Optimal control in fluid mechanics by finite elements with symmetric stabilization

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1. Motivation

Optimization problem

Build Karush-Kuhn-Tucker (KKT) system

Solve KKT system
Two possibilities for optimization with PDE

1. Optimization problem with PDE
   - Build KKT system
   - Discretize KKT system
   - Solve discrete KKT system

2. Discretize PDE
   - Discretize PDE
   - Build discretize KKT system
Two possibilities for optimization with PDE

Optimization problem with PDE

Build KKT system

Discretize PDE

Discretize KKT system

Build discretize KKT system

Solve discrete KKT system

Optimize-discretize

Discretize-optimize
**Model problem:** Linearized Navier-Stokes with control $q$

$$ -\mu \Delta v + (\beta \cdot \nabla) v + \sigma v + \nabla p + Bq = f \quad \text{in } \Omega , $$
$$ \text{div } v = 0 \quad \text{in } \Omega , $$
$$ v = 0 \quad \text{on } \partial \Omega , $$

**Objective functional:**

$$ J(u, q) := \frac{1}{2} \| Cu - C\hat{u} \|^2 + \frac{\alpha}{2} \| q \|^2 \rightarrow \text{min!} $$

$C\hat{u} = \hat{v} \text{ observationes}$
Linear flow problem:

\[ Au + Bq = f \]

state variable \( u = (v, p) \), and control \( q \)

Optimal control problem:

\[
\text{arg min} \left\{ J(u, q) : Au + Bq = f \text{ for control } q \in Q \right\}.
\]
Augmented Lagrangian

\[ L(u, q, z) := J(u, q) + \langle z, Au + Bq - f \rangle \]

Unrestricted minimization problem

\[ \min_{u, q, z} L(u, q, z) \]

Necessary conditions for saddle point of \( L \)

\[ d_q L(u, q, z) = 0 \iff d_q J(u, q) + B^* z = 0 \]
\[ d_u L(u, q, z) = 0 \iff d_u J(u, q) + A^* z = 0 \]
\[ d_z L(u, q, \lambda) = 0 \iff Au + Bq = f \]
Continuous Karush-Kuhn-Tucker (KKT) system

\[
\begin{pmatrix}
\alpha I & 0 & B^* \\
0 & C & A^* \\
B & A & 0
\end{pmatrix}
\begin{pmatrix}
q \\
u \\
z
\end{pmatrix}
=
\begin{pmatrix}
0 \\
C \hat{u} \\
f
\end{pmatrix}
\]
What is an appropriate discretization of...

**Primal equation**

\[-\mu \Delta v + (\beta \cdot \nabla)v + \sigma v + \nabla p + Bq = f \quad \text{in } \Omega\]

\[\text{div } v = 0 \quad \text{in } \Omega\]

\[v = 0 \quad \text{on } \partial \Omega\]

**Adjoint equation**

\[-\mu \Delta z_v - (\beta \cdot \nabla)z_v + \sigma z_v - \nabla z_p = \hat{v} - v \quad \text{in } \Omega\]

\[-\text{div } z_v = 0 \quad \text{in } \Omega\]

\[z_v = 0 \quad \text{on } \partial \Omega\]
2. Finite element discretization

Bilinear form for \( u = (v, p) \in X := [H^1_0(\Omega)]^d \times L^2_0(\Omega) \)

\[
a(u, \varphi) := (\text{div } v, \xi) + (\sigma v, \phi) + (\beta \cdot \nabla v, \phi) + (\mu \nabla v, \nabla \phi) - (p, \text{div } \phi)
\]

Influence of the control by \( b : Q \times X \rightarrow \mathbb{R} \) for \( q \in Q \subset L^2(\Omega) \).

Variational formulation:

\[
u \in X : \quad a(u, \varphi) + b(q, \varphi) = (f, \varphi) \quad \forall \varphi \in X
\]

Galerkin formulation:

\[
u_h \in X_h : \quad a(u_h, \varphi) + b(q_h, \varphi) = (f, \varphi) \quad \forall \varphi \in X_h
\]
**SUPG+PSPG, Grad-div stabilization for Oseen**

- Inf-sup condition not fulfilled for equal-order elements
- Dominant convective terms

\[ s_h(u_h)(\varphi) = \sum_{T \in T_h} \int_T \left\{ \rho_{mom} \cdot [\delta_T (\beta \cdot \nabla) \phi + \alpha_T \nabla \xi] + (\text{div} \, v) \gamma_T (\text{div} \phi) \right\} \, dx \]

(Hughes, Johnson, Lube, Tobiska, Glowinski, Le Tallec,..)
SUPG+PSPG, Grad-div stabilization for Oseen

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(Hughes, Johnson, Lube, Tobiska, Glowinski, Le Tallec,..)

