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Analysis of commutation errors for functions with low regularity

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Abstract

Commutation errors arise in the derivation of the space averaged Navier–Stokes equations, the basic equations for the large eddy simulation of turbulent flows, if the filter is non-uniform or asymmetric (skewed) with non-constant skewness. These errors need to be analyzed for turbulent flow fields, where one expects a limited regularity of the solution. This paper studies the order of convergence of commutation errors, as the filter width tends to zero, for functions with low regularity. Several convergence results are proved and it is also shown that convergence may fail (or its order decreases) if the functions become less smooth. The main results are those dealing with Hölder–continuous functions and with functions having singularities. The sharpness of the analytic results is confirmed with numerical illustrations.

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1. Commutation errors in large eddy simulation (LES)—the motivation for the studies

Large eddy simulation (LES) is currently one of the most popular approaches for the simulation of turbulent flows. Turbulent incompressible flows are governed by the incompressible Navier–Stokes equations

$$\begin{aligned} &\mathbf{u}_t - Re^{-1}\Delta\mathbf{u} + \nabla(\mathbf{u}\,\mathbf{u}^{\mathrm{T}}) + \nabla p = \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\ &\nabla\cdot\mathbf{u} = 0 \quad \text{in } [0,T]\times\Omega, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1, \ldots, u_d)$ is the velocity, p the pressure, $Re > 0$ the Reynolds number, $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) a given domain, T a positive time and $\mathbf{f}$ the external force. Eqs. (1) must be equipped with boundary and initial conditions. Often, the no-slip boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ is considered. A straightforward discretization of (1) by a finite element or finite difference method, a so-called direct numerical simulation (DNS), seeks to simulate the behavior of all persistent flow structures. However, the richness of scales in turbulent flows is by far too large to be handled by

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present day computers and a DNS is in general not possible. The basic goal of LES is to simulate only the behavior of "large" flow structures $(\overline{\mathbf{u}}, \overline{p})$, defined by a space filtering of the unknowns $(\mathbf{u}, p)$. This filtering (or space averaging) is intended to act as a low-pass filter in order to capture only the large eddies of the flow and is generally realized by the convolution of $(\mathbf{u}, p)$ with an appropriate filter function G:

$$\overline{\mathbf{u}}(\mathbf{y}) = \frac{1}{\delta^d}\int_{\mathbb{R}^d} G\left(\frac{\mathbf{y}-\mathbf{x}}{\delta}\right)\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \overline{p}(\mathbf{y}) = \frac{1}{\delta^d}\int_{\mathbb{R}^d} G\left(\frac{\mathbf{y}-\mathbf{x}}{\delta}\right)p(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{2}$$

Roughly speaking, the parameter $\delta \in \mathbb{R}^+$, the *filter width*, describes how much narrow (near to a Dirac's delta function) the filtering is. Furthermore, in the limit $\delta \to 0$, the operation of filtering must reconstruct the original functions. For an introduction to LES see Aldama [2], Sagaut [24], [4] or [18].

In order to simulate $(\overline{\mathbf{u}}, \overline{p})$ and to describe their dynamical behavior, one must derive a set of partial differential equations satisfied by these unknowns. The basic idea in the first step of the derivation of such equations is to filter the Navier–Stokes equations (1) which gives, by using the linearity of the convolution operator,

$$\begin{aligned}
&\overline{\mathbf{u}_t} - \frac{1}{Re}\overline{\Delta\mathbf{u}} + \overline{\nabla\cdot(\mathbf{u}\,\mathbf{u}^{\mathrm{T}})} + \overline{\nabla p} = \overline{\mathbf{f}} \quad \text{in } (0,T]\times\Omega,\\
&\overline{\nabla\cdot\mathbf{u}} = 0 \quad \text{in } [0,T]\times\Omega.
\end{aligned} \tag{3}$$

The derivation proceeds by *assuming* that convolution and differentiation commute, arriving to the so-called space averaged Navier–Stokes equations

$$\begin{aligned}
&\overline{\mathbf{u}}_t - \frac{1}{Re}\Delta\overline{\mathbf{u}} + \nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\mathrm{T}}) + \nabla\overline{p} = \overline{\mathbf{f}} + \nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\mathrm{T}} - \overline{\mathbf{u}\mathbf{u}^{\mathrm{T}}}) \quad \text{in } (0,T]\times\Omega,\\
&\nabla\cdot\overline{\mathbf{u}} = 0 \quad \text{in } [0,T]\times\Omega.
\end{aligned} \tag{4}$$

Now, the last and mostly studied step of LES-modeling consists in modeling the subgrid scale stress tensor $\mathscr{R}(\mathbf{u},\mathbf{u}) = \overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\mathrm{T}} - \overline{\mathbf{u}\mathbf{u}^{\mathrm{T}}}$ in terms of $(\overline{\mathbf{u}}, \overline{p})$. In this paper, we want to focus on the derivation of the space averaged Navier–Stokes equations. First, we note that the integrals in (2) are well-defined for functions defined in the whole space and the definition could be easily adapted also in the periodic case. Additional problems with convolutions arise when solid boundaries, which is the usual situation in applications, are present or a non-uniform filter width is used. These problems are well-known, however they are usually neglected in practice. The topic of this paper is to analyze the step from (3) to (4) if $\delta = \delta(\mathbf{y})$, more precisely the assumption of the *commutation property*. This property holds in special cases, e.g., if $\Omega = \mathbb{R}^d$ and the filter width δ is constant. In a general setting, a *commutation error*, whose ith component is defined by

$$\mathscr{E}_{\mathrm{c},i}(\mathbf{y}) := \partial_i\overline{u}(\mathbf{y}) - \overline{\partial_i u}(\mathbf{y}), \tag{5}$$

is committed, where u is the function (scalar or vector valued) which is filtered. Nowadays, this fact is well-known [8,10,13,16–18,29,31], even if a rigorous analysis is still not complete. Note, if commutation errors do not vanish, the divergence-free constraint may not be true anymore, i.e., $\nabla\cdot\overline{\mathbf{u}} \neq 0$, and the space averaged Navier–Stokes equations may be (slightly, if the commutation errors are small) compressible. In particular, current mathematical research on LES is starting to focus on this topic, see Layton [19] for currents trends in mathematics of LES, and also on problems of consistency and sensitivity of equations with respect to δ, see Anitescu et al. [3].

The commitment of an error in going from (3) to (4) may have several reasons:

(1) The convolution with a filter kernel with a constant δ requires the extension of all functions outside Ω for the integral in (2) to be well-defined. The extended functions will have, in general, a lack of smoothness at the boundary of Ω. The theory of distributions shows that an additional term, see (6), appears in this case.
(2) The assumption of the commutation property fails if the filter width $\delta = \delta(\mathbf{y})$ is non-constant but it is a function. Here, $\mathbf{y}$ denotes the point at which a given function is filtered. In fact, the filtering is not described by a simple convolution, but by a more complicated integral transform, see (10). The study of such *non-uniform filters* have been started in Ghosal and Moin [16].

(3) A commutation error might appear if the filter is not applied in a symmetric way. This means that $\mathbf{y}$ is the point in which the function u is filtered and the center of the filter kernel G is not in $\mathbf{y}$ but it is shifted away from $\mathbf{y}$ by a given quantity, possibly depending on $\mathbf{y}$ itself. This type of filters is called asymmetric or skewed [15,29]. Only for constant filter width and constant asymmetry, there is no commutation error for skewed filters.

Possible advantages of using non-uniform and skewed filters are in the study of complex geometry problems and flows with very different scales in different parts of the computational domain, etc., see [29] for a detailed discussion.

The mathematical study of questions arising in the application of the filter with constant δ, after having trivially extended outside Ω all functions appearing in the Navier–Stokes equations started in Dunca et al. [8], see also [18]. In particular, if the convolution of the terms $\Delta\mathbf{u}$ and ∇p is correctly defined in the sense of distributions, commutation of convolution and differentiation leads to the term

$$\int_{\partial\Omega} G(\mathbf{x}-\mathbf{s})\, \mathbb{S}(\mathbf{u}, p)(t, \mathbf{s})\mathbf{n}(\mathbf{s})\, \mathrm{d}\mathbf{s} \tag{6}$$

in the right-hand side of the space averaged Navier–Stokes equations (4), where $\mathbb{S}(\mathbf{u}, p) = Re^{-1}\nabla\mathbf{u} - p\,\mathbb{I}$ is the stress tensor, $\mathbb{I}$ is the unit tensor and $\mathbf{n}$ is the outward unit normal on $\partial\Omega$. The study of the convergence of (6) in $L^p(\mathbb{R}^d)$, as δ vanishes, reveals that (6) goes to zero if and only if the normal stress $\mathbb{S}(\mathbf{u}, p)(t, \mathbf{s})\mathbf{n}(\mathbf{s})$ is zero almost everywhere on the boundary, for almost every time. This implies that there is no interaction between the (turbulent) flow and the boundary of the domain—a situation which is rather unlikely in applications. A consequence of this result is that using a discretization which is based on the strong form of the space averaged Navier–Stokes equations, the term (6) cannot be neglected. For discretizations which are based on a weak or variational form, the effect of neglecting (6) can be measured in the $H^{-1}(\Omega)$-norm and it is shown in [8] that this norm of (6) tends to zero as $\delta \to 0$, with order of convergence at least $\frac{1}{2}$.

The choice of a filter width function $\delta(\mathbf{y})$ vanishing at the boundary overcomes the challenging problem of boundary conditions for LES. We recall that the boundary conditions that complete (4) are not known. This is the problem of “Near Wall Modeling.” The study of a filter width function such that

$$\delta(\mathbf{y}) \to 0 \quad \text{as } \mathbf{y} \to \partial\Omega \tag{7}$$

has been pioneered by the van Driest damping [30] and filters with variable width in the whole domain are applied in the dynamic method of Germano et al. [14]. The mathematical properties of the system arising with this variable filter are much more delicate and their treatment requires deeper functional analysis tools. First steps in the analysis of a Smagorinsky model with variable $\delta(\mathbf{y})$ have been performed recently by Świerczewska [28].

A theoretical question which arises now is the following: What happens with the commutation errors (5) if the filter width tends to zero? In applications, the filter width is in general proportional to the mesh size. If the mesh becomes finer and finer, one likes to simulate smaller and smaller flow structures and asymptotically all flow structures governed by the Navier–Stokes equations (1) should be simulated. This implies that the commutation errors must vanish as the filter width tends to zero. The question is if this really happens. Furthermore, if the answer is positive, a knowledge of the order of convergence is of interest. In view of applications, it is also of interest to find out how large the commutation errors are in a given situation and how does neglecting the commutation errors influence the numerical results.

