On discontinuity–capturing methods for convection–diffusion equations

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Summary. This paper is devoted to the numerical solution of two–dimensional steady scalar convection–diffusion equations using the finite element method. If the popular streamline upwind/Petrov–Galerkin (SUPG) method is used, spurious oscillations usually arise in the discrete solution along interior and boundary layers. We review various finite element discretizations designed to diminish these oscillations and we compare them computationally.

1 Introduction

This paper is devoted to the numerical solution of the scalar convection–diffusion equation

$$-\varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u = f \quad \text{in } \Omega, \qquad\qquad u = u_b \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a polygonal boundary $\partial\Omega$, $\varepsilon > 0$ is constant and $\boldsymbol{b}$, f and u_b are given functions.

If convection strongly dominates diffusion, the solution of (1) typically contains interior and boundary layers and solutions of Galerkin finite element discretizations are usually globally polluted by spurious oscillations. To enhance the stability and accuracy of these discretizations, various stabilization strategies have been developed during the past three decades. One of the most efficient procedures is the SUPG method developed by Brooks and Hughes [2]. Unfortunately, the SUPG method does not preclude spurious oscillations localized in narrow regions along sharp layers and hence various terms introducing artificial crosswind diffusion in the neighbourhood of layers have been proposed to be added to the SUPG formulation. This procedure is often referred to as discontinuity capturing (or shock capturing). The literature on discontinuity–capturing methods is rather extended and numerical tests published in the literature do not allow to draw conclusions concerning their advantages and drawbacks. Therefore, the aim of this paper is to provide a review of various discontinuity–capturing methods and to compare these methods computationally.

The plan of the paper is as follows. In the next section, we recall the Galerkin discretization of (1) and, in Section 3, we formulate the SUPG method. Section 4 contains a review and a computational comparison of discontinuity–capturing methods and, in Section 5, we present our conclusions.

2 Galerkin's finite element discretization

We introduce a triangulation $\mathcal{T}_h$ of the domain Ω consisting of a finite number of open polygonal elements K. We assume that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$ and that the elements of $\mathcal{T}_h$ satisfy the usual compatibility conditions. Further, we introduce a finite element space V_h approximating the space $H_0^1(\Omega)$ and satisfying

$$\mathrm{V}_h \subset \{v \in L^2(\Omega)\,;\; v|_K \in C^\infty(\overline{K}) \;\;\forall\, K \in \mathcal{T}_h\}\,.$$

Since the functions from V_h may be discontinuous across edges of the triangulation $\mathcal{T}_h$, we define the 'discrete' operators ∇_h and Δ_h by

$$(\nabla_h\, v)|_K = \nabla(v|_K)\,, \qquad (\Delta_h\, v)|_K = \Delta(v|_K) \qquad \forall\, K \in \mathcal{T}_h\,.$$

Finally, let $u_{bh} \in L^2(\Omega)$ be a piecewise smooth function whose trace on $\partial\Omega$ approximates u_b. Then a discrete solution of (1) can be defined as a function $u_h \in L^2(\Omega)$ satisfying $u_h - u_{bh} \in \mathrm{V}_h$ and $a_h(u_h, v_h) = (f, v_h)\;\forall\, v_h \in \mathrm{V}_h$, where

$$a_h(u, v) = \varepsilon\,(\nabla_h\, u, \nabla_h\, v) + (\boldsymbol{b}\cdot\nabla_h\, u, v)$$

and $(\cdot,\cdot)$ denotes the inner product in the space $L^2(\Omega)$ or $L^2(\Omega)^2$.

3 The SUPG method

Brooks and Hughes [2] enriched the Galerkin method by a stabilization term yielding the streamline upwind/Petrov–Galerkin (SUPG) method. The discrete solution $u_h \in L^2(\Omega)$ satisfies $u_h - u_{bh} \in \mathrm{V}_h$ and

$$a_h(u_h, v_h) + (R_h(u_h), \tau\,\boldsymbol{b}\cdot\nabla_h\, v_h) = (f, v_h) \qquad \forall\, v_h \in \mathrm{V}_h\,, \tag{2}$$

where $R_h(u) = -\varepsilon\,\Delta_h\, u + \boldsymbol{b}\cdot\nabla_h\, u - f$ is the residual and τ is a nonnegative stabilization parameter. As we see, the SUPG method introduces numerical diffusion along streamlines in a consistent manner. A delicate question is the choice of the parameter τ which may dramatically influence the accuracy of the discrete solution. Here we shall use the formula (cf. Galeão *et al.* [8])

