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Berlin, 17.06.2019

Numerical Mathematics III – Partial Differential Equations

Exercise Problems 09

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Local basis.* Let $P(K)$ be unisolvent with respect to the functionals $\{\Phi_{K,i}\}_{i=1}^{N_K}$. Show that the set $\{\phi_{K,i}\}_{i=1}^{N_K}$ with $\Phi_{K,i}(\varphi_{K,j}) = \delta_{ij}$ forms a basis of $P(K)$.

2. *Local basis of $P_2(\hat{K})$.* Consider the reference triangle $\widehat{K}$ with the vertices $(0,0)$, $(1,0)$, and $(0,1)$. The space of polynomials of degree two is spanned by
$$1,\ \widehat{x},\ \widehat{y},\ \widehat{x}\widehat{y},\ \widehat{x}^2,\ \widehat{y}^2.$$
Use as functionals the values of the functions in the vertices and the barycenters of the edges. Compute the local basis with respect to these functionals.

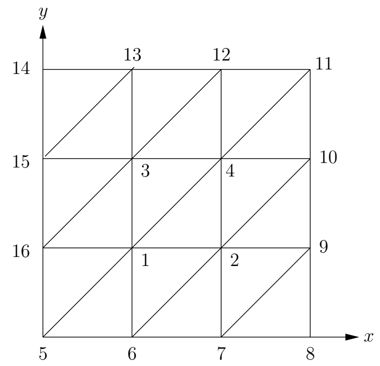
3. *Local basis of $Q_1^{\mathrm{rot}}(\hat{K})$.* Consider the reference square $\widehat{K} = [-1,1]^2$ and the space that is spanned by the functions
$$1,\ \widehat{x},\ \widehat{y},\ \widehat{x}^2 - \widehat{y}^2.$$
Let the integral mean values on the edges of $\widehat{K}$ be the functionals of the finite element
$$\Phi(v) = \frac{1}{\mathrm{meas}(E)} \int_E v(s)\ ds.$$
Compute the local basis with respect to these functionals.

4. *Affine transform.* Let K be a triangle with in $\mathbb{R}^2$ with the vertices $(x_i, y_i), i = 1,2,3$. Compute the affine transform of the reference triangle $\hat{K}$ to K, which maps $(0,0)$ to (x_1,y_1), $(1,0)$ to (x_2,y_2), and $(0,1)$ to (x_3,y_3). Which geometric property of K is connected to the determinant of the matrix of the transform?

5. **UNTIL July 04 !!!**
Code with P_1 finite elements. Write a code (MATLAB or some other language) for the numerical solution of
$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0,1)^2, \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$
with the finite element method.

   The right-hand side and the Dirichlet boundary conditions should be chosen such that
$$u(x,y) = x^4 y^5 - 17 \sin(xy)$$
is the solution of the boundary value problem.

Use for discretizing the problem the P_1 finite element method on the following grid



Choose the mesh width to be

$$h_x = h_y = h = 2^{-n} \quad n = 2, 3, 4, \ldots, 8.$$

Store the matrix in `sparse` format. The vertices should be enumerated analogously as in the sketch, i.e., the interior nodes are enumerated lexicographically and then follow the vertices on the boundary, counter clockwise and starting with $(0, 0)$.

Evaluate

$$\|u - u_h\|_{L^2(\Omega)} \quad \text{and} \quad \|\nabla u - \nabla u_h\|_{L^2(\Omega)}.$$

Hint: the FEM problem is: Find $u^h \in P_1 +$ boundary condition with

$$(\nabla u^h, \nabla v^h) = (f, v^h) \quad \forall\ v^h \in P_1 + \text{ zero boundary condition.}$$

One can write the integrals as a sum over the mesh cells, for instance

$$(\nabla u^h, \nabla v^h) = \sum_{K \in \mathcal{T}^h} (\nabla u^h, \nabla v^h)_K.$$

For this reason, one should use in FEM an approach for assembling the matrices and the right-hand side which is based on a loop over the mesh cells (and not over the vertices as in finite difference methods):

- write a loop over the mesh cells,
- compute for each mesh cell K the numbers of the degrees of freedom (unknowns), which are for the P_1 finite element the numbers of the vertices,
- compute the local update of the matrix entries

  $$(\nabla \phi_j, \nabla \phi_i)_K$$

  and add this update to the global matrix

  $$a_{ij} := a_{ij} + (\nabla \phi_j, \nabla \phi_i)_K.$$

  Do the same for the right-hand side.

Concerning the matrix, one compute alternatively the entries by hand and just set them in the correct positions.

In the rows, which correspond to the nodes on the boundary, replace the diagonal entry with one and set all other entries to be zero. The respective entry on the right-hand side gets the value of the boundary condition in this node.

In this way, one has obtained the linear system of equations whose solution gives the coefficient of the finite element solution u^h.

For the computation of the errors, use the same approach as for assembling the matrix:

- write a loop over the mesh cells,
- compute for each mesh cell K the numbers of the degrees of freedom (unknowns), which are for the P_1 finite element the numbers of the vertices,
- compute the squares of the local errors

$$l2 := (u - u^h, u - u^h), \quad h1 := (\nabla u - \nabla u^h, \nabla u - \nabla u^h),$$

  (formula for ∇u^h see previous problem)
- update the square of the global errors

$$L2 := L2 + l2, \quad H1 := H1 + h1.$$

Finally, take the roots of $L2$ and $H1$.

For the numerical quadrature in assembling the matrix and for computing the errors, use the edge midpoint rule

$$\int_K f(\boldsymbol{x})\ d\boldsymbol{x} \approx \frac{|K|}{3}\left(f(\boldsymbol{x}_1) + f(\boldsymbol{x}_2) + f(\boldsymbol{x}_3)\right),$$

where $|K|$ is the volume (area) of the triangle K and $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are the midpoints of the edges of K.

The exercise problems should be solved in groups of two or three students. The written parts have to be submitted until **Thursday, June 27, 2019**. The executable codes have to be send by email to A. Jha.