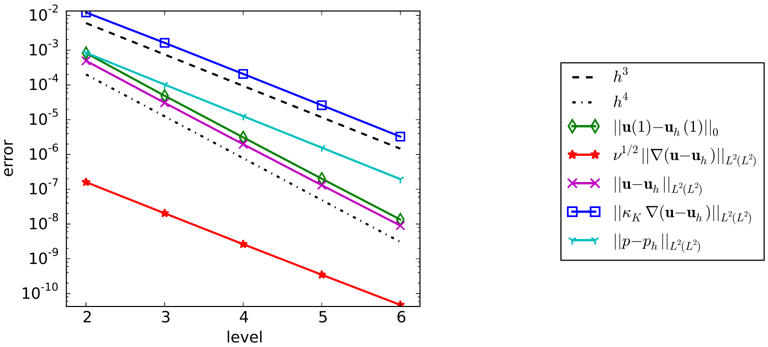

Figure 1: Example 9: Convergence of various errors with respect to the spatial mesh width.

Table 1: Example 9: Various errors with respect to the spatial mesh width, $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$, the order is computed from the results of the finest levels.

level	$\|\boldsymbol{e}_h(1)\|_0$	$\nu^{1/2}\|\nabla\boldsymbol{e}\|_{L^2(L^2)}$	$\|\boldsymbol{e}\|_{L^2(L^2)}$	$\|\kappa_K\nabla\boldsymbol{e}\|_{L^2(L^2)}$	$\|p-p_h\|_{L^2(L^2)}$
2	8.261071e-04	1.599153e-07	4.936612e-04	3.846468e-03	8.430718e-04
3	4.890464e-05	2.035870e-08	3.063295e-05	5.139051e-04	1.026164e-04
4	3.104466e-06	2.625447e-09	1.969092e-06	6.548987e-05	1.263452e-05
5	2.010103e-07	3.471540e-10	1.308868e-07	8.241557e-06	1.567482e-06
6	1.344915e-08	4.718591e-11	9.000207e-09	1.032959e-06	1.952184e-07
order	3.90	2.88	3.86	3.00	3.01

In this example, the spaces $V_h/Q_h = \mathbb{Q}_2^{\text{bubble}}/\mathbb{P}_1^{\text{disc}}$ and the projection space $D(K) = \mathbb{P}_1(K)$ were considered. The mesh consisted of 16×16 squares. Note that for any time t the solution can be represented exactly by functions from the finite element spaces V_h and Q_h. Hence, all occurring errors will result from the temporal discretization.

Figure 2 and Table 2 report the order of convergence for the methods cGP(k), $k \in \{2,3,4\}$, in combination with the LPS method. One can observe the predicted convergence order $k+1$ for the errors estimated in (47) and (64). Also for the pressure, order $k+1$ can be seen although estimate (67) predicts only order k.

Example 11. *Comparison with a grad-div stabilized method.* This example provides a brief comparison of a LPS/cGP method with another stabilized scheme for the evolutionary Oseen problem that was studied in the literature.

The method for comparison is the grad-div stabilization which is analyzed for the evolutionary Oseen equations in [19]. The same example as in [19] was used with the analytical solution

$$\begin{aligned} \boldsymbol{u} &= \cos(t)\begin{pmatrix}\sin(\pi x - 0.7)\sin(\pi y + 0.2)\\ \cos(\pi x - 0.7)\cos(\pi y + 0.2)\end{pmatrix}, \\ p &= \cos(t)(\sin(x)\cos(y) + (\cos(1)-1)\sin(1)). \end{aligned}$$

Numerical results for $\nu = 10^{-6}$, $\boldsymbol{b} = \boldsymbol{u}$, $\sigma = 1$, $\Omega = (0,1)^2$, and $T = 5$ are presented.