$$\tau|_K = \frac{h_K}{2\,|\boldsymbol{b}|\,p_K}\left(\coth(\mathrm{Pe}_K) - \frac{1}{\mathrm{Pe}_K}\right) \qquad \text{with} \qquad \mathrm{Pe}_K = \frac{|\boldsymbol{b}|\,h_K}{2\,\varepsilon\,p_K}\,, \tag{3}$$

where h_K is the diameter of $K \in \mathcal{T}_h$ in the direction of $\boldsymbol{b}$, p_K is the order of approximation of V_h on K (usually the maximum degree of polynomials in V_h on K), $|\cdot|$ is the Euclidean norm and Pe_K is the local Péclet number.

4 Methods diminishing spurious oscillations in layers

In this section, we present a review and a computational comparison of most of the methods introduced during the last two decades to diminish the oscillations arising in discrete solutions of the problem (1). These methods can be divided into upwinding techniques and into methods adding additional artificial diffusion to the SUPG discretization (2). The artificial diffusion may be either isotropic, or orthogonal to streamlines, or based on edge stabilizations. These four classes of methods will be discussed in the following subsections. It is not possible to describe here thoroughly the ideas on which the design of the methods relies, see [11] for a more comprehesive description. Generally, one can say that the methods are based either on convergence analyses or on investigations of the discrete maximum principle (called DMP in the following) or on heuristic arguments. As we shall see, most of the methods will be nonlinear. The computational comparison of the methods will be performed by means of two test problems specified by the following data of (1):

Example 1. $\Omega = (0,1)^2$, $\varepsilon = 10^{-7}$, $\boldsymbol{b} = (\cos(-\pi/3), \sin(-\pi/3))^T$, $f = 0$, $u_b(x,y) = 0$ for $x = 1$ or $y \leq 0.7$, $u_b(x,y) = 1$ otherwise.

Example 2. $\Omega = (0,1)^2$, $\varepsilon = 10^{-7}$, $\boldsymbol{b} = (1,0)^T$, $f = 1$, $u_b = 0$.

The solution of Ex. 1 possesses an interior layer and exponential boundary layers whereas the solution of Ex. 2 possesses parabolic and exponential boundary layers but no interior layers. All results were computed on uniform $N \times N$ triangulations of the type depicted in Fig. 1. Unless stated otherwise, we used the conforming linear finite element P_1, $N = 20$ for Ex. 1 and $N = 10$ for Ex. 2. The SUPG solutions of Ex. 1 and 2 are shown in Fig. 2 and 5, respectively. It is important that the parameter τ is optimal for the P_1 element in the sense that the SUPG method approximates the boundary layers at $y = 0$ in Ex. 1 and at $x = 1$ in Ex. 2 sharply and without oscillations.

4.1 Upwinding techniques

Initially, stabilizations of the Galerkin discretization of (1) imitated upwind finite difference techniques. However, like in the finite difference method, the upwind finite element discretizations remove the unwanted oscillations but the accuracy attained is often poor since too much numerical diffusion is introduced. According to our experiences, one of the most successful up-

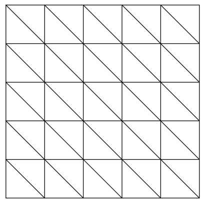

Fig. 1. Type of triangulations ($N = 5$)

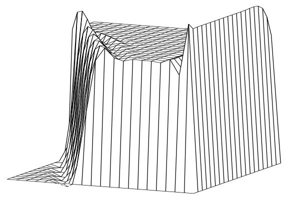

Fig. 2. Ex. 1, SUPG

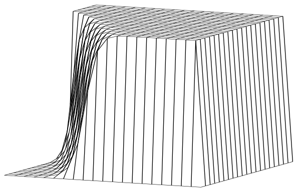

Fig. 3. Ex. 1, IMH [14]

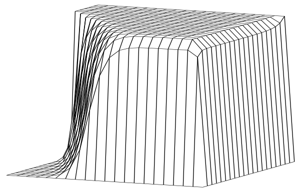

Fig. 4. Ex. 1, do Carmo, Galeão [6]

winding techniques is the improved Mizukami–Hughes (IMH) method, see Knobloch [14]. It is a nonlinear Petrov–Galerkin method for P_1 elements which satisfies the DMP on weakly acute meshes. In contrast with many other upwinding methods for P_1 elements satisfying the DMP, the IMH method adds much less numerical diffusion and provides rather accurate solutions, cf. Knobloch [15]. The IMH solution for Ex. 1 is depicted in Fig. 3. For Ex. 2, it is even nodally exact.