Numerical studies of the size of commutation errors for different filters at a turbulent mixing layer flow are presented in van der Bos and Geurts [29]. It was found that—in particular—the size of the commutation error for asymmetric filters might be so large that its modeling becomes necessary. A priori tests and modeling are also presented in [17]. Another observation in [17,29,5] is that modeling of commutation errors may be as important as that of the subgrid scale stress tensor. Using so-called second order filters (first moment of the filter kernel vanishes but second moment does not), many LES models lead to models of the subgrid scale stress tensor formally correct up to terms of $\mathcal{O}(\delta^2)$, e.g., the Smagorinsky model [26], the gradient model [20,6], the rational model [12], or the dynamic subgrid scale model [14,21]. Then, the divergence of second order models for $\mathcal{R}(\mathbf{u}, \mathbf{u})$ (see (4)) introduces terms that are formally $\mathcal{O}(\sum_{l=1}^{d} \partial_i \delta_l(\mathbf{y})\delta_l(\mathbf{y}))$—which is the same order as for commutation errors for symmetric filters!

In this paper, we will study commutation errors for the most common filters (Gaussian filter, box filter) which are applied to functions with (relatively) low regularity. The study of such functions is motivated by the available analysis, in particular that for the weak solutions of the Navier–Stokes equations, one can prove only very low regularity results.

Our approach is complementary to the available literature which considers smooth functions ($C^\infty(\Omega)$) and constructs filters satisfying certain properties, e.g., [16,22]. The consideration of functions with low regularity is, in our opinion, necessary for applications.

The paper is organized as follows. Section 2 introduces the filters which will be studied and reviews the estimates for commutation errors for smooth functions. Unified representation formulae for the commutation errors are derived. Section 3 studies commutation errors for symmetric filters with compact support applied to Hölder–continuous functions. It is shown that the order of convergence for the commutation error decreases if the function to be filtered possesses less smoothness. Additionally, an estimate in the $L^p(\Omega)$-norm will be given. In Section 4, commutation errors near the boundary in the cases that the mean velocity obeys the $1/\alpha$th power law and that the flow field has a singularity are investigated. It is shown that the convergence of the commutation errors requires the filter width to tend to zero sufficiently fast near the boundary. The resulting smallness of the filter width requires in practice the resolution of the flow field near the boundary. The analytical results of Sections 3 and 4 are supported with numerical illustrations. Finally, Section 5 summarizes the results of the paper.

2. Classes of filters, review of filtering smooth functions

This section introduces the classes of filters with and without compact support. The filtering of smooth functions with non-constant filter width and asymmetric filter kernels is reviewed. The derivation of the estimates is straightforward and the results are in principle already known, see [22] for the case of assuming infinitely smooth functions. Below, the regularity conditions to obtain these estimates are stated precisely. It turns out that the representation of the commutation error is the same for both classes of filters, see Lemmas 2.1 and 2.3. To our best knowledge, the expressions (14) and (19) has not been presented in the literature so far although alternative representations of the commutation error can be found, e.g., see [29]. These expressions will be used in the following sections. In higher dimensions, we consider the case that the filter operator is a tensor product of one-dimensional filters, which is the common approach in LES [2,24].

2.1. Filters without compact support

First, we suppose that the function u which should be filtered and the filter G are both given on $\mathbb{R}^d$. It will be allowed that the filter width function $\delta = \delta(\mathbf{y})$ depends on the point $\mathbf{y}$ in which the function is filtered and the center of the filter kernel is moved away from this point by a vector $\mathbf{t}(\mathbf{y}) = (t_1(\mathbf{y}), \ldots, t_d(\mathbf{y}))$.

Let $G \geqslant 0$ be a filter kernel with $\operatorname{supp}(G) = \mathbb{R}$. For the analysis, we assume that the filter is "normalized", that the first moment of the filter kernel exists and vanishes and that the second moment of the filter kernel exists, i.e.,

$$\int_{-\infty}^{+\infty} G(x)\,\mathrm{d}x = 1, \quad \int_{-\infty}^{+\infty} G(x)x\,\mathrm{d}x = 0, \quad \int_{-\infty}^{+\infty} G(x)x^2\,\mathrm{d}x = M_2 < +\infty. \tag{8}$$

The existence of the above moments implies that $G(x)$ tends sufficiently fast to zero as $|x| \to +\infty$. In particular, for any bounded function u it holds

$$\lim_{|x|\to\infty} G\left(\frac{x+t}{\delta}\right) x^k u(x-y) = 0 \quad \text{for } k \in \{0,1,2\} \text{ and } t, y, \delta \in \mathbb{R}. \tag{9}$$

The most popular example for such a filter kernel is the Gaussian $G(x) = \sqrt{6/\pi}\exp(-6x^2)$. For the Gaussian filter, all moments exist and all odd moments vanish:

$$\int_{-\infty}^{+\infty} G(x)x^k\,\mathrm{d}x = \begin{cases} 0 & \text{if } k \text{ is odd},\\ \dfrac{1}{2^k}\,\dfrac{1}{3^{k/2}}3\;5\;\cdots\;(k-1) & \text{if } k \text{ is even}. \end{cases}$$

In the scalar case, the filtered function $\overline{u}(y)$ of $u(x)$ is defined by

$$\overline{u}(y) = \frac{1}{\delta(y)}\int_{-\infty}^{+\infty} G\left(\frac{x+t(y))}{\delta(y)}\right) u(y-x)\,\mathrm{d}x. \tag{10}$$

The generalization to the multi-dimensional case requires the introduction of some function spaces. Given any open set $A \subset \mathbb{R}^d$, we shall denote by $C^k(A)$, $k \in \mathbb{N}$, the space of functions which are continuous together with all partial derivatives up to order k and by $C_b^k(A)$ the functions belonging to $C^k(A)$ which are bounded together with all partial derivatives up to order k. The norm in $C_b^k(A)$ is defined by $\|f\|_{C_b^k(A)} = \sum_{|\alpha|=0}^k \sup_{\mathbf{x}\in A} |\partial^\alpha f|$, where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index and $|\alpha| = \sum_{k=1}^d \alpha_i$.

Let $u \in C^1(\mathbb{R}^d) \cap C_b^0(\mathbb{R}^d)$ be a given function, let $\delta_k(\mathbf{y}) \in C_b^1(\mathbb{R}^d)$ denote the width function of the filter with respect to the direction x_k and let the translation $\mathbf{t}(\mathbf{y})$ belong to $(C_b^1(\mathbb{R}^d))^d$. The filter $\overline{u}$ of u is then defined by a tensor product of one-dimensional filters:

$$\overline{u}(\mathbf{y}) = \prod_{k=1}^d \frac{1}{\delta_k(\mathbf{y})} \int_{\mathbb{R}^d} \prod_{l=1}^d G\left(\frac{x_l + t_l(\mathbf{y})}{\delta_l(\mathbf{y})}\right) u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{11}$$

To keep the notations concise, we define $\mathscr{A}(\mathbf{y}) = \prod_{k=1}^d \delta_k(\mathbf{y})$, and

$$\mathscr{G}(\mathbf{x},\mathbf{y}) = \prod_{k=1}^d G\left(\frac{x_k + t_k(\mathbf{y})}{\delta_k(\mathbf{y})}\right), \quad \mathscr{G}_l(\mathbf{x},\mathbf{y}) = \prod_{k=1, k\neq l}^d G\left(\frac{x_k + t_k(\mathbf{y})}{\delta_k(\mathbf{y})}\right)$$

such that

$$\overline{u}(\mathbf{y}) = \frac{1}{\mathscr{A}(\mathbf{y})} \int_{\mathbb{R}^d} \mathscr{G}(\mathbf{x},\mathbf{y}) u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{12}$$

A direct calculation shows

$$\begin{aligned}\partial_i \overline{u}(\mathbf{y}) = &- \frac{1}{\mathscr{A}(\mathbf{y})}\left[\left(\sum_{k=1}^d \frac{\partial_i \delta_k(\mathbf{y})}{\delta_k(\mathbf{y})}\right) \int_{\mathbb{R}^d} \mathscr{G}(\mathbf{x},\mathbf{y}) u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\right.\\ &+ \int_{\mathbb{R}^d}\left(\sum_{l=1}^d \mathscr{G}_l(\mathbf{x},\mathbf{y}) G'\left(\frac{x_l + t_l(\mathbf{y})}{\delta_l(\mathbf{y})}\right) \frac{\partial_i t_l(\mathbf{y})\delta_l(\mathbf{y}) - (x_l + t_l(\mathbf{y}))\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})^2}\right) u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &\left.+ \int_{\mathbb{R}^d} \mathscr{G}(\mathbf{x},\mathbf{y}) \partial_i u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\right],\end{aligned} \tag{13}$$

where $\partial_i = \partial/\partial y_i$. The last term is just $\overline{\partial_i u}$; consequently the ith component of the commutation error is the sum of the other two terms. The commutation error is now transformed using integration by parts

$$\frac{1}{\delta_l(\mathbf{y})} \int_{\mathbb{R}^d} \mathscr{G}_l(\mathbf{x},\mathbf{y}) G'\left(\frac{x_l + t_l(\mathbf{y})}{\delta_l(\mathbf{y})}\right) u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \mathscr{G}(\mathbf{x},\mathbf{y}) \partial_l u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x},$$

$$\frac{1}{\delta_l(\mathbf{y})} \int_{\mathbb{R}^d} \mathscr{G}_l(\mathbf{x},\mathbf{y}) G'\left(\frac{x_l + t_l(\mathbf{y})}{\delta_l(\mathbf{y})}\right) x_l u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^d} \mathscr{G}(\mathbf{x},\mathbf{y}) u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} \mathscr{G}(\mathbf{x},\mathbf{y}) x_l \partial_l u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

The vanishing of the boundary terms (at infinity) follows from (9). Inserting these expressions into (13) shows that the first term of (13) cancels out and it gives the following lemma.

Lemma 2.1. *Let $u \in C^1(\mathbb{R}^d) \cap C_b^0(\mathbb{R}^d)$, $\delta_l \in C_b^1(\mathbb{R}^d)$, $t_l \in C_b^1(\mathbb{R}^d)$, $l = 1, \ldots, d$. Then the ith component of the commutation error has the form*

$$\mathscr{E}_{\mathrm{c},i}(\mathbf{y}) = \sum_{l=1}^d \left[\frac{\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \left(\overline{x_l \partial_l u} - y_l \overline{\partial_l u}\right)(\mathbf{y}) + \frac{\partial_i t_l(\mathbf{y})\delta_l(\mathbf{y}) - t_l(\mathbf{y})\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \overline{\partial_l u}(\mathbf{y})\right]. \tag{14}$$

It can be observed that there is no commutation error for skewed filters with both a constant skewness, i.e., $\partial_i t_l(\mathbf{y}) = 0$, and with a constant filter width, i.e., $\partial_i \delta_l(\mathbf{y}) = 0$, $l = 1, \ldots, d$.