**Discretized primal problem:**

\[
(A_h + S^u_h)u_h + (B_h + S^q_h)q_h = f_h
\]

- Forget for a while the parameter dependence: \( S^u_h, S^q_h \) are linear.
- Otherwise: \( S^u_h, S^q_h, f_h \) may depend on \( u_h \).
**Adjoint equation:**

\[-\mu \Delta z_v - (\beta \cdot \nabla) z_v + \sigma z_v - \nabla z_p = \hat{\nu} - \nu \quad \text{in } \Omega \]
\[-\text{div } z_v = 0 \quad \text{in } \Omega \]
\[z_v = 0 \quad \text{on } \partial \Omega \]

is also of Ossen type and need to be stabilized.

**Discretized adjoint problem:**

\[(A_h^* + S_h^z)z_h + Cu_h = C\hat{u} \]

For residual based stabilization: \(S_h^z\) depend on the full adjoint residual.
Discrete KKT system (optimize-discretize):

\[
\begin{pmatrix}
\alpha I & 0 & B_h^* \\
0 & C_h & A_h^* + S_h^z \\
B_h + S_h^q & A_h + S_h^u & 0 \\
\end{pmatrix}
\begin{pmatrix}
q_h \\
u_h \\
z_h \\
\end{pmatrix} = \begin{pmatrix}
0 \\
C\hat{u} \\
f_h \\
\end{pmatrix}
\]

The other way round (discretize-optimize): cf. Collis & Heinkenschloss [2002]

Build KKT system of discretized PDE:

\[(A_h + S_h^u)u + (B_h + S_h^q)q_h = f_h\]

In general: \(S_h^q \neq 0\) and \(S_h^z \neq (S_h^u)^*\).
Streamline diffusion & pressure stabilized Petrov Galerkin

\[
(S_h^u)^* - S_h^z \equiv \sum_K \{(\hat{v}_h - v_h + \sigma z^v + (\beta \cdot \nabla)z^v - \mu \Delta z^v, \delta^p \nabla \xi)_K \} \\
+ \sum_K \{(\sigma \phi + (\beta \cdot \nabla)\phi - \mu \Delta \phi, \delta^p \nabla z^p)_K \}
\]

\[
(\hat{v}_h - v_h - \nabla z^p, \delta^v (\beta \cdot \nabla) \phi) \\
+ (\nabla \xi, \delta^v (\beta \cdot \nabla) z^v)
\]

Numerical tests by Collis & Heinkenschloss [2002]:

- D-O has better convergence properties than O-D for SUPG;
- large differences in \(z_h\) between D-O and O-D.
comparison of *do* and *od* with different settings of stabilization constants:

- **diag:** $h_K := \text{max. element length}$
- **adv:** $h_K := \sum_i |(\beta_K \cdot \nabla)\phi_i|_K / \|\beta\|_{K,\infty}$ (Tezduyar, Park (1986))
Consider linear stabilization:

\[ a(u_h, \varphi) + b(q_h, \varphi) + s_h(u_h, \varphi) = (f, \varphi) \quad \forall \varphi \in X_h \]

First requirement **Symmetry**:

(P1) \[ s_h(u, \varphi) = s_h(\varphi, u) \quad \forall u, \varphi \in X \]

**Lemma**: For linear and symmetric stabilization (P1), discretization and optimization commutes.
We will show an a priori estimate in a (semi) norm:

\[ \| \cdot \|_h : X \rightarrow \mathbb{R}_0^+ \]

**Second requirement Coercivity:**

\[(P2) \quad \| u_h \|_h^2 \lesssim a_h(u_h, u_h) + s_h(u_h, u_h) \quad \forall u_h \in X_h \]

This is the case e.g. for

\[ \| u \|_h := (a_h(u, u) + s_h(u, u))^{1/2} \]

if \( s_h(u, u) \geq 0. \)
Third requirement:
\[ \| u_h \|_h \text{ stronger than } L^2\text{-norm of velocities:} \]

\[ (P3) \quad \| v \| \lesssim \| u \|_h \quad \forall u = (v, p) \in X \]

For example:

\[ \| u \|_h^2 = \sigma \| v \|^2 + \mu \| \nabla v \|^2 + s_h(u, u) \]
Fourth requirement: a priori estimate for fixed control.