4.2 Methods adding isotropic artificial diffusion

Hughes *et al.* [10] came with the idea to change the upwind direction in the SUPG term of (2) by adding a multiple of the function $\boldsymbol{b}_h^{\|}$ which is the projection of $\boldsymbol{b}$ into the direction of ∇u_h. This leads to the additional term

$$(R_h(u_h), \sigma\, \boldsymbol{b}_h^{\|} \cdot \nabla_h\, v_h) \tag{4}$$

on the left–hand side of (2), where σ is a nonnegative stabilization parameter. Since $\boldsymbol{b}_h^{\|}$ depends on u_h, the resulting method is nonlinear. Hughes *et al.* [10] proposed to set $\sigma = \max\{0, \tau(\boldsymbol{b}_h^{\|}) - \tau(\boldsymbol{b})\}$ where we use the notation $\tau(\boldsymbol{b}^\star)$ for τ defined by (3) with $\boldsymbol{b}$ replaced by $\boldsymbol{b}^\star$. Other definitions of σ in (4) were proposed by Tezduyar and Park [17]. Since the term (4) equals to

$$(\widetilde{\varepsilon}\, \nabla_h\, u_h, \nabla_h\, v_h) \tag{5}$$

with $\widetilde{\varepsilon} = \sigma\, R_h(u_h)\, \boldsymbol{b}\cdot\nabla u_h/|\nabla u_h|^2$, it introduces an isotropic artificial diffusion.

Another stabilization strategy was introduced by Galeão and do Carmo [9] who proposed to replace the flow velocity $\boldsymbol{b}$ in the SUPG stabilization term by an approximate upwind direction. This gives rise to the additional term

$$(R_h(u_h), \sigma\, \boldsymbol{z}_h \cdot \nabla_h\, v_h) \tag{6}$$

on the left–hand side of (2), where $\boldsymbol{z}_h = R_h(u_h)\, \nabla u_h/|\nabla u_h|^2$ and $\sigma = \max\{0, \tau(\boldsymbol{z}_h) - \tau(\boldsymbol{b})\}$. If $f = 0$ and $\Delta_h\, u_h = 0$, we have $\boldsymbol{z}_h = \boldsymbol{b}_h^{\|}$ and hence the method of Galeão and do Carmo [9] is identitical with the method of Hughes *et al.* [10]. Do Carmo and Galeão [6] proposed to simplify σ to

$$\sigma = \tau(\boldsymbol{b})\, \max\left\{0, \frac{|\boldsymbol{b}|}{|\boldsymbol{z}_h|} - 1\right\}\,. \tag{7}$$

Almeida and Silva [1] suggested to replace (7) by

$$\sigma = \tau(\boldsymbol{b}) \max\left\{0, \frac{|\boldsymbol{b}|}{|\boldsymbol{z}_h|} - \zeta_h\right\} \quad \text{with} \quad \zeta_h = \max\left\{1, \frac{\boldsymbol{b}\cdot\nabla_h u_h}{R_h(u_h)}\right\}, \tag{8}$$

which reduces the amount of artificial diffusion along the $\boldsymbol{z}_h$ direction.

Do Carmo and Galeão [6] also introduced a feedback function which should minimize the influence of the term (6) in regions where the solution u of (1) is smooth. Since this approach was rather involved, do Carmo and Alvarez [5] introduced another procedure (still defined using several formulas) suppressing the addition of the artificial diffusion in regions where u is smooth.

Again, the term (6) can be written in the form (5), now with $\widetilde{\varepsilon} = \sigma\,|R_h(u_h)|^2/|\nabla u_h|^2$. To prove error estimates, Knopp *et al.* [16] proposed to replace this $\widetilde{\varepsilon}$, on any $K \in \mathcal{T}_h$, by

$$\widetilde{\varepsilon}|_K = \sigma_K(u_h)\,|Q_K(u_h)|^2 \qquad \text{with} \qquad Q_K(u_h) = \frac{\|R_h(u_h)\|_{0,K}}{S_K + \|u_h\|_{1,K}}, \tag{9}$$

where $\sigma_K(u_h) \ge 0$ and $S_K > 0$ are appropriate constants.