We now investigate how the representation formula in the above proposition can be used to obtain a pointwise estimate of the commutation error.

Proposition 2.2. *Let $u \in C_b^2(\mathbb{R}^d)$, let the first moment of the filter kernel vanish and the second moment exist and let $\delta_l \in C_b^1(\mathbb{R}^d)$, $t_l \in C_b^1(\mathbb{R}^d)$, $l = 1, \ldots, d$. Then,*

$$|\mathscr{E}_{c,i}(\mathbf{y})| \leqslant \left|\sum_{l=1}^{d} \partial_i t_l(\mathbf{y}) \partial_l u(\mathbf{y})\right| + \|u\|_{C^2(\mathbb{R}^d)} \left(\sum_{k,l=1}^{d} |\partial_i t_l(\mathbf{y}) t_k(\mathbf{y})| + |M_2| \sum_{l=1}^{d} |\partial_i \delta_l(\mathbf{y}) \delta_l(\mathbf{y})|\right). \tag{15}$$

Proof. We start by investigating the first term of (14). By using the Taylor formula for u at $\mathbf{y}$ with the Lagrange formula for the remainder and by applying the following identities:

$$\int_{-\infty}^{+\infty} G\left(\frac{x + t(y)}{\delta(y)}\right) \mathrm{d}x = \delta(y), \quad \int_{-\infty}^{+\infty} G\left(\frac{x + t(y)}{\delta(y)}\right) x \, \mathrm{d}x = -\delta(y) t(y),$$

$$\int_{-\infty}^{+\infty} G\left(\frac{x + t(y)}{\delta(y)}\right) x^2 \, \mathrm{d}x = M_2 \delta(y)^3 + \delta(y) t(y)^2,$$

that follow from (8), one gets

$$\begin{aligned}
&\sum_{l=1}^{d} \frac{\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \left(\overline{x_l \partial_l u} - y_l \overline{\partial_l u}\right)(\mathbf{y}) \\
&\quad = \sum_{l=1}^{d} \frac{\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \left(t_l(\mathbf{y}) \partial_l u(\mathbf{y}) + \sum_{k=1}^{d} \partial_k \partial_l u(\mathbf{y} + \eta_k \mathbf{e}_k) t_l(\mathbf{y}) t_k(\mathbf{y}) + M_2 \delta_l(\mathbf{y})^2 \partial_l \partial_l u(\mathbf{y} + \eta_l \mathbf{e}_l)\right),
\end{aligned}$$

where $\mathbf{e}_k$ is the unit vector in x_k direction and $\eta_k \in \mathbb{R}$. For the second term, one obtains in the same way

$$\begin{aligned}
&\sum_{l=1}^{d} \frac{\partial_i t_l(\mathbf{y}) \delta_l(\mathbf{y}) - t_l(\mathbf{y}) \partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \overline{\partial_l u}(\mathbf{y}) \\
&\quad = \sum_{l=1}^{d} \left(\partial_i t_l(\mathbf{y}) \partial_l u(\mathbf{y}) - \frac{t_l(\mathbf{y}) \partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \partial_l u(\mathbf{y})\right) + \sum_{k,l=1}^{d} \partial_i t_l(\mathbf{y}) t_k(\mathbf{y}) \partial_k \partial_l u(\mathbf{y} + \eta_k \mathbf{e}_k) \\
&\qquad - \sum_{k,l=1}^{d} \frac{\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \partial_k \partial_l u(\mathbf{y} + \eta_k \mathbf{e}_k) t_l(\mathbf{y}) t_k(\mathbf{y}).
\end{aligned}$$

By collecting all terms and applying the triangle inequality, the statement of the proposition follows. □

Remark 2.1. Estimate (15) shows that the ith component of the commutation error depends not only on the derivative with respect to y_i of $\delta_i(\mathbf{y})$ and $t_i(\mathbf{y})$, but it depends also on the y_i-derivative of all components of the filter width functions and all translation directions.

Remark 2.2. Consider first the case that there is no translation of the filter kernel, $t_l(\mathbf{y}) \equiv 0$ for all l. Then, the leading term in (15) is

$$\|u\|_{C^2(\mathbb{R}^d)} |M_2| \sum_{l=1}^{d} |\partial_i \delta_l(\mathbf{y}) \delta_l(\mathbf{y})|,$$

which shows that the commutation error vanishes as $\delta_l(\mathbf{y}) \to 0$ for all l since $|\partial_i \delta_l(\mathbf{y})|$ is bounded. The order of convergence is at least linear. If $\delta_l(\mathbf{y})$ are constants but the translations of the filter kernel do not vanish, the requirement of the asymptotic vanishing of the commutation error is that $\partial_i t_l(\mathbf{y}) \to 0$ as $\delta_l(\mathbf{y}) \to 0$ for all l. Thus, asymptotically the translation has to be the same for all $\mathbf{y} \in \mathbb{R}^d$, i.e., the translation has to tend to a constant vector. Note that in the case that all $\delta_l(\mathbf{y})$ and $t_l(\mathbf{y})$ are constants, no commutation error is committed. In the general case, one has to require for a linear convergence of the commutation error that $\max_l |\partial_i t_l(\mathbf{y})| = \mathcal{O}(\max_l \delta_l(\mathbf{y}))$, i.e., the translation is allowed locally only to vary slowly and this variation has to tend to zero in the same way as $\max_l |\delta_l(\mathbf{y})|$.

Remark 2.3. There are two situations where a higher order convergence of the commutation error can be obtained than proved in Proposition 2.2. The first one requires additional properties of the filter, namely that the filter is chosen in such a way that estimates of the form $|\partial_i t_l(\mathbf{y})| \leqslant C\delta_l(\mathbf{y})^{\alpha+1}$, $|\partial_i \delta_l(\mathbf{y})| \leqslant C\delta_l(\mathbf{y})^{\alpha}$ with $\alpha > 0$ hold. That means, the translation and the filter width function are allowed only to vary slowly.

The second situation requires, besides a condition on the filter, also a condition on the function u. Consider for simplicity a symmetric filtering, the condition on the filter is that not only the first moment but all moments of the filter kernel (starting from the first one) up to a certain order vanish and the condition on the function is a higher regularity. This case has been studied in [22] for $u \in C^\infty(\mathbb{R})$ using also a Taylor series expansion of u like in the proof of Proposition 2.2. It is demonstrated that such filter kernels can be constructed. However, this construction is rather involved. With the two assumptions from above, the pointwise commutation error can be made as small as desired. Note, the use of such higher order filters leads to an equally strong decrease of the contribution of the divergence of the subgrid scale stress tensor. Since the construction of an appropriate filter kernel can be done in principle, the crucial assumption is the smoothness of u. In applications, u is the velocity field or the pressure of a turbulent flow. To expect a high regularity is not realistic in this case.

2.2. Filters with compact support

In this section, we study filters with compact kernel which are applied to functions u which are defined in a bounded domain Ω. A main feature which is required is that the application of the filter leads to integrals whose domain of integration is a subset of $\overline{\Omega}$, i.e., the filter width at a point $\mathbf{y}$ in any direction is not allowed to be larger than the distance of $\mathbf{y}$ to the boundary in that direction. This situation has the appealing property that an extension of u outside Ω is not necessary. It was shown in [8] that such an extension leads to additional terms in the commutation. But this requirement also implies that the filter width has to tend to zero (at least in one direction) as the point $\mathbf{y}$ in which u is filtered tends to the boundary of Ω. Thus, necessarily, the filter width is a function of $\mathbf{y}$. We refer to [16] for a different approach which avoids extensions of functions by using a map from the bounded domain onto $\mathbb{R}^d$.

Let G be a filter kernel with support $[-\frac{1}{2}, \frac{1}{2}]$ (without loss of generality) which is normalized. Moreover, we assume that the first moment of $G(x)$ exists and vanishes and the second moment of $G(x)$ exists

$$\int_{-1/2}^{1/2} G(x)\,\mathrm{d}x = 1, \qquad \int_{-1/2}^{1/2} G(x)x\,\mathrm{d}x = 0, \qquad \int_{-1/2}^{1/2} G(x)x^2\,\mathrm{d}x = M_2 \tag{16}$$

and $G \in C^1(-\frac{1}{2}, \frac{1}{2})$. The most popular filter which fits into this framework is the box or the top hat filter $G(x) = 1$ if $|x| \leqslant \frac{1}{2}$ and $G(x) = 0$ elsewhere.

For simplicity of presentation, we show the derivation of the formula for the commutation error for symmetric filters. For asymmetric filters, the derivation uses the same techniques. The final result will be given for the general case. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $u \in C^1(\overline{\Omega})$, $\delta_l(\mathbf{y})$ be scalar filter width functions with $\delta_l(\mathbf{y}) \in C^1(\overline{\Omega})$, $\delta_l(\mathbf{y}) > 0$ for all $\mathbf{y} \in \Omega$, $l = 1, \ldots, d$. We denote by $\mathscr{B}(\mathbf{y}) = [-\delta_1(\mathbf{y}), \delta_1(\mathbf{y})] \times \cdots \times [-\delta_d(\mathbf{y}), \delta_d(\mathbf{y})]$ and we assume that

$$\mathbf{y} + \mathscr{B}(\mathbf{y}) := [y_1 - \delta_1(\mathbf{y}), y_1 + \delta_1(\mathbf{y})] \times \cdots \times [y_d - \delta_d(\mathbf{y}), y_d + \delta_d(\mathbf{y})] \subset \overline{\Omega}$$

for all $\mathbf{y} = (y_1, \ldots, y_d) \in \overline{\Omega}$. Denoting by $\mathscr{A}(\mathbf{y}) = \prod_{l=1}^d (2\delta_l(\mathbf{y}))$, the filter of u is defined by

$$\overline{u}(\mathbf{y}) = \frac{1}{\mathscr{A}(\mathbf{y})} \int_{\mathscr{B}(\mathbf{y})} \prod_{l=1}^{d} G\left(\frac{x_l}{2\delta_l(\mathbf{y})}\right) u(\mathbf{y} - \mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{17}$$