Both schemes were applied with respective standard configurations that are comparable. For the LPS/cGP method, cGP(2) was used as temporal discretization. Since this is a third

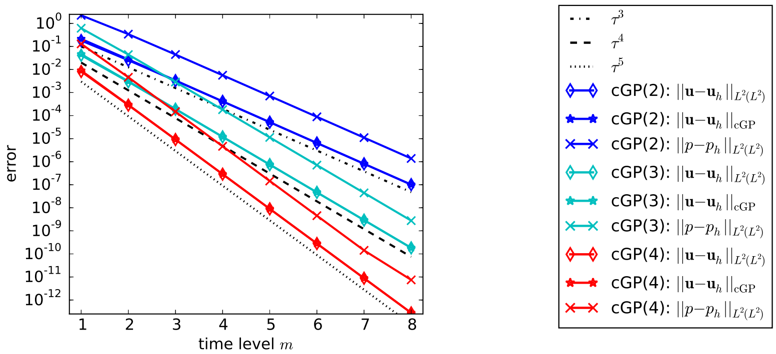

Figure 2: Example 10: Convergence of various errors with respect to the time step, where the time step is given by $\tau = 0.1 \cdot 2^{-m+1}$.

Table 2: Example 10: $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{cGP}}$, the order is computed from the results of the finest levels.

time level	cGP(2)	cGP(3)	cGP(4)
1	2.123580e-01	4.621311e-02	8.992460e-03
2	2.754030e-02	3.103225e-03	2.953547e-04
3	3.364046e-03	1.938513e-04	9.254991e-06
4	4.162566e-04	1.202551e-05	2.887537e-07
5	5.191342e-05	7.495265e-07	9.017116e-09
6	6.485677e-06	4.681011e-08	2.817272e-10
7	8.106020e-07	2.925068e-09	8.803837e-12
8	1.013219e-07	1.828079e-10	2.773163e-13
order	3.00	4.00	4.99

order method, it is natural to use also a third order spatial discretization. Our choice was $V_h/Q_h = \mathbb{Q}_3^{\mathrm{bubble}}/\mathbb{P}_2^{\mathrm{disc}}$, like in Example 9. Also, the same stabilization parameter $\mu_K = 0.1$ was utilized. Since spatial and temporal discretization are of the same order, convergence studies require the time step to be halved if the grid is refined once. The grad-div stabilization is used in practice with standard pairs of finite element spaces and simple temporal discretizations. This method does not require such special spaces as LPS methods. We used also a third order method in space, namely the Taylor–Hood pair $\mathbb{Q}_3/\mathbb{Q}_2$. The spatial discretizations of both stabilized methods possess a similar number of degrees of freedom on the same meshes. With respect to the temporal discretization, the Crank–Nicolson scheme was used. This scheme is only of second order. To account for this order difference, the time step was refined by the factor of $\sqrt{8}$ whenever the grid was refined once (factor 2 of the mesh width). The time step on level 0 (one mesh cell) was set to be 0.4 for both methods. The stabilization parameter for the grad-div stabilization was chosen to be the same as in the numerical studies in [19].

Results with respect to some standard errors are presented in Tables 3 and 4. It can be observed that both methods possess a similar accuracy and the pressure errors as well as the dominant velocity errors are of at least third order. In particular, the LPS/cGP method is competitive.

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Table 3: Example 11: Various errors for the LPS method with cGP(2), $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$, the order is computed from the results of the finest levels.

level	$\|\boldsymbol{e}_h(5)\|_0$	$\nu^{1/2}\|\nabla \boldsymbol{e}\|_{L^2(L^2)}$	$\|\boldsymbol{e}\|_{L^2(L^2)}$	$\|p - p_h\|_{L^2(L^2)}$
2	2.441646e-04	4.592660e-05	1.532429e-03	7.070311e-04
3	1.817895e-05	5.706725e-06	9.696640e-05	6.520649e-05
4	1.348730e-06	7.203769e-07	6.006353e-06	6.120355e-06
5	1.001092e-07	9.481435e-08	3.900740e-07	6.056901e-07
6	7.185082e-09	1.287943e-08	2.633239e-08	6.435478e-08
order	3.80	2.88	3.89	3.23

Table 4: Example 11: Various errors for the grad-div stabilization with Crank–Nicolson scheme from [19], $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$, the order is computed from the results of the finest levels.

level	$\|\boldsymbol{e}_h(5)\|_0$	$\nu^{1/2}\|\nabla \boldsymbol{e}\|_{L^2(L^2)}$	$\|\boldsymbol{e}\|_{L^2(L^2)}$	$\|p - p_h\|_{L^2(L^2)}$
2	9.066868e-05	1.433633e-05	4.850142e-04	1.893259e-04
3	6.413237e-06	1.388954e-06	2.120568e-05	1.978418e-05
4	7.558467e-07	2.084255e-07	1.752358e-06	2.464377e-06
5	9.698209e-08	5.209587e-08	2.396720e-07	3.076402e-07
6	1.138595e-08	8.287498e-09	2.151598e-08	3.860506e-08
order	3.09	2.65	3.48	2.99

compressible flow problems is beyond the scope of the present paper and a future topic of research.