The stabilization term (5) was also used by Johnson [12], who considered

$$\widetilde{\varepsilon}|_K = \max\{0, \alpha\,[\mathrm{diam}(K)]^\nu\,|R_h(u_h)| - \varepsilon\} \qquad \forall\, K \in \mathcal{T}_h$$

with some constants α and $\nu \in (3/2, 2)$. He suggested to take $\nu \sim 2$.

If the above methods are applied to Ex. 1, the discrete solution improves in comparison to the SUPG method. However, most of the methods do not remove the spurious oscillations completely and/or lead to an excessive smearing of the layers. The best methods are the methods of do Carmo and Galeão [6] and Almeida and Silva [1] which are identical in this case, see Fig. 4.

4.3 Methods adding artificial diffusion orthogonally to streamlines

Since the streamline diffusion introduced by the SUPG method seems to be enough along the streamlines, an alternative approach to the above methods is to modify the SUPG discretization (2) by adding artificial diffusion in the crosswind direction only as considered by Johnson *et al.* [13]. A straightforward generalization of their approach leads to the additional term

$$(\widetilde{\varepsilon}\,D\,\nabla_h u_h, \nabla_h v_h) \tag{10}$$

on the left–hand side of (2), where $\widetilde{\varepsilon}|_K = \max\{0, |\boldsymbol{b}|\,h_K^{3/2} - \varepsilon\}\ \forall\, K \in \mathcal{T}_h$ and $D = I - \boldsymbol{b}\otimes\boldsymbol{b}/|\boldsymbol{b}|^2$ is the projection onto the line orthogonal to $\boldsymbol{b}$, I being the identity tensor.

Investigating the validity of the DMP for several model problems, Codina [7] came to the conclusion that the artificial diffusion $\widetilde{\varepsilon}$ in (10) should be defined, for any $K \in \mathcal{T}_h$, by

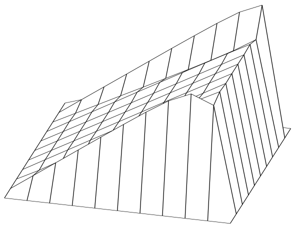

Fig. 5. Ex. 2, SUPG

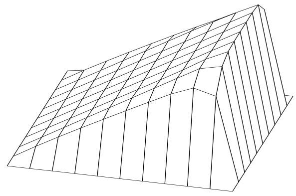

Fig. 6. Ex. 2, MBE

$$\widetilde{\varepsilon}|_K = \frac{1}{2}\,\max\left\{0, C - \frac{2\,\varepsilon}{|\boldsymbol{b}_h^{\|}|\,\mathrm{diam}(K)}\right\}\,\mathrm{diam}(K)\,\frac{|R_h(u_h)|}{|\nabla u_h|}\,, \tag{11}$$

where C is a suitable constant (we use $C = 0.6$ for linear elements and $C = 0.35$ for quadratic elements). Motivated by assumptions and results of general a priori and a posteriori error analyses, Knopp *et al.* [16] changed (11) to

$$\widetilde{\varepsilon}|_K = \frac{1}{2}\,\max\left\{0, C - \frac{2\,\varepsilon}{Q_K(u_h)\,\mathrm{diam}(K)}\right\}\,\mathrm{diam}(K)\,Q_K(u_h)\,, \tag{12}$$

where $Q_K(u_h)$ is defined in (9) (the constants S_K equal to 1 in numerical experiments of [16]). Combining the above two definitions of $\widetilde{\varepsilon}$, we further propose to use (10) with $\widetilde{\varepsilon}$ defined by (12) where $Q_K(u_h) = |R_h(u_h)|/|\nabla u_h|$. This modified method of Codina is called MC method in the following. It is equivalent to (11) if $f = 0$ and $\Delta_h\, u_h = 0$.