For \( u \in [H^{r+1}(\Omega)]^{d+1} \) and finite elements of order \( r \):

\[
(P4) \quad \| u(q) - u_h(q) \|_h \lesssim h^s \| u \|_{r+1}
\]

\( u(q), u_h(q) = \) solutions of continuous and discrete problems for given control \( q \in Q \).

convergence order \( s \leq r + 1 \) (optimal \( s = r + 1/2 \))

**Lemma:** If (P4) holds for the primal problem, then it holds for the adjoint problems with given velocity field \( w \) in the rhs:

\[
\| z(w) - z_h(w) \|_h \lesssim h^s \| z \|_{r+1} \quad \text{if} \ z \in [H^{r+1}(\Omega)]^{d+1}
\]
4. A convergence result

Theorem

Under the following conditions:

- (P1), (P2), (P3), (P4)
- approximation property of the discrete control space:
  \[ \|q - i_h q\| \lesssim h^s \|q\|_{r+1} \]
- regularity of the solutions: \( u, z \in [H^{r+1}(\Omega)]^{d+1}, q \in H^{r+1}(\Omega) \)

it holds the convergence result:

\[ \|q - q_h\| \lesssim h^s (\|u\|_{r+1} + \|z\|_{r+1} + \|q\|_{r+1}) \]
**Principle of proof:**

Since the reduced functional \( j_h(q) := J(u_h(q), q) \) is at most quadratic:

\[
\alpha \left\Vert i_h q - q_h \right\Vert^2 \leq j''_h(q_h)(\delta q_h) = j'_h(q_h + \delta q_h)(\delta q_h) - j'_h(q_h)(\delta q_h) = 0 \Rightarrow j'(q)(\delta q_h)
\]

Expressing \( j' \) and \( j'_h \) and continuity of \( b(\cdot, \cdot) \) gives (\( \hat{z}_h := z_h(u_h(i_h q)) \)):

\[
\alpha \left\Vert i_h q - q_h \right\Vert^2 \leq b(i_h q - q_h, \hat{z}_h^\gamma - z^\gamma) + (\alpha (i_h q - q), \delta q_h)
\]

\[
\leq c \left\Vert \hat{z}_h^\gamma - z^\gamma \right\Vert \cdot \left\Vert i_h q - q_h \right\Vert + \alpha \left\Vert i_h q - q \right\Vert \cdot \left\Vert i_h q - q_h \right\Vert
\]

\[
\left\Vert \hat{z}_h^\gamma - z^\gamma \right\Vert \leq \left\Vert z_h^\gamma(u_h(i_h q)) - z_h^\gamma(u(q)) \right\Vert + \left\Vert z_h^\gamma(u(q)) - z^\gamma(u(q)) \right\Vert
\]

stabilization & adjoint & primal pb.

prev. Lemma
Theorem

Under the same conditions as the previous theorem with $s = r + \frac{1}{2}$:

$$\| u - u_h \|_h^2 \lesssim h^{r+\frac{1}{2}} (\| u \|_{r+1} + \| z \|_{r+1} + \| q \|_{r+1})$$

Proof.

$$\| u - u_h \|_h \leq \| u(q) - u_h(q) \|_h + \| u_h(q) - u_h(q_h) \|_h$$

$$h^{r+\frac{1}{2}} \| u \|_{r+1} \text{ due to (P4)}$$

Coercivity (P2) for $w_h := u_h(q) - u_h(q_h)$:

$$\| w_h \|_h^2 \lesssim a(w_h, w_h) + s_h(w_h, w_h) = -(B(q - q_h), w_h^\nu)$$

Cauchy-Schwarz, (P3) and continuity of $B$:

$$\| w_h \|_h \lesssim \| B(q - q_h) \| \lesssim \| q - q_h \| \quad \square$$
5. Examples of symmetric stabilization techniques

**Edge oriented stabilization (EOS)** [Burman, Hansbo]

Jumps across edges:

\[
[u(x)] := u(x)|_K - u(x)|_{K'}.
\]

Stabilization terms:

\[
s_{h}^{es}(u, \varphi) := s_{h}^{es,p}(p, \xi) + s_{h}^{es,v}(v, \phi)
\]

\[
s_{h}^{es,p}(p, \xi) := \sum_{K \in \mathcal{T}_h} \int_{\partial K} \alpha_K \|\nabla p\| \cdot \|\nabla \xi\| \, ds
\]

\[
s_{h}^{es,v}(v, \phi) := \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left\{ \delta_K \|n \cdot \nabla v\| \cdot \|n \cdot \nabla \phi\| + \gamma_K \|\text{div} \, v\| \cdot \|\text{div} \, \phi\| \right\} \, ds
\]