Note, with this definition, the filter width in $\mathbf{y}$, in the direction x_l, is $2\delta_l(\mathbf{y})$. Using the notations

$$\mathscr{G}(\mathbf{x},\mathbf{y}) = \prod_{k=1}^{d} G\left(\frac{x_k}{2\delta_k(\mathbf{y})}\right), \quad \mathscr{G}_l(\mathbf{x},\mathbf{y}) = \prod_{k=1,k\neq l}^{d} G\left(\frac{x_k}{2\delta_k(\mathbf{y})}\right),$$

$$\mathscr{B}_l(\mathbf{y}) = [-\delta_1(\mathbf{y}),\delta_1(\mathbf{y})]\times\cdots\times[-\delta_{l-1}(\mathbf{y}),\delta_{l-1}(\mathbf{y})]\times[-\delta_{l+1}(\mathbf{y}),\delta_{l+1}(\mathbf{y})]\times\cdots\times[-\delta_d(\mathbf{y}),\delta_d(\mathbf{y})],$$

$$(\mathrm{d}\mathbf{x})_l = \mathrm{d}x_1\cdots\mathrm{d}x_{l-1}dx_{l+1}\cdots\mathrm{d}x_d$$

and $\mathbf{e}_l$ for the unit vector in x_l direction, a straightforward calculation gives

$$\begin{aligned}
\partial_i\overline{u}(\mathbf{y}) = \frac{1}{\mathscr{A}(\mathbf{y})}\Bigg[& -\left(\sum_{k=1}^{d}\frac{\partial_i\delta_k(\mathbf{y})}{\delta_k(\mathbf{y})}\right)\int_{\mathscr{B}(\mathbf{y})}\mathscr{G}(\mathbf{x},\mathbf{y})u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x} \\
& -\sum_{l=1}^{d}\frac{2\partial_i\delta_l(\mathbf{y})}{(2\delta_l(\mathbf{y}))^2}\int_{\mathscr{B}(\mathbf{y})}\mathscr{G}_l(\mathbf{x},\mathbf{y})G'\left(\frac{x_l}{2\delta_l(\mathbf{y})}\right)x_l u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x} \\
& + G\left(\frac{1}{2}\right)\sum_{l=1}^{d}\partial_i\delta_l(\mathbf{y})\int_{\mathscr{B}_l(\mathbf{y})}\mathscr{G}_l(\mathbf{x},\mathbf{y})u\left(\mathbf{y}-\mathbf{x}+(x_l-\delta_l(\mathbf{y}))\mathbf{e}_l\right)(\mathrm{d}\mathbf{x})_l \\
& + G\left(-\frac{1}{2}\right)\sum_{l=1}^{d}\partial_i\delta_l(\mathbf{y})\int_{\mathscr{B}_l(\mathbf{y})}\mathscr{G}_l(\mathbf{x},\mathbf{y})u\left(\mathbf{y}-\mathbf{x}+(x_l+\delta_l(\mathbf{y}))\mathbf{e}_l\right)(\mathrm{d}\mathbf{x})_l \\
& + \int_{\mathscr{B}(\mathbf{y})}\mathscr{G}(\mathbf{x},\mathbf{y})\partial_i u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\Bigg]. \qquad (18)
\end{aligned}$$

For $d=1$, the integral over $\mathscr{B}_l(\mathbf{y})$ is to be understood as the difference of the value of the function u at the boundary points $x=\delta(y)$ and $x=-\delta(y)$. The last term is just $\overline{\partial_i u}$ such that (18) without this term is the ith component of the commutation error. Observing that $\mathscr{G}_l(\mathbf{x},\mathbf{y})$ does not depend on x_l, integration by parts gives

$$\begin{aligned}
& -\frac{2\partial_i\delta_l(\mathbf{y})}{(2\delta_l(\mathbf{y}))^2}\int_{\mathscr{B}(\mathbf{y})}\mathscr{G}_l(\mathbf{x},\mathbf{y})G'\left(\frac{x_l}{2\delta_l(\mathbf{y})}\right)x_l u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x} \\
& = \frac{\partial_i\delta_l(\mathbf{y})}{\delta_l(\mathbf{y})}\Bigg[\int_{\mathscr{B}(\mathbf{y})}\mathscr{G}(\mathbf{x},\mathbf{y})\left(u(\mathbf{y}-\mathbf{x})-x_l\partial_l u(\mathbf{y}-\mathbf{x})\right)\,\mathrm{d}\mathbf{x} \\
& \quad - G\left(\frac{1}{2}\right)\delta_l(\mathbf{y})\int_{\mathscr{B}_l(\mathbf{y})}\mathscr{G}_l(\mathbf{x},\mathbf{y})u\left(\mathbf{y}-\mathbf{x}+(x_l-\delta_l(\mathbf{y}))\mathbf{e}_l\right)(\mathrm{d}\mathbf{x})_l \\
& \quad - G\left(-\frac{1}{2}\right)\delta_l(\mathbf{y})\int_{\mathscr{B}_l(\mathbf{y})}\mathscr{G}_l(\mathbf{x},\mathbf{y})u\left(\mathbf{y}-\mathbf{x}+(x_l+\delta_l(\mathbf{y}))\mathbf{e}_l\right)(\mathrm{d}\mathbf{x})_l\Bigg].
\end{aligned}$$

Inserting this expression into (18), one observes that the first term of (18) cancels as well as all sums with the factors $G(\frac{1}{2})$ and $G(-\frac{1}{2})$. Performing the same calculations for the asymmetric case proves the following lemma.

Lemma 2.3. *Let $u \in C^1(\overline{U(\mathbf{y})})$, where $U(\mathbf{y})$ is a neighborhood of $\mathbf{y}$ such that $\mathbf{y}+\mathscr{B}(\mathbf{y})\subset U(\mathbf{y})$, $\delta_l^+(\mathbf{y})\in C^1(\overline{U(\mathbf{y})})$ and $\delta_l^-(\mathbf{y})\in C^1(\overline{U(\mathbf{y})})$, $l=1,\ldots,d$. Then, the ith component of the commutation error has the form*

$$\mathscr{E}_{c,i}(\mathbf{y}) = \sum_{l=1}^{d}\left[\frac{\partial_i\delta_l^+(\mathbf{y})+\partial_i\delta_l^-(\mathbf{y})}{\delta_l^+(\mathbf{y})+\delta_l^-(\mathbf{y})}(\overline{x_l\partial_l u}-y_l\overline{\partial_l u})(\mathbf{y}) + \frac{\partial_i\delta_l^+(\mathbf{y})\delta_l^-(\mathbf{y})-\partial_i\delta_l^-(\mathbf{y})\delta_l^+(\mathbf{y})}{\delta_l^+(\mathbf{y})+\delta_l^-(\mathbf{y})}\overline{\partial_l u}(\mathbf{y})\right]. \qquad (19)$$

The second term in (19) vanishes for symmetric filters and the commutation error vanishes for constant filter width functions, i.e., $\partial_i\delta_l^-(\mathbf{y})=\partial_i\delta_l^+(\mathbf{y})=0$, $l=1,\ldots,d$, regardless of the skewness. Comparing (19) with the commutation error (14) for filters without compact support, one finds that both formulas are in principle identical. The role of the filter width function $\delta_l(\mathbf{y})$ in (14) is played by $(\delta_l^+(\mathbf{y}) + \delta_l^-(\mathbf{y}))/2$ in (19) and the role of the translation $t_l(\mathbf{y})$ in (14) is played by $(\delta_l^+(\mathbf{y}) - \delta_l^-(\mathbf{y}))/2$ in (19). Next, a pointwise error estimate for the commutation error is proved. With the same substitutions as above, it has a form similar to that of the error estimate (15).

Proposition 2.4. *Let* $u \in C^2(\overline{U(\mathbf{y})})$ *where* $U(\mathbf{y})$ *is defined in Lemma* 2.3, *let the first moment of the filter kernel vanish and the second moment exists and let* $\delta_l^+(\mathbf{y}) \in C^1(\overline{U(\mathbf{y})})$ *and* $\delta_l^-(\mathbf{y}) \in C^1(\overline{U(\mathbf{y})})$,$l = 1, \ldots, d$. *Then*

$$
\begin{aligned}
|\mathscr{E}_{c,i}(\mathbf{y})| \leqslant & \left| \sum_{l=1}^{d} \frac{\partial_i \delta_l^+(\mathbf{y}) - \partial_i \delta_l^-(\mathbf{y})}{2} \partial_l u(\mathbf{y}) \right| \\
& + \|u\|_{C^2(\overline{U(\mathbf{y})})} \left[\sum_{k,l=1}^{d} \frac{|\delta_k^+(\mathbf{y}) - \delta_k^-(\mathbf{y})| \, |\partial_i \delta_l^+(\mathbf{y}) - \partial_i \delta_l^-(\mathbf{y})|}{4} \right. \\
& \left. + \sum_{l=1}^{d} |M_2| |\partial_i \delta_l^+(\mathbf{y}) + \partial_i \delta_l^-(\mathbf{y})| (\delta_l^+(\mathbf{y}) + \delta_l^-(\mathbf{y})) \right].
\end{aligned} \tag{20}
$$

Proof. From the normalization of the filter kernel and the values of the first two moments, it follows that for $\delta^+, \delta^- \in \mathbb{R}$

$$
\begin{aligned}
&\int_{-\delta^+}^{\delta^-} G\left(\frac{2x + \delta^+ - \delta^-}{2(\delta^+ + \delta^-)}\right) \mathrm{d}x = \delta^+ + \delta^-, \\
&\int_{-\delta^+}^{\delta^-} G\left(\frac{2x + \delta^+ - \delta^-}{2(\delta^+ + \delta^-)}\right) x \, \mathrm{d}x = -\frac{(\delta^+ - \delta^-)(\delta^+ + \delta^-)}{2}, \\
&\int_{-\delta^+}^{\delta^-} G\left(\frac{2x + \delta^+ - \delta^-}{2(\delta^+ + \delta^-)}\right) x^2 \, \mathrm{d}x = M_2(\delta^+ + \delta^-)^3 + \frac{1}{4}(\delta^+ - \delta^-)^2(\delta^+ + \delta^-).
\end{aligned}
$$

Now, the proof is readily obtained by using the Taylor formula for $\partial_l u$ at the point $\mathbf{y}$ with the Lagrange formula for the remainder and by applying the above formulas. □

Since the error estimate (20) has in principle the same form as the error estimate (15), the remarks after Proposition 2.2 will apply also here. However, there are two important differences. First, in (20), the C^2-regularity of u is needed only in a neighborhood of $\mathbf{y}$ and not on the whole domain. Thus, using a compact filter, an irregularity of u which is far enough away from $\mathbf{y}$ with respect to the filter width does not effect the pointwise convergence of the commutation error in $\mathbf{y}$. Second, the commutation error vanishes if the filter width functions $\delta_l^-(\mathbf{y}) = \delta_l^+(\mathbf{y})$, $l = 1, \ldots, d$, are constant, see (19). These conditions cannot be fulfilled (or only trivially) if one considers a bounded domain and requires that the filter kernel should always be inside the closure of this domain. For simplicity, let $\Omega = (a, b) \subset \mathbb{R}$. If the filter kernel should be contained in $[a, b]$ then necessarily $\delta^-(y) \to 0$ as $y \to a$ and $\delta^+(y) \to 0$ as $y \to b$. Thus, either we have $\delta^-(y) = \delta^+(y) = 0$ (no filtering) or the filter width functions are not constant.