Based on investigations of the DMP for strictly acute meshes and linear simplicial finite elements, Burman and Ern [3] suggested to use (10) with $\widetilde{\varepsilon}$ defined, on any $K \in \mathcal{T}_h$, by

$$\widetilde{\varepsilon}|_K = \frac{\tau(\boldsymbol{b})\,|\boldsymbol{b}|^2\,|R_h(u_h)|}{|\boldsymbol{b}|\,|\nabla_h\, u_h| + |R_h(u_h)|}\;\frac{|\boldsymbol{b}|\,|\nabla_h\, u_h| + |R_h(u_h)| + \tan\alpha_K\,|\boldsymbol{b}|\,|D\,\nabla_h\, u_h|}{|R_h(u_h)| + \tan\alpha_K\,|\boldsymbol{b}|\,|D\,\nabla_h\, u_h|}\,.$$

Here, α_K is equal to $\pi/2$ minus the largest angle of K (if K is a triangle). In case of right triangles, it is recommended to set $\alpha_K = \pi/6$.

Our numerical experiments indicate that the above value of $\widetilde{\varepsilon}$ is too large and therefore we also consider (10) with $\widetilde{\varepsilon}$ defined, on any $K \in \mathcal{T}_h$, by

$$\widetilde{\varepsilon}|_K = \frac{\tau(\boldsymbol{b})\,|\boldsymbol{b}|^2\,|R_h(u_h)|}{|\boldsymbol{b}|\,|\nabla_h\, u_h| + |R_h(u_h)|}\,. \tag{13}$$

This modified Burman–Ern method is called MBE method in the following.

If we apply the methods of this subsection to Ex. 2, then only the MC and MBE methods give satisfactory results (and they are comparable), see Fig. 6. For Ex. 1, these methods provide comparable results to the solution in Fig. 4. On the other hand, the two best methods of the previous subsection (do Carmo and Galeão [6], Almeida and Silva [1]) give almost the same results for Ex. 2, which are comparable to the results of the MC and MBE methods.

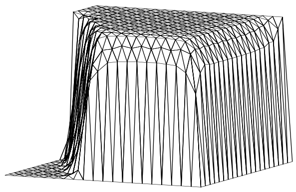

Fig. 7. Ex. 1, MC, P_2, $N = 10$

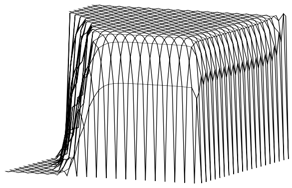

Fig. 8. Ex. 1, do Carmo, Galeão [6], P_1^{nc}

4.4 Edge stabilizations

Another stabilization strategy for linear simplicial finite elements was introduced by Burman and Hansbo [4]. The term to be added to the left–hand side of (2) is defined by

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \Psi_K(u_h)\,\mathrm{sign}(\boldsymbol{t}_{\partial K}\cdot\nabla(u_h|_K))\,\boldsymbol{t}_{\partial K}\cdot\nabla(v_h|_K)\,\mathrm{d}\sigma\,,$$

where $\boldsymbol{t}_{\partial K}$ is a unit tangent vector to the boundary ∂K of K, $\Psi_K(u_h) = \mathrm{diam}(K)\,(C_1\,\varepsilon + C_2\,\mathrm{diam}(K))\,\max_{E\subset\partial K}|\,[|\boldsymbol{n}_E\cdot\nabla u_h|]_E\,|$, $\boldsymbol{n}_E$ are normal vectors to edges E of K, $[|v|]_E$ denotes the jump of a function v across the edge E and C_1, C_2 are appropriate constants. Burman and Hansbo proved that, using an edge stabilization instead of the SUPG term, the DMP is satisfied. Other choices of $\Psi_K(u_h)$ based on investigations of the DMP were recently proposed by Burman and Ern. However, all these edge stabilizations add much more artificial diffusion than the best methods of the previous subsections.

5 Conclusions

Our computations indicate that, among the methods mentioned in this paper, the best ones are: the IMH method [14], the method of do Carmo, Galeão [6] defined by (6), (7), the method of Almeida and Silva [1] defined by (6), (8), the MC method introduced below (12) and the MBE method defined by (10), (13). The IMH method can be used for the P_1 element only but gives best results in this case. The other methods can be successfully also applied to other finite elements as Figs. 7 and 8 show (for the conforming quadratic element P_2 and the nonconforming Crouzeix–Raviart element P_1^{nc}). However, much more comprehensive numerical studies are still necessary to obtain clear conclusions of the advantages and drawbacks of the discontinuity–capturing methods.

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