6 Summary

This paper analyzed a combination of higher order continuous Galerkin–Petrov schemes in time with the one-level variant of the LPS method in space applied to the transient Oseen equations. The continuous-in-time case and the fully discrete situation were considered. Optimal error bounds for velocity and pressure were obtained with constants that do not depend on the viscosity parameter ν. The theoretical results were confirmed by numerical simulations.

References

[1] N. Ahmed, S. Becher, and G. Matthies. Higher-order discontinuous Galerkin time stepping and local projection stabilization techniques for the transient Stokes problem. *Comput. Methods Appl. Mech. Engrg.*, 313:28–52, 2017.

[2] N. Ahmed, T. Chacón Rebollo, V. John, and S. Rubino. Analysis of a full space-time discretization of the Navier-Stokes equations by a local projection stabilization method. *IMA J. Numer. Anal.*, 37(3):1437–1467, 2017.

[3] N. Ahmed and V. John. Adaptive time step control for higher order variational time discretizations applied to convection-diffusion-reaction equations. *Comput. Methods Appl. Mech. Engrg.*, 285:83–101, 2015.

[4] N. Ahmed and G. Matthies. Numerical studies of variational-type time-discretization techniques for transient Oseen problem. In *Algoritmy 2012. 19th conference on scientific computing, Vysoké Tatry, Podbanské, Slovakia, September 9–14, 2012. Proceedings of contributed papers and posters.*, pages 404–415. Bratislava: Slovak University of Technology, Faculty of Civil Engineering, Department of Mathematics and Descriptive Geometry, 2012.

[5] N. Ahmed and G. Matthies. Higher order continuous Galerkin-Petrov time stepping schemes for transient convection-diffusion-reaction equations. *ESAIM Math. Model. Numer. Anal.*, 49(5):1429–1450, 2015.

[6] N. Ahmed and G. Matthies. Numerical study of SUPG and LPS methods combined with higher order variational time discretization schemes applied to time-dependent linear convection-diffusion-reaction equations. *J. Sci. Comput.*, 67(3):988–1018, 2016.

[7] N. Ahmed, G. Matthies, L. Tobiska, and H. Xie. Discontinuous Galerkin time stepping with local projection stabilization for transient convection-diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 200(21-22):1747–1756, 2011.

[8] D. Arndt, H. Dallmann, and G. Lube. Local projection FEM stabilization for the time-dependent incompressible Navier-Stokes problem. *Numer. Methods Partial Differential Equations*, 31(4):1224–1250, 2015.

[9] B. Ayuso, B. García-Archilla, and J. Novo. The postprocessed mixed finite-element method for the Navier-Stokes equations. *SIAM J. Numer. Anal.*, 43(3):1091–1111, 2005.

[10] A. K. Aziz and P. Monk. Continuous finite elements in space and time for the heat equation. *Math. Comp.*, 52(186):255–274, 1989.

[11] G. R. Barrenechea, V. John, and P. Knobloch. A local projection stabilization finite element method with nonlinear crosswind diffusion for convection-diffusion-reaction equations. *ESAIM Math. Model. Numer. Anal.*, 47(5):1335–1366, 2013.

[12] R. Becker and M. Braack. A finite element pressure gradient stabilization for the Stokes equations based on local projections. *Calcolo*, 38(4):173–199, 2001.

[13] R. Becker and M. Braack. A two-level stabilization scheme for the Navier-Stokes equations. In *Numerical mathematics and advanced applications*, pages 123–130. Springer, Berlin, 2004.

[14] M. Braack and E. Burman. Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method. *SIAM J. Numer. Anal.*, 43(6):2544–2566 (electronic), 2006.

[15] M. Braack, E. Burman, V. John, and G. Lube. Stabilized finite element methods for the generalized Oseen problem. *Comput. Methods Appl. Mech. Engrg.*, 196(4-6):853–866, 2007.

[16] A. N. Brooks and T. J. R. Hughes. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 32(1-3):199–259, 1982. FENOMECH '81, Part I (Stuttgart, 1981).