3. The box filter applied to functions with low regularity

This section contains the main results of this paper. The commutation error will be analyzed for functions with low regularity, in particular for Hölder-continuous functions. Standard notations for Lebesgue and Sobolev spaces are used, see [1].

3.1. Motivations for considering functions with low regularity

In this section, we study the application of the box filter to functions which possess less smoothness than required in the analysis of Section 2.2. Our interest for finding upper bounds on the commutation error for non-smooth functions is two-fold. There are mathematical and physical motivations.

From the mathematical point of view, it is still unknown whether the three-dimensional Navier–Stokes equations have smooth global-in-time solutions, or not. Even if for practical purposes it is often assumed that the solution is of class C^2 (at least when viewed at certain scales), more than 70 years of efforts of mathematicians have not been enough for solving this important question. A both detailed and introductory review of the main results about this problem can be found in Galdi [11]. We like to recall that "Leray–Hopf weak solutions", those for which we have rigorous global existence results, belong just to $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. Their regularity is so weak that the problem of their uniqueness is still open.

Another mathematical question arises when considering domains with corners. We recall that only in special cases (e.g., small data, small time intervals, or small Reynolds number, and smooth domains) it is known that weak solutions of the Navier–Stokes equations are "strong", e.g., $\mathbf{u} \in W^{2,2}(\Omega)$, see Sohr [27]. In the case that the domain $\Omega \subset \mathbb{R}^3$ is a polyhedral domain, this result is not proved. A regularity result (in the time-dependent case) has been proved in Deuring and von Wahl [7]: $\mathbf{u} \in W^{3/2-\varepsilon,2}(\Omega)$, $p \in W^{1/2-\varepsilon,2}(\Omega)$, for all $\varepsilon > 0$, while an "interior regularity" result still holds and $\mathbf{u} \in W^{2,2}(\Omega')$ and $p \in W^{1,2}(\Omega')$ for each Ω' such that $\overline{\Omega'} \subset \Omega$. We observe that the Sobolev embedding theorem implies $p \in L^\gamma(\Omega)$ for $2 \leqslant \gamma < 3$. This means that, near a possible singular point $\mathbf{x}_0$, the pressure may have a behavior of the form $|p| \sim \|\mathbf{x} - \mathbf{x}_0\|^{-\alpha}$ for $\alpha < 1$. In the presence of re-entrant corners, like a flow facing a step, the behavior could be even worse.

Another motivation to study functions with low regularity comes from the classical theory of homogeneous turbulence. From the physical point of view, we recall that within Kolmogorov's K41 theory, the first hypothesis, that of small-scale homogeneity, predicts that $\Delta\mathbf{u}(\mathbf{r}, \ell) = \mathbf{u}(\mathbf{r} + \ell) - \mathbf{u}(\mathbf{r})$, the velocity increment at point $\mathbf{r}$, equals $\Delta\mathbf{u}(\mathbf{r} + \boldsymbol{\rho}, \ell)$ (that at point $\mathbf{r} + \boldsymbol{\rho}$) in a certain statistical sense and for all increments ℓ and displacements $\boldsymbol{\rho}$ smaller than the integral scale, e.g., see Frisch [9, Section 6.1]. The second hypothesis, that of self-similarity at small scales, implies a unique scaling exponent $h \in \mathbb{R}$ such that, in a proper statistical sense, $\Delta\mathbf{u}(\mathbf{r}, \lambda\ell) = \lambda^h \Delta\mathbf{u}(\mathbf{r}, \ell)\ \forall \lambda \in \mathbb{R}^+$, for all $\mathbf{r}$ and increments ℓ and $\lambda\ell$ small compared to the integral scale. Finally, this unique scaling exponent h equals $\frac{1}{3}$ by assuming that the kinetic energy dissipated per unit mass is given by $\varepsilon = 0.5 C_{\mathrm{D}} U^3 L^{-1}$, where C_{D} is the drag coefficient (approximatively constant for high Reynolds numbers), U is a typical velocity, and L a typical length. Several authors rediscovered or contributed to the K41 theory (Obukhov, von Weizsäcker, Heisenberg, Onsager et al., see [9, Section 6.5]). Onsager [23] was the first to point out that the $|\ell|^{1/3}$ law for velocity increments means that, in some sense, the velocity is not smooth but only Hölder-continuous of exponent $\frac{1}{3}$ and he also pointed out the issue of possible singularities.

We will consider from now only functions in a bounded domain, since this is the most interesting case in applications, and filters with compact support which do not require extending the functions of the Navier–Stokes equations off the domain. Concretely, we will study the box filter because it is the most widely used filter with compact support. For simplicity, we consider only the symmetric box filter, i.e., $\delta_l^+(\mathbf{y}) = \delta_l^-(\mathbf{y}) = \delta_l(\mathbf{y})$, $l = 1, \ldots, d$. For smooth functions, it was proved in Proposition 2.4 that the commutation error vanishes as $\delta_l(\mathbf{y}) \to 0$ for all l and that the rate of convergence is at least linear. We will see that the behavior becomes worse for less smooth functions.

We use the notations of Section 2.2. From (18) follows that the ith component of the commutation error in the case of the symmetric box filter has the form

$$\begin{aligned}
\mathscr{E}_{\mathrm{c},i}(\mathbf{y}) = & -\frac{1}{\mathscr{A}(\mathbf{y})}\left(\sum_{l=1}^{d} \frac{\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})}\right) \int_{\mathscr{B}(\mathbf{y})} u(\mathbf{y} - \mathbf{x})\, \mathrm{d}\mathbf{x} \\
& + \frac{1}{\mathscr{A}(\mathbf{y})}\left[\sum_{l=1}^{d} \partial_i \delta_l(\mathbf{y}) \int_{\mathscr{B}_l(\mathbf{y})} u\,(\mathbf{y} - \mathbf{x} + (x_l - \delta_l(\mathbf{y}))\mathbf{e}_l)\,(\mathrm{d}\mathbf{x})_l \right. \\
& \left. + \sum_{l=1}^{d} \partial_i \delta_l(\mathbf{y}) \int_{\mathscr{B}_l(\mathbf{y})} u\,(\mathbf{y} - \mathbf{x} + (x_l + \delta_l(\mathbf{y}))\mathbf{e}_l)\,(\mathrm{d}\mathbf{x})_l \right].
\end{aligned} \tag{21}$$

This formula was originally derived under the assumption $u \in C^1(\overline{U(\mathbf{y})})$, where $U(\mathbf{y})$ is defined in Lemma 2.3, but it is clear that the derivation proceeds analogously if $\partial_i u$ is defined only in a weak sense such that $\partial_i u$ may be undefined on a set of Lebesgue measure zero. However, the case that $\partial_i u$ includes Dirac distributions is excluded. Note that also the representation (19) of the commutation error is still valid.

3.2. *Pointwise estimate of the commutation error for Hölder-continuous functions*

The space $C^{0,\alpha}(\Omega)$, $\alpha \in (0, 1]$, of α-Hölder-continuous functions is defined by

$$C^{0,\alpha}(\Omega) = \left\{ u : \; \Omega \to \mathbb{R} : \sup_{\mathbf{x},\mathbf{y}\in\Omega,\; \mathbf{x}\neq\mathbf{y}} \frac{|u(\mathbf{x}) - u(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|^{\alpha}} := M_\alpha < +\infty \right\}.$$

The property of Hölder-continuity is stronger than continuity, e.g., for $\alpha = 1$ the functions are Lipschitz-continuous. On the other hand, Hölder-continuous functions may not be differentiable at all. An upper bound of the pointwise commutation error is given in the following proposition.

Proposition 3.1. *Let $u \in C^{0,\alpha}(\Omega)$, $\alpha \in (0, 1]$ and $\delta_l(\mathbf{y}) \in C^1(\overline{U(\mathbf{y})})$, $l = 1, \ldots, d$, then*

$$|\mathscr{E}_{\mathrm{c},i}(\mathbf{y})| \leqslant C_\alpha \left| \sum_{l=1}^{d} \frac{\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \right| \left(\sum_{l=1}^{d} 4\delta_l(\mathbf{y})^2 \right)^{\alpha/2}, \tag{22}$$

where $C_\alpha > 0$ is a constant depending on α.

Proof. Let us fix $\mathbf{y} \in \mathscr{B}(\mathbf{y})$. By the mean value theorem of integral calculus there are $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathbf{y} + \mathscr{B}(\mathbf{y})$ such that

$$\int_{\mathscr{B}(\mathbf{y})} u(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathscr{A}(\mathbf{y})u(\mathbf{y}_1),$$

$$\int_{\mathscr{B}_l(\mathbf{y})} u\,(\mathbf{y}-\mathbf{x}+(x_l-\delta_l(\mathbf{y}))\mathbf{e}_l)\,(\mathrm{d}\mathbf{x})_l = \frac{\mathscr{A}(\mathbf{y})}{2\delta_l(\mathbf{y})}u(\mathbf{y}_2),$$

$$\int_{\mathscr{B}_l(\mathbf{y})} u\,(\mathbf{y}-\mathbf{x}+(x_l+\delta_l(\mathbf{y}))\mathbf{e}_l)\,(\mathrm{d}\mathbf{x})_l = \frac{\mathscr{A}(\mathbf{y})}{2\delta_l(\mathbf{y})}u(\mathbf{y}_3).$$

The l-component of $\mathbf{y}_2$ is $y_l - \delta_l(\mathbf{y})$ and of $\mathbf{y}_3$ is $y_l + \delta_l(\mathbf{y})$. Inserting these formulas into the expression for the commutation error (21) gives

$$\mathscr{E}_{\mathrm{c},i}(\mathbf{y}) = \sum_{l=1}^{d} \frac{\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \left(-u(\mathbf{y}_1) + \frac{1}{2}u(\mathbf{y}_2) + \frac{1}{2}u(\mathbf{y}_3) \right).$$

From the Hölder-continuity of u, it follows

$$|\mathscr{E}_{\mathrm{c},i}(\mathbf{y})| \leqslant \frac{M_\alpha}{2} \left| \sum_{l=1}^{d} \frac{\partial_i \delta_l(\mathbf{y})}{\delta_l(\mathbf{y})} \right| (\|\mathbf{y}_2-\mathbf{y}_1\|_2^\alpha + \|\mathbf{y}_3-\mathbf{y}_1\|_2^\alpha).$$

Since $\mathbf{y}_1,\ \mathbf{y}_2,\ \mathbf{y}_3$ are points in the box $\mathbf{y} + \mathscr{B}(\mathbf{y})$, the last factor can be finally estimated by twice the diameter of the filter box, raised to the power of α. □

To interpret the result of the proposition, we consider the simplest case that $\delta_l(\mathbf{y}) = \delta(\mathbf{y})$, $l = 1, \ldots, d$. Then (22) becomes

$$|\mathscr{E}_{\mathrm{c},i}(\mathbf{y})| \leqslant C_\alpha 4^{\alpha/2} d^{1+\alpha/2} \frac{|\partial_i \delta(\mathbf{y})|}{\delta(\mathbf{y})^{1-\alpha}}.$$

We observe that the condition $\delta(\mathbf{y}) \to 0$ is not longer sufficient for the pointwise convergence of the commutation error. There is now an additional condition of the form $\partial_i \delta(\mathbf{y}) \to 0$ needed. In the case $\alpha < 1$, one has even to ensure that $\partial_i \delta(\mathbf{y}) \to 0$ tends sufficiently fast to zero in comparison to $\delta(\mathbf{y})$. The latter has to be valid in particular as $\mathbf{y}$ approaches the boundary $\partial\Omega$ of Ω. Let $\delta(\mathbf{y})$ be of the form $\delta(\mathbf{y}) = (\mathrm{distance}(\mathbf{y}, \partial\Omega))^{1+\beta}$. A straightforward calculations shows that β has to fulfill the condition $\beta > (1-\alpha)/\alpha$ in order to ensure the pointwise convergence of the commutation error.

