Department for Mathematics and Computer Science
Free University of Berlin
Prof. Dr. V. John, john@wias-berlin.de
Hanne Hardering, harderin@zedat.fu-berlin.de

Berlin, 27.05.2013

Numerical Mathematics III – Partial Differential Equations

Exercise Problems 07

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. Show that
$$a(u,v) = \int_0^\infty e^{-x} u(x) g(x)\ dx$$
defines a (real) inner product in $L^2(0,\infty)$.
Hint: A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a (real) inner product if it is symmetric and coercive:
   i) $a(\alpha u+\beta v, w) = \alpha a(u,v)+\beta a(w,v)$, $a(u, \alpha v+\beta w) = \alpha a(u,v)+\beta a(u,w)$, $\forall u,v,w \in V, \alpha, \beta \in \mathbb{R}$,
   ii) $a(u,v) = a(v,u)\ \forall u,v \in V$,
   iii) $a(u,u) \geq 0\ \forall u \in V$ and $a(u,u) = 0 \iff u = 0$.

2. Let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the bilinear form
$$a(u,v) = \int_\Omega \nabla u(\mathbf{x})^T A(\mathbf{x}) \nabla v(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x})\ d\mathbf{x}.$$
with
$$m\|\mathbf{y}\|_2^2 \leq \mathbf{y}^T A(\mathbf{x})\mathbf{y} \leq M\|\mathbf{y}\|_2^2 \quad \forall\ \mathbf{y} \in \mathbb{R}^d, \quad \forall\ \mathbf{x} \in \Omega,$$
and $c \in L^\infty(\Omega)$, $c \geq 0$. Show that this bilinear form is bounded, i.e., there is a constant c such that
$$|a(u,v)| \leq c\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad \forall\ u,v \in H^1(\Omega).$$

3. Let
$$A = (a_{ij}) = a(\phi_j, \phi_i),$$
where $\{\phi_i\}_{i=1}^k$ is the basis of a finite dimensional space V_k. Show that
$$A = A^T \iff a(v,w) = a(w,v)\ \forall v,w \in V_k,$$
$$\mathbf{x}^T A\mathbf{x} > 0 \text{ for } \mathbf{x} \neq \mathbf{0} \iff a(v_k,v_k) > 0 \text{ for } v_k \neq 0.$$

The exercise problems should be solved in groups of two or three students. They have to be submitted until **Tuesday, June 04, 2013** either before or after one of the lectures or directly at the office of Mrs. Hardering. The executable codes have to be send by email to Mrs. Hardering.