[17] P. G. Ciarlet. *The finite element method for elliptic problems.* North-Holland Publishing Co., Amsterdam, 1978. Studies in Mathematics and its Applications, Vol. 4.

[18] H. Dallmann, D. Arndt, and G. Lube. Local projection stabilization for the Oseen problem. *IMA J. Numer. Anal.*, 36(2):796–823, 2016.

[19] J. de Frutos, B. García-Archilla, V. John, and J. Novo. Grad-div stabilization for the evolutionary Oseen problem with inf-sup stable finite elements. *J. Sci. Comput.*, 66(3):991–1024, 2016.

[20] Javier de Frutos, Bosco García-Archilla, Volker John, and Julia Novo. Analysis of the grad-div stabilization for the time-dependent Navier-Stokes equations with inf-sup stable finite elements. *Adv. Comput. Math.*, 44:195–225, 2018.

[21] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[22] T. J. R. Hughes and A. Brooks. A multidimensional upwind scheme with no crosswind diffusion. In *Finite element methods for convection dominated flows (Papers, Winter Ann. Meeting Amer. Soc. Mech. Engrs., New York, 1979)*, volume 34 of *AMD*, pages 19–35. Amer. Soc. Mech. Engrs. (ASME), New York, 1979.

[23] S. Hussain, F. Schieweck, and S. Turek. A note on accurate and efficient higher order Galerkin time stepping schemes for the nonstationary Stokes equations. *Open Numer. Methods J.*, 4:35–45, 2012.

[24] S. Hussain, F. Schieweck, and S. Turek. An efficient and stable finite element solver of higher order in space and time for nonstationary incompressible flow. *Internat. J. Numer. Methods Fluids*, 73(11):927–952, 2013.

[25] V. John. *Finite element methods for incompressible flow problems*, volume 51 of *Springer Series in Computational Mathematics*. Springer, Cham, 2016.

[26] V. John and G. Matthies. MooNMD—a program package based on mapped finite element methods. *Comput. Vis. Sci.*, 6(2-3):163–169, 2004.

[27] V. John, G. Matthies, and J. Rang. A comparison of time-discretization/linearization approaches for the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 195(44-47):5995–6010, 2006.

[28] V. John and J. Novo. Error analysis of the SUPG finite element discretization of evolutionary convection-diffusion-reaction equations. *SIAM J. Numer. Anal.*, 49(3):1149–1176, 2011.

[29] V. John and J. Novo. Analysis of the pressure stabilized Petrov-Galerkin method for the evolutionary Stokes equations avoiding time step restrictions. *SIAM J. Numer. Anal.*, 53(2):1005–1031, 2015.

[30] V. John and J. Rang. Adaptive time step control for the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 199(9-12):514–524, 2010.

[31] V. John and E. Schmeyer. Finite element methods for time-dependent convection-diffusion-reaction equations with small diffusion. *Comput. Methods Appl. Mech. Engrg.*, 198(3-4):475–494, 2008.

[32] P. Knobloch. On the application of local projection methods to convection-diffusion-reaction problems. In *BAIL 2008—boundary and interior layers*, volume 69 of *Lect. Notes Comput. Sci. Eng.*, pages 183–194. Springer, Berlin, 2009.

[33] P. Knobloch. A generalization of the local projection stabilization for convection-diffusion-reaction equations. *SIAM J. Numer. Anal.*, 48(2):659–680, 2010.

[34] G. Matthies, P. Skrzypacz, and L. Tobiska. A unified convergence analysis for local projection stabilisations applied to the Oseen problem. *M2AN Math. Model. Numer. Anal.*, 41(4):713–742, 2007.

[35] G. Matthies, P. Skrzypacz, and L. Tobiska. Stabilization of local projection type applied to convection-diffusion problems with mixed boundary conditions. *Electron. Trans. Numer. Anal.*, 32:90–105, 2008.

[36] G. Matthies and L. Tobiska. Local projection type stabilization applied to inf-sup stable discretizations of the Oseen problem. *IMA J. Numer. Anal.*, 35(1):239–269, 2015.

[37] F. Schieweck. A-stable discontinuous Galerkin-Petrov time discretization of higher order. *J. Numer. Math.*, 18(1):25–57, 2010.