3.3. Estimates for the $L^p(\Omega)$-norm of the commutation error

In this section, we study the commutation error for functions u, given in an interval (a, b), which are bounded and continuous, $u \in C_b^0(a, b)$, and continuously differentiable but in a given point $\lambda \in \Omega = (a, b)$ such that approaching λ, the derivatives are not bounded. We suppose that the derivatives can be estimated by rational functions

$$|u'(y)| \leqslant \begin{cases} \dfrac{C_3}{|y-\lambda|^{\gamma}}, & y \in U_0(\lambda)\backslash\{\lambda\}, \\ C_4 & \text{elsewhere}, \end{cases} \qquad |u''(y)| \leqslant \begin{cases} \dfrac{C_5}{|y-\lambda|^{1+\gamma}}, & y \in U_0(\lambda)\backslash\{\lambda\}, \\ C_6 & \text{elsewhere}, \end{cases}$$

where $\gamma \in [0, 1)$, $U_0(\lambda)$ is a fixed neighborhood of λ and the constants $C_3, \ldots, C_6$ are fixed.

Definition 3.2. A family of filter width functions $\{\delta(y)\}_{\delta_0}$ is called a regular δ_0-family, if the filter width functions depend on a parameter δ_0 such that

- $\delta(y) \in C^1(\overline{\Omega})$ for all $\delta(y) \in \{\delta(y)\}_{\delta_0}$;
- $\|\delta(y)\|_{L^\infty(\Omega)} \to 0$ if and only if $\delta_0 \to 0$;
- there is a constant C_0 independent of $\delta(y)$ such that $\max_{y\in\overline{\Omega}} |\delta(y)| \leqslant C_0\delta_0$ for all $\delta(y) \in \{\delta(y)\}_{\delta_0}$.

By considering a regular δ_0-family of filter width functions, one can study the asymptotic behavior of the commutation error by taking the limit $\delta_0 \to 0$.

Proposition 3.3. *Let $\Omega = (a, b)$, u be a function with the properties described above and $\{\delta(y)\}_{\delta_0}$ a regular δ_0-family of filter width functions. Then*

$$\|\mathscr{E}_{\mathrm{c}}\|_{L^p(\Omega)} \leqslant C(\varepsilon, \gamma)\|\delta'(y)\|_{L^\infty(\Omega)}\, \delta_0^{1/p-\gamma(1+\varepsilon)}, \quad p \in [1, \infty), \tag{23}$$

for sufficiently small δ_0, where $\varepsilon > 0$ is any positive real number.

Proof. The regular δ_0-family of filter width functions satisfies the following property: if δ_0 is sufficiently small, then there exists $U(\lambda)$, a neighborhood of λ, such that

$$(\lambda - 2C_0\delta_0, \lambda + 2C_0\delta_0) \subset U(\lambda) \subset (\lambda - C_1\delta_0, \lambda + C_1\delta_0) \subset (a, b)$$

with C_1 independent of $\delta(y)$. In addition, one can choose a constant $C_2 \in \mathbb{R}$, $C_2 > 0$, independent of δ_0, such that $C_2\delta_0 \leqslant \delta(y)$ for all $y \in (\lambda - 3C_0\delta_0, \lambda + 3C_0\delta_0)$.

The idea of the proof is to treat the neighborhood where the derivatives of u are singular separately from the rest of the domain. To perform this, the $L^p(\Omega)$-norm of the commutation error can be split as follows

$$\int_\Omega |\mathscr{E}_{\mathrm{c}}(y)|^p\,\mathrm{d}y = \int_{\Omega\backslash(U(\lambda)\cup U_1(\lambda))} |\mathscr{E}_{\mathrm{c}}(y)|^p\,\mathrm{d}y + \int_{U_1(\lambda)\backslash U(\lambda)} |\mathscr{E}_{\mathrm{c}}(y)|^p\,\mathrm{d}y + \int_{U(\lambda)} |\mathscr{E}_{\mathrm{c}}(y)|^p\,\mathrm{d}y,$$

where $U_1(\lambda)$ is a neighborhood of λ such that for all $y \in \Omega\backslash U_1(\lambda)$ we have $y \pm \delta(y) \in \Omega\backslash U_0(\lambda)$. If δ_0 is small enough then $U(\lambda) \subset U_0(\lambda) \subset U_1(\lambda)$. Hence, $U(\lambda) \cup U_1(\lambda) = U_1(\lambda)$, see Fig. 1 for an illustration. A technical difficulty arises in the following estimates from the commutation error being inside the integrals. The commutation error at a point y of the domain of integration is bounded by values of the functions in a neighborhood of y, e.g., see (20). But this neighborhood may be extended outside the domain of integration and therefore we have to consider in the following estimates always a domain which is slightly larger than the domain of integration. The extension of the domain which we have to consider is bounded by $C_0\delta_0$.

To estimate the first integral, one can use the estimate (20) for the commutation error and the uniform boundedness of u'':

$$\int_{\Omega\backslash U_1(\lambda)} |\mathscr{E}_{\mathrm{c}}(y)|^p\,\mathrm{d}y \leqslant C \int_{\Omega\backslash U_1(\lambda)} \left|\delta'(y)\delta(y)\|u\|_{C^2(\overline{U(y)})}\right|^p \mathrm{d}y$$
$$\leqslant C\|\delta'(y)\|^p_{L^\infty(\Omega)}\|\delta(y)\|^p_{L^\infty(\Omega)} \leqslant C\|\delta'(y)\|^p_{L^\infty(\Omega)}\delta_0^p.$$

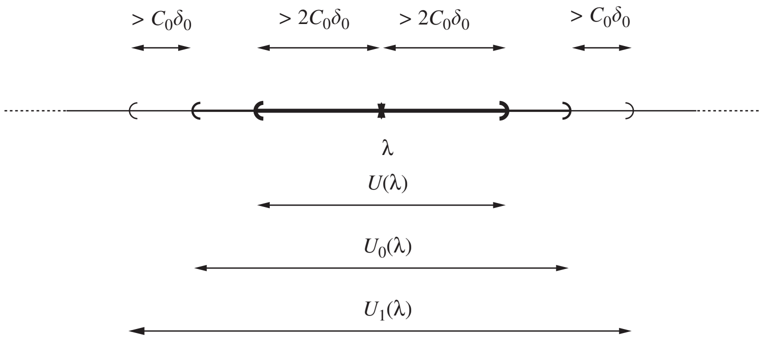

Fig. 1. Illustration to the proof of Proposition 3.3.

Choosing C_5 large enough and using (20), the second integral can be estimated in the following way:

$$\begin{aligned}
&\int_{U_1(\lambda)\setminus U(\lambda)} |\mathscr{E}_{\mathrm{c}}(y)|^p \,\mathrm{d}y \\
&\quad\leqslant C\|\delta'(y)\|^p_{L^\infty(\Omega)}\|\delta(y)\|^p_{L^\infty(\Omega)} \int_{\Omega\setminus(\lambda-C_0\delta_0,\lambda+C_0\delta_0)} \frac{1}{|y-\lambda|^{p(1+\gamma)}}\,\mathrm{d}y \\
&\quad\leqslant C\|\delta'(y)\|^p_{L^\infty(\Omega)}\|\delta(y)\|^p_{L^\infty(\Omega)} \left(C-\frac{1}{(C_0\delta_0)^{p(1+\gamma)-1}}\right) \\
&\quad\leqslant C\|\delta'(y)\|^p_{L^\infty(\Omega)}\delta_0^p + C\|\delta'(y)\|^p_{L^\infty(\Omega)}\delta_0^{1-p\gamma}.
\end{aligned}$$

To estimate the last term, the representation (19) of the commutation error is used. Choosing C_3 large enough and using Hölder's inequality, one gets

$$\begin{aligned}
\int_{U(\lambda)} |\mathscr{E}_{\mathrm{c}}(y)|^p \,\mathrm{d}y &= \int_{U(\lambda)} \left| \frac{\delta'(y)}{2\delta(y)^2} \int_{y-\delta(y)}^{y+\delta(y)} (y-x)u'(x)\,\mathrm{d}x \right|^p \mathrm{d}y \\
&\leqslant C\|\delta'(y)\|^p_{L^\infty(\Omega)} \int_{(\lambda-3C_0\delta_0,\lambda+3C_0\delta_0)} \left[\frac{1}{\delta(y)^{2p}} \|x\|^p_{L^{\gamma_1'}(-\delta(y),\delta(y))} \right. \\
&\quad \left. \times \left\| \frac{1}{d(\lambda,x)} \right\|^p_{L^{\gamma_1}(y-\delta(y),y+\delta(y))} \right] \mathrm{d}y.
\end{aligned}$$

From the assumptions on u', the maximal exponent γ_1 which can be chosen is $\gamma_1 = 1/(\gamma+\varepsilon\gamma)$ with an arbitrary $\varepsilon > 0$. The exponent γ_1' is the dual exponent of γ_1 and it is $\gamma_1' = 1/(1-\gamma-\varepsilon\gamma)$ with an arbitrary $\varepsilon > 0$. By using the inequality $0 < C_2\delta_0 \leqslant \delta(y) \leqslant C_0\delta_0$ with C_2 independent of δ_0, the fact that the length of the domain of integration is $6C_0\delta_0$, the fact that the second factor is bounded since $\gamma < 1$ and the formula for γ_1', one obtains

$$\begin{aligned}
\int_{U(\lambda)} |\mathscr{E}_{\mathrm{c}}(y)|^p \,\mathrm{d}y &\leqslant C\|\delta'(y)\|^p_{L^\infty(\Omega)} \int_{(\lambda-3C_0\delta_0,\lambda+3C_0\delta_0)} \frac{\delta(y)^{p(\gamma_1'+1)/\gamma_1'}}{\delta(y)^{2p}}\,\mathrm{d}y \\
&\leqslant C\|\delta'(y)\|^p_{L^\infty(\Omega)}\delta_0^{1-p\gamma(1+\varepsilon)}.
\end{aligned}$$

Comparing all estimates, one finds that the last one dominates the other ones. □

An inspection of the proof shows that if $\gamma = 0$ then $\gamma_1 = \infty$ and $\gamma_1' = 1$. In this case, the constant in (23) depends only on γ but not on some parameter ε. If $\gamma > 0$, then $C(\varepsilon,\gamma) \to \infty$ as $\varepsilon \to 0$.