Fulfill (P1), (P2), (P3) and (P4).
5. Examples of symmetric stabilization techniques

**Edge oriented stabilization (EOS)** [Burman, Hansbo]

Jumps across edges:

$$\left[ u(x) \right] := u(x)|_K - u(x)|_{K'}.$$  

Stabilization terms:

$$s_{es}^p (u, \varphi) := s_{es,p}^p (p, \xi) + s_{es,v}^v (v, \phi)$$

$$s_{es,p}^p (p, \xi) := \sum_{K \in T_h} \int_{\partial K} \alpha_K \left[ \nabla p \right] \cdot \left[ \nabla \xi \right] \, ds$$

$$s_{es,v}^v (v, \phi) := \sum_{K \in T_h} \int_{\partial K} \left\{ \delta_K \left[ n \cdot \nabla v \right] \cdot \left[ n \cdot \nabla \phi \right] + \gamma_K \left[ \text{div } v \right] \cdot \left[ \text{div } \phi \right] \right\} \, ds$$

Fulfill (P1), (P2), (P3) and (P4). Hence: optimal order of convergence.
Local projection stabilization (LPS) [Becker, Br., Burman, Tobiska, Matthies, Lube, Rapin]

Step 1 - Definition of fluctuation operator:
- $D_{2h}^{r-1} =$ discontinuous, patchwise polynomial order $r - 1$.

- Patchwise $L^2$-projection
  \[ \pi_h : L^2(\Omega) \to D_{2h}^{r-1} \]

- Fluctuation operator
  \[ \kappa_h = i - \pi_h \]

Example $r = 1$: Patch-wise projection on constants:
\[ \kappa_h \nabla p|_K = \nabla p - \frac{1}{|K|} \int_K \nabla p \, dx, \quad K \in T_{2h} \]
Step 2 - Definition of stabilization terms

- Pressure stabilization (Br. & Becker ’00)

\[ S_h(u, \varphi) = \left( \kappa_h(\nabla p), \alpha \kappa_h(\nabla \xi) \right) \]

- stabilization of convective terms by the full gradient

\[ \ldots + \left( \kappa_h(\nabla v), \delta \kappa_h(\nabla \phi) \right) \]

- or streamline derivatives + stabilization of divergence-free condition

\[ \ldots + \left( \kappa_h((\beta \cdot \nabla)v), \delta \kappa((\beta \cdot \nabla)\phi) \right) + \left( \kappa_h(\text{div } v), \gamma \kappa((\text{div } \phi)) \right) \]

But: nonlinear for Navier-Stokes.
Fulfill (P1), (P2), (P3) and (P4).
Step 2 - Definition of stabilization terms

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Fulfill (P1), (P2), (P3) and (P4). Hence: optimal order of convergence.
6. Numerical validation

Navier-Stokes:

\[-\mu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p + Bq = f \quad \text{in } \Omega,\]
\[\text{div } \mathbf{v} = 0 \quad \text{in } \Omega,\]
\[\mathbf{v} = \mathbf{v}_0 \quad \text{on } \partial \Omega,\]

Discretized with local projection stabilization.

DFG benchmark: (uncontroled solution at $Re = 100$)
Objective functional:

\[ J(v, q) := \frac{1}{2} \| v - \hat{v} \|^2 \rightarrow \min! \]

\( \hat{v}(x, y) = \text{double-Poiseuille flow (parabolic)} \)

Initial sol.  optimiz. sol.
Comparison of convergence:

LPS = local projection stabilization (symmetric)
GLS = PSPG / SUPG optimize-discretize
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LPS = local projection stabilization (symmetric)
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Further optimization results with LPS: Becker, Meidner, Vexler
Type of discretization is important for flow control.
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Finite element schemes may provide consistent KKT systems when symmetric stabilization is used.
Summary

- Type of discretization is important for flow control.
- Finite element schemes may provide consistent KKT systems when symmetric stabilization is used.
- Convergence proof for Oseen with general symmetric stabilization (LPS, EOS,...)
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Convergence proof for Oseen with general symmetric stabilization (LPS,EOS,...)

First numerical test problem indicate the benefit of symmetric stabilization.
Summary

- Type of discretization is important for flow control.
- Finite element schemes may provide consistent KKT systems when symmetric stabilization is used.
- Convergence proof for Oseen with general symmetric stabilization (LPS,EOS,...)
- First numerical test problem indicate the benefit of symmetric stabilization.

Thanks a lot!