Chapter 8

Outlook

Remark 8.1 *More general problems.*

- There are only few contributions to the analysis of multigrid methods for problems which are not symmetric positive definite or a slight perturbation of such problems. One example where it is nothing proved are linear convection-diffusion equations which are convection-dominated. However, the practical experience is that multigrid solvers, with appropriate preconditioners, work reasonably well for such problems.
- The key for the efficiency of the multigrid method is generally the smoother. There is a lot of experience for scalar problems, e.g., for convection-diffusion problems often SSOR or ILU work reasonably well, see Example 8.2. For coupled problems, sometimes the construction of smoothers is already complicated. For instance, many discretizations for the Navier–Stokes equations lead to matrices where a number of diagonal entries are zero. In this case, one cannot apply classical iterative schemes since these schemes require the division by the diagonal entries.
- Algebraic multigrid methods are usually applied to scalar problems. There are only few proposals of algebraic multigrid methods for coupled problems.
- The extension of the multigrid idea to nonconforming finite element discretizations is possible.

□

Example 8.2 *Convection-diffusion problem in two dimensions.* A standard convection-diffusion test problem in two dimensions has the form

$$\begin{aligned} -\varepsilon\Delta u + (1,0)^T \cdot \nabla u &= 1 \quad \text{in } \Omega = (0,1)^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

see the lecture notes of the course on numerical methods for convection-dominated problems. Considering $\varepsilon = 10^{-8}$ with the Q_1 finite element method and the SUPG stabilization, then one obtains the iterations and computing times as shown in Table 8.1. In these simulations, the multigrid methods were applied with the F-(ν,ν)-cycle, where ν is the number of pre and post smoothing steps. In the geometric multigrid method, a SSOR smoother was used and in the algebraic multigrid method, a ILU smoother.

Table 8.1: Example 8.2. Number of iterations and computing times (14/01/23 on a HP BL460c Gen8 2xXeon, Eight-Core 2700MHz). The number of degrees of freedom (d.o.f.) includes the Dirichlet values.

level	h	d.o.f.	FGMRES+MG, F(3,3)		FGMRES+MG F(10,10)		FGMRES+AMG F(3,3)		FGMRES+AMG F(5,5)		UMFPACK	
			ite	time	ite	time	ite	time	ite	time	ite	time
0	1/16	289	1	0	1	0	6	0	4	0	1	0
1	1/32	1089	6	0.03	2	0.01	8	0.05	5	0.05	1	0.01
2	1/64	4225	9	0.10	3	0.05	11	0.42	7	0.41	1	0.02
3	1/128	16641	15	0.44	5	0.25	17	3.19	12	4.64	1	0.16
4	1/256	66049	26	2.68	9	1.73	30	34.32	23	42.20	1	1.35
5	1/512	263169	47	20.09	16	8.59	no conv.		149	866.58	1	10.27
6	1/1024	1050625	145	252.80	29	66.69			no conv.		1	75.17
7	1/2048	4198401	302	2057.39	76	838.18						

One can see in Table 8.2 that none of the solvers behaves optimal, i.e,. for none of the solvers, the computing time scales with the number of degrees of freedom. The most efficient solvers in this example are the direct solver (note that this is a two-dimensional problem) and the geometric multigrid as preconditioner with sufficiently many smoothing steps. On the finest grid, only the geometric multigrid approaches work since the direct solver terminates because of internal memory limitations. In the multigrid methods, one can well observe the effect of increasing the number of smoothing steps.

Altogether, the linear systems obtained for convection-dominated problem are usually hard to solve and so far an optimal solver is not known. □

Remark 8.3 *Multigrid methods with different finite element spaces.* One can apply the multigrid idea also with different (finite element) spaces. For instance, consider just one grid. As coarse grid space, one can use P_1 finite elements and as fine grid space P_2 finite elements. With these two finite element spaces, one can perform a two-level method.

This idea has been used in the construction of finite element spaces for higher order finite elements. It is known from numerical studies that multigrid methods with the same finite element space on all levels might become inefficient for higher order elements because it is hard to construct good smoothers. On the other hand, multigrid methods are usually more efficient for lower order elements. The idea consists in using on the fine grid the higher order finite element space as the finest level of the multigrid hierarchy and using as next coarser level of this hierarchy a first order finite element space on the same geometric grid. On the coarser geometric grids, one uses also low order finite elements. In this way, one has a multigrid method for the higher order discretization which uses low order discretizations on the coarser grids. Some good experience with this approach is reported in the literature. □

Remark 8.4 *Simulations in practice.* The great difficulty of the application of multigrid methods for problems from practice comes from the situation that in practice the domains are often complicated. A good initial triangulation of a complicated domain leads already to a fine mesh. Often, the computational resources can just afford this mesh such that there is no mesh hierarchy available. Also, generally (in industrial codes) there is only one type of discrete space, e.g., P_1 finite elements, available. Altogether, in this situation one has to use a different solver. □