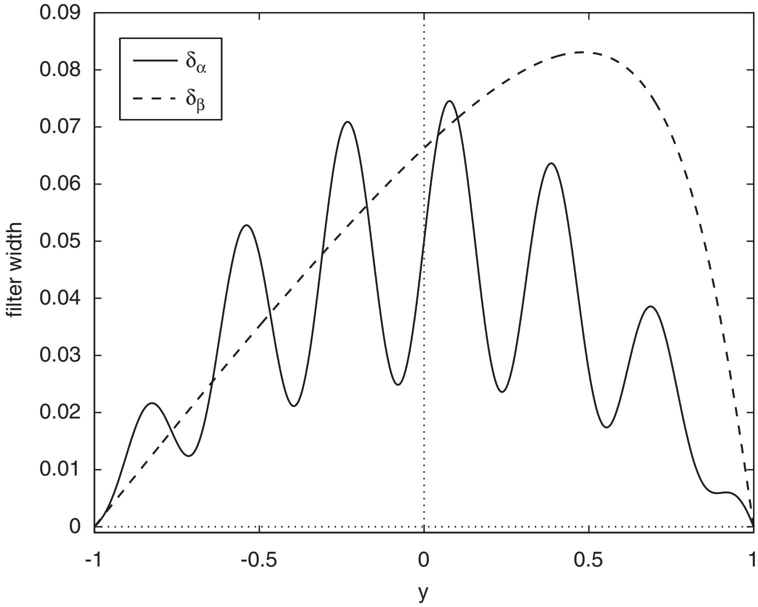

Fig. 2. The filter width functions δ_α and δ_β for $\delta_0 = 0.05$.

In can be shown by a straightforward computation that a function with the properties described at the beginning of this section belongs to the Hölder space $C^{0,1-\gamma}$. Thus, (22) with $\alpha = 1 - \gamma$ can be used for estimating the commutation error in $L^\infty(\Omega)$.

3.4. Numerical examples

The sharpness of the estimates given in this section is shown by Examples 3.1 and 3.2. In these examples, the interval $[-1, 1]$ was divided by an equi-distant grid into 20 000 sub-intervals. The commutation error was computed in all grid points. The maximum of the absolute values in the grid points is given below as its $L^\infty(\Omega)$-norm. Numerical quadrature was used to compute the other $L^p(\Omega)$-norms. The order of convergence was computed using the results corresponding to the two smallest values of the parameter δ_0.

Example 3.1. We consider $\Omega = (-1, 1)$ and $u = |x|$. Apart from $x = 0$, the derivative of u is defined in the classical sense and u is Lipschitz-continuous, i.e., $u \in C^{0,1}(\Omega)$. A straightforward calculation of the commutation error using either its definition or the representation (19) shows

$$\mathscr{E}_c(y) = \begin{cases} 0 & \text{if } |y| \geqslant \delta(y), \\ \dfrac{\delta'(y)}{2}\left(1 - \dfrac{y^2}{\delta(y)^2}\right) & \text{if } |y| < \delta(y). \end{cases}$$

The commutation error vanishes outside $(-\delta(y), \delta(y))$ since u is linear if $x \neq 0$.

We will study the commutation error for two filter width functions. The first one is

$$\delta_\alpha(y) = \delta_0\left(1 + \frac{1}{2}\sin\left(\frac{y}{\delta_0}\right)\right)(1 - y^2)$$

with $\delta_0 > 0$, see Fig. 2, and it follows that

$$\frac{\delta_0}{2}(1 - y^2) \leqslant \delta_\alpha(y) \leqslant \frac{3\delta_0}{2}\left(1 - y^2\right) \quad \forall\, y \in [-1, 1].$$

The left inequality shows that δ_α is positive in $(-1, 1)$ and the right one shows that $y - \delta_\alpha(y) \in [-1, 1]$ and $y + \delta_\alpha(y) \in [-1, 1]$ if $\delta_0 \leqslant \frac{2}{3}$. Clearly, $\delta_\alpha \in C^1([-1, 1])$ such that all requirements on the filter width function which were made

Table 1
Example 3.1, computed norms of the commutation error

δ_0	Filter width function δ_α				Filter width function δ_β		
	$\mathscr{E}_c(0)$	$\|\mathscr{E}_c\|_{L^1}$	$\|\mathscr{E}_c\|_{L^2}$	$\|\mathscr{E}_c\|_{L^\infty}$	$\|\mathscr{E}_c\|_{L^1}$	$\|\mathscr{E}_c\|_{L^2}$	$\|\mathscr{E}_c\|_{L^\infty}$
0.4	0.25	0.0956	0.1376	0.2615	0.12423	0.14731	0.22524
0.2	0.25	0.0557	0.1037	0.2509	0.03698	0.05648	0.11023
0.1	0.25	0.0299	0.0757	0.2501	0.00956	0.02038	0.05469
0.05	0.25	0.0153	0.0540	0.2500	0.00241	0.00724	0.02728
0.025	0.25	0.0077	0.0384	0.2500	6.021E−4	0.00256	0.01363
0.0125	0.25	0.0038	0.0271	0.2500	1.506E−4	9.061E−4	0.00682
Order	0	0.998	0.499	0	1.999	1.5	1
Theory	0	1	0.5	0	2	1.5	1

Table 2
Example 3.2, computed norms of the commutation error

δ_0	Filter width function δ_α			Filter width function δ_β		
	$\|\mathscr{E}_c\|_{L^1}$	$\|\mathscr{E}_c\|_{L^2}$	$\|\mathscr{E}_c\|_{L^\infty}$	$\|\mathscr{E}_c\|_{L^1}$	$\|\mathscr{E}_c\|_{L^2}$	$\|\mathscr{E}_c\|_{L^\infty}$
0.4	0.09045	0.11244	0.23265	0.103927	0.110421	0.181423
0.2	0.05718	0.09857	0.28480	0.040079	0.051558	0.115233
0.1	0.04100	0.09654	0.38779	0.016174	0.026757	0.076259
0.05	0.03032	0.09625	0.54319	0.006120	0.013557	0.051955
0.025	0.02222	0.09621	0.76635	0.002250	0.006805	0.036022
0.0125	0.01613	0.09621	1.08313	8.1558E−4	0.003407	0.025215
0.00625	0.01161	0.09621	1.53155	2.9309E−4	0.001704	0.017739
Order	0.474	0	−0.5	1.476	1	0.507
Theory	$0.5-\varepsilon$	$-\varepsilon$	−0.5	$1.5-\varepsilon$	$1-\varepsilon$	0.5

in the analysis are fulfilled. Obviously, $\delta_\alpha(y) \to 0$ as $\delta_0 \to 0$. The derivative δ'_α is well-defined for every $\delta_0 > 0$ and uniformly bounded with respect to δ_0. But its limit for $\delta_0 \to 0$ does not exist due to the infinite number of oscillations that occur. It is $\delta_\alpha(0) = \delta_0$ and $\delta'_\alpha(0) = \|\delta'_\alpha\|_{L^\infty(\Omega)} = 0.5$ for all $\delta_0 > 0$.

The second filter width function which was used is

$$\delta_\beta(y) = \delta_0(1+y)\arctan(4(1-y)),$$

see Fig. 2. This filter width function fulfills all requirements made in the analysis. In contrast to δ_α, it has the property $\delta'_\beta(y) = \mathscr{O}(\delta_0)$ in a neighborhood of $y = 0$.

The numerical results are presented in Table 1. For the convergence of the commutation error in $L^p(\Omega)$, $p \in [1, \infty)$, estimate (23) with $\gamma = 0$ applies. The commutation error in $L^\infty(\Omega)$ is estimated by (22) with $\alpha = 1$. Table 1 shows that all theoretical expectations are fulfilled and that the estimates are sharp.

Example 3.2. We now consider the function

$$u(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

in $\Omega = (-1, 1)$. The commutation error has the form

$$\mathscr{E}_c(y) = \begin{cases} 0 & \text{if } y < -\delta(y), \\ -\dfrac{\delta'(y)}{6\delta(y)^2}\left(\sqrt{y+\delta(y)}(2y-\delta(y))\right) & \text{if } |y| \leqslant \delta(y), \\ -\dfrac{\delta'(y)}{6\delta(y)^2}\left(\sqrt{y+\delta(y)}(2y-\delta(y)) - \sqrt{y-\delta(y)}(2y+\delta(y))\right) & \text{if } y > \delta(y). \end{cases}$$

The function is Hölder-continuous with the Hölder exponent $\frac{1}{2}$ and it fulfills the assumptions of Proposition 3.3 with $\gamma = \frac{1}{2}$. Thus, the order of convergence in $L^\infty(\Omega)$ is estimated by (22), the order of convergence in $L^p(\Omega)$, $p \in [1, \infty)$, by (23). The numerical results in Table 2 show that the analytical bounds are sharp.

4. Convergence of the commutation error near boundaries

The error estimate (20) for a filter kernel with compact support applied to a function in a bounded domain provides an estimate of the commutation error provided $\|u\|_{C^2(\overline{U(\mathbf{y})})}$ is bounded. If $\|u\|_{C^2(\overline{U(\mathbf{y})})}$ is not bounded, the commutation error might be unbounded in Ω as well. We will consider two cases, which are relevant in practice, where this can happen and formulate the conditions on the filter width function to ensure convergence of the commutation error. For simplicity of presentation, the analysis is performed for a symmetric application of the filter kernel, $\delta^-(y) = \delta^+(y) = \delta(y)$.

The results are also connected with the previous section and in particular with the intent of understanding the role of the commutation error for non-smooth solutions to the Navier–Stokes equations. Analytical studies of commutation errors for model situations in the turbulent channel flow problem can be found in [5].

4.1. The $1/\alpha$th power wall law

Let $\Omega = (0, a)$. We study the situation that the mean velocity obeys the $1/\alpha$th power wall law at $x = 0$. That means, the corresponding component u of the velocity has the form

$$u(x) = \begin{cases} U_\infty \left(\dfrac{x}{\eta}\right)^{1/\alpha}, & 0 \leqslant x \leqslant \eta, \\ U_\infty, & \eta < x, \end{cases}$$

where U_∞ is the free stream velocity, $\alpha > 1$ and η is the boundary layer thickness, see [25]. Often, $\alpha = 7$ is used. This function fulfills the assumptions of Proposition 2.4.

We are interested in the behavior inside the boundary layer, i.e., $0 < x < \eta$. Inside this interval, there is

$$u''(x) = U_\infty \eta^\alpha \frac{1}{\alpha} \left(\frac{1}{\alpha} - 1\right) x^{1/\alpha - 2}.$$

Thus, $u''(x)$ is not bounded as $x \to 0$. Approaching the boundary, it is required in Section 2.2 that the filter width function tends to zero. Let $\delta(y) = \mathcal{O}(y^q)$, $q > 0$. Then, $\delta'(y) = \mathcal{O}(y^{q-1})$ and the error estimate (20) has near the boundary $y = 0$ the form

$$|\mathscr{E}_c(y)| \leqslant C y^{2q+1/\alpha-3}. \tag{24}$$

Thus, the requirement for the convergence of the commutation error in a neighborhood of zero is $q > (3 - 1/\alpha)/2$. The following numerical example shows the sharpness of estimate (24).

Example 4.1. The commutation errors presented in Fig. 3 are computed using $U_\infty = 1$, $\eta = 1$, $\alpha = 7$ and $\delta(y) = 0.1(y(1-y))^q$. It can be seen for $q = \frac{10}{7}$ that the commutation error is bounded but it does not converge to zero. This is exactly the value of q which makes the power of y in (24) equal to zero. For larger values of q, convergence of the commutation error can be observed and for smaller values divergence. The order of convergence and divergence is exactly the order which is given in the estimate (24), compare with the asymptotics presented in Fig. 3.

The convergence of the commutation error in the case that the mean flow obeys the $1/\alpha$th power law at the wall is only given if the filter width function tends to zero sufficiently fast near the boundary. Since in computations the value of the filter width function has to be larger than the mesh width, this implies that the mesh has to be very fine at the wall. In practice, the convergence of the commutation error requires that the boundary layer must be resolved in the numerical simulation. The use of higher order filters does not present a remedy in this situation since estimates for higher order filters involve higher derivatives of u which are even more singular than the second derivative.

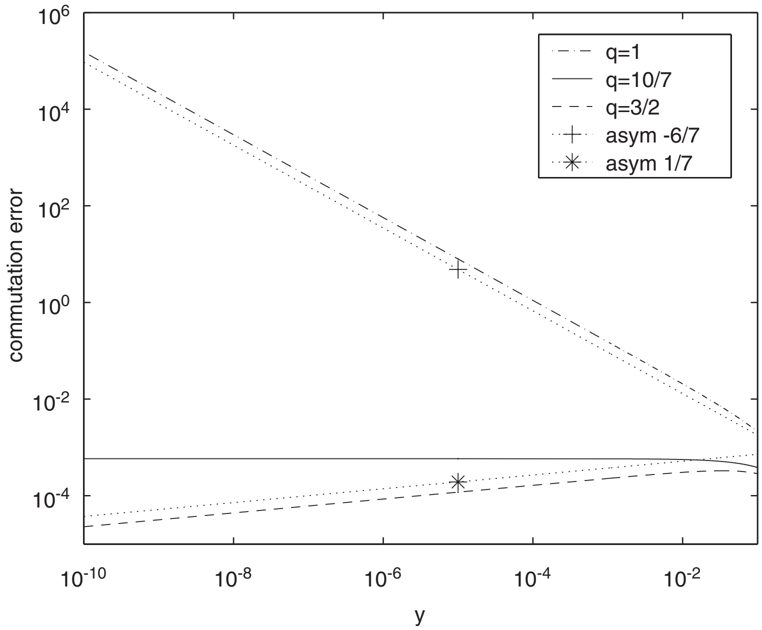

Fig. 3. Example 4.1, the absolute value of the commutation error in a neighborhood of $y = 0$ for different values of q.

4.2. Singularities at the boundary

In this section, we analyze the role of singularities at the boundary on the convergence of the commutation error. In particular, we have in mind the application to the situation of having a singularity in the velocity or pressure field at a corner on the boundary, cf., the discussion in Section 3.1.

Proposition 4.1. *Consider $u = x^{-a}$, $a > 0$ in $(0, b]$ and let the filter width function be of the form $\delta(y) = cy^q$ if y is sufficiently small, $q \geqslant 1$, with $c > 0$ such that $y - \delta(y) > 0$ for $y \in (0, b]$. Then*

$$|\mathscr{E}_{\mathrm{c}}(y)| \leqslant C y^{2q-3-a} \tag{25}$$

if y is sufficiently small.

Proof. The commutation error (19) reduces in one dimension and for the symmetric box filter to

$$\mathscr{E}_{\mathrm{c}}(y) = \frac{\delta'(y)}{2\delta(y)} \left(u(y + \delta(y)) + u(y - \delta(y)) - \frac{1}{\delta(y)} \int_{y-\delta(y)}^{y+\delta(y)} u(x)\, \mathrm{d}x \right).$$

Let $a \neq 1$. A straightforward calculation gives

$$\mathscr{E}_{\mathrm{c}}(y) = \frac{\delta'(y)}{2(1-a)\delta(y)^2} \left(-\frac{y + a\delta(y)}{(y + \delta(y))^a} + \frac{y - a\delta(y)}{(y - \delta(y))^a} \right).$$

Using the special form of $\delta(y)$ leads to

$$\mathscr{E}_{\mathrm{c}}(y) = \frac{q}{4c(1-a)} y^{-(q+a)} \frac{-(1 + acy^{q-1})(1 - cy^{q-1})^a + (1 - acy^{q-1})(1 + cy^{q-1})^a}{(1 + cy^{q-1})^a (1 - cy^{q-1})^a}.$$

Inserting the Taylor series expansion of $(1 \pm cy^{q-1})^a$ at $y = 0$ into this formula reveals that all terms up to the power $y^{2(q-1)}$ vanish. Hence $\mathscr{E}_{\mathrm{c}}(y) = \mathscr{O}(y^{-(q+a)+3(q-1)})$.

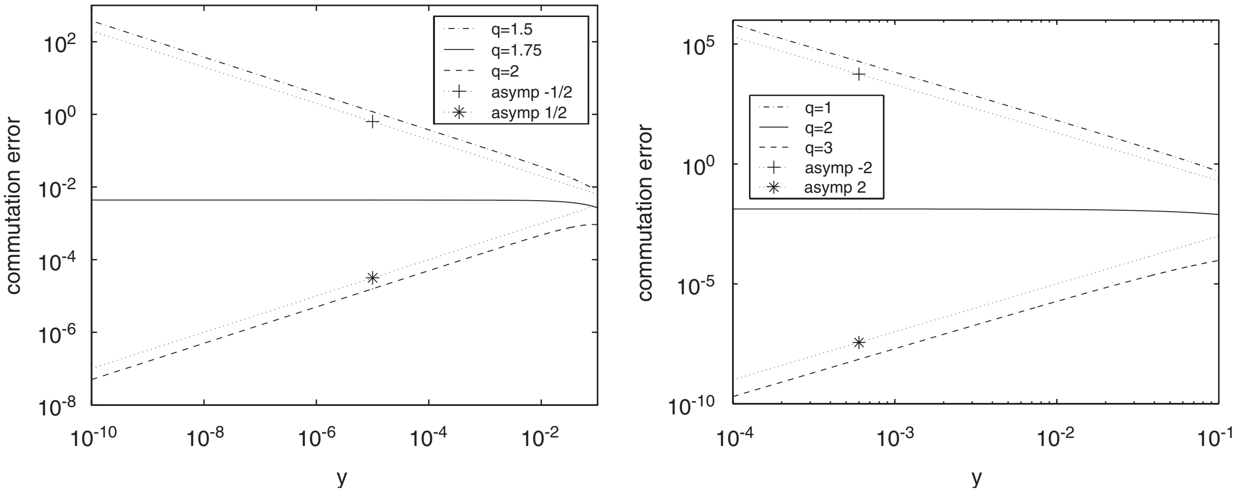

Fig. 4. Example 4.2, the absolute value of the commutation error at $y = 0$ for different values of q, left $u(x) = 1/\sqrt{x}$, right $u(x) = 1/x$.

For $a = 1$, proceeding in the same way, one obtains

$$\mathscr{E}_{\mathrm{c}}(y) = \frac{q}{4y^{q+1}}\left(\frac{2cy^{q-1} - (1 - c^2y^{2(q-1)})(\ln(1 + cy^{q-1}) - \ln(1 - cy^{q-1}))}{(1 + cy^{q-1})(1 - cy^{q-1})}\right).$$

Using the Taylor series expansion for $\ln(1 \pm cy^{q-1})$ at $y = 0$ shows that all terms with powers of y less than $3(q - 1)$ cancel out in the numerator. This leads to the desired result. □

Example 4.2. In this example, we consider $u(x) = 1/\sqrt{x}$ and $u(x) = 1/x$, $x \in (0, 1]$, and $\delta(y) = 0.1(y(1 - y))^q$, for different values of q. The assumptions of Proposition 4.1 are fulfilled in this example and the behavior of the commutation error near $y = 0$ is estimated by (25). The results presented in Fig. 4 show the sharpness of this estimate in the asymptotic limit.

Like in the case of the $1/\alpha$th power wall law, the convergence of the commutation error requires that the values of the filter width function $\delta(y)$ must become sufficiently small near the singularity. In this region, it has to be asymptotically even smaller than in the region where the power law holds. The connection of the value of the filter width function and the mesh size requires—in practice—a mesh near the singularity which allows the resolution of the singularity.

5. Summary

In this paper, commutation errors were studied for functions with low regularity and non-uniform and asymmetric filters. Commutation errors are committed by deriving the space averaged Navier–Stokes equations and they are one source of error in each LES model. The significance of commutation errors has been realized only recently.

The main novelty of our studies is that we did not try to find good asymptotics of the errors by making ad hoc assumptions on the filter kernel and on the smoothness of the functions. On the contrary, we tried to understand the role of commutation errors for widely used filters, applied to functions with realistic regularity. As first step, a unified representation of the commutation errors has been derived. Estimates of the commutation errors were proved analytically for Hölder-continuous functions and for functions with rational singularities in their derivatives. Furthermore, commutation errors for functions obeying the $1/\alpha$th power wall law and functions which are singular at the boundary have been analyzed analytically and numerically.

A main result in this paper is that the asymptotic vanishing of the commutation errors requires in all cases some strict connection between the regularity of the function and the value of the filter width function. This connection may lead to such a small filter width that in practice the resolution of the flow becomes necessary.

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