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Topological Characterization of the Approximate Subdifferential in the Finite-Dimensional Case[1]

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Abstract: Topological properties of the approximate subdifferential introduced by Mordukhovich are studied. Apart from formulating a sufficient condition for connectedness, it is shown that, up to homeomorphy, each compact subset of $\mathbb{R}^p$ may occur as the approximate subdifferential of some Lipschitz function. Furthermore, even an exact result is possible when considering the partial approximate subdifferential, which was introduced as a parametric extension by Jourani and Thibault: Given any compact subset of $\mathbb{R}^p$, there is a locally Lipschitzian function realizing this set as its partial approximate subdifferential at some predefined point.

Key Words: Approximate subdifferential, topological properties, partial approximate subdifferential, nonsmooth optimization.

1 Introduction

The approximate subdifferential introduced by Mordukhovich [10] has been extensively studied by Ioffe both in the finite-dimensional [5] and Banach space setting [7]. Being minimal within a family of reasonable subdifferentials ([5], Theorem 9), this concept has attracted much attention in some recent papers. As examples, the derivation of Fritz-John or Kuhn-Tucker conditions as well as metric regularity results by Jourani and Thibault [8], [9], Glover and Craven [2] and Glover, Craven and Flåm [3] may be cited. Fruitful applications can be expected in stochastic and semi-infinite programming (see e.g. [3]).

As one of the most important features the approximate subdifferential is contained in that of Clarke [1]. On the other hand it preserves upper semicontinuity. This makes it preferable to, for instance, the Dini subdifferential [4]

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which is also a subset of Clarke's subdifferential. Not being defined via support functions, the approximate subdifferential fails to be convex in general. It may even happen to become disconnected as can be seen from the simple example $f(x) = -|x|$.

This paper deals with the quesiton, which topological types may occur in the finite dimensional case. It turns out that, for locally Lipschitzian functions, there are no restrictions apart from compactness. Nevertheless, it is possible to impose specific conditions on the Dini subdifferential – being the essential ingredient of the approximate subdifferential – which ensure certain topological properties like convexity, star-shapedness or connectedness.

2 Basic Definitions and Properties

For topological spaces X, X^* consider a multifunction $F: X \to 2^{X^*}$.

Definition 2.1 (limits of multifunctions): For $z \in X$ put

$$\limsup_{x\to z} F(x) = \{x^* \in X^* \,|\, \textit{there exist sequences } x_n \to z\ (x_n \in X),\ x_n^* \to x^*\ (x_n^* \in X^*) \textit{ such that } x_n^* \in F(x_n)\}$$

(upper limit of F at z) and

$$\liminf_{x\to z} F(x) = \{x^* \in X^* \,|\, \textit{for all sequences } x_n \to z\ (x_n \in X) \textit{ there exists a sequence } x_n^* \to x^*\ (x_n^* \in X^*) \textit{ such that } x_n^* \in F(x_n)\}$$

(lower limit of F at z)

Sometimes the convergence $x_n \to z$ in the above definitions is restricted by additional conditions to some subset of X. This will be indicated below the 'limsup' and 'liminf' signs.

Now, let $f: \mathbb{R}^p \to \mathbb{R} \cup \{-\infty, \infty\}$ be an extended-valued function.

Definition 2.2 (Dini subdifferential): For $x \in \mathbb{R}^p$ put

$$\partial^- f(x) = \begin{cases} \{x^* \in \mathbb{R}^p \,|\, x^{*T}h \le d^- f(x; h)\,\forall h \in \mathbb{R}^p\} & \textit{if } |f(x)| < \infty \\ \varnothing & \textit{else} \end{cases}$$

where

$$d^-f(x;h) = \liminf_{\substack{u\to h\\ t\downarrow 0}} \frac{f(x+tu)-f(x)}{t}$$

is the lower Dini directional derivative of f at x in direction h.

The definition of the approximate subdifferential will be given in the notation of [5].

Definition 2.3 (approximate subdifferential): For $x \in \mathbb{R}^p$ put

$$\partial_a f(z) = \begin{cases} \limsup\limits_{\substack{x\to z\\ f(x)\to f(z)}} \partial^- f(x) & \text{if } |f(z)| < \infty \\ \varnothing & \text{else} \end{cases}$$

For continuous functions the condition $f(x) \to f(z)$ may be omitted, of course.

Example 2.1: Consider the lower semicontinouos function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x+y+1 & \text{if } x>0 \text{ and } y>0 \\ 0 & \text{else} \end{cases}$$

Then

$$\partial^- f(x,y) = \begin{cases} \{(0,0)\} & \text{if } x<0 \text{ or } y<0 \text{ or } x=y=0 \\ [0,\infty)\times\{0\} & \text{if } x=0 \text{ and } y>0 \\ \{0\}\times[0,\infty) & \text{if } y=0 \text{ and } x>0 \\ \{(1,1)\} & \text{if } y>0 \text{ and } x>0 \end{cases}$$

When computing $\partial_a f(0,0)$ here, one may exclude those sequences $(x_n, y_n) \to (0,0)$ with $x_n > 0$ and $y_n > 0$ because of $f(x_n, y_n) \to 1 \neq 0 = f(0,0)$ (see definition 2.3). Therefore

$$\partial_a f(0,0) = [[0,\infty) \times \{0\}] \cup [\{0\} \times [0,\infty)]$$

(union of two half-rays).

The following simple properties may be stated (see also [5], Theorem 2):

1. $\partial^- f(x)$ is a convex and closed set, $\partial_a f(x)$ is a closed set. It always holds $\partial^- f(x) \subseteq \partial_a f(x)$.
2. $\partial_a f(z) = \limsup_{\substack{x \to z \\ f(x) \to f(z)}} \partial_a f(x)$ (upper semicontinuity).
3. For lower semicontinuous functions the relation $\partial^- f(x) \subseteq \partial_a f(x) \subseteq \partial_c f(x)$ is valid, where ∂_c refers to Clarke's subdifferential. For convex functions the three subdifferentials coincide with the classical subdifferential of convex analysis.
4. For locally Lipschitzian functions $\partial^- f(x)$, $\partial_c f(x)$ and $\partial_a f(x)$ are compact and $\partial_c f(x) = clco\ \partial_a f(x)$, where 'co' refers to convex hull and 'cl' to closure. In particular, $\partial_a f(x) \neq \emptyset\ \forall x \in \mathbb{R}^p$.

In the context of metric regularity investigations for functions depending on a parameter, Jourani and Thibault [8] introduced a parametric extension of the approximate subdifferential which will be marked by an upper index '*' for better distinction.

Definition 2.4 (partial approximate subdifferential): For a metric space U, an extended-valued function $f\colon \mathbb{R}^p \times U \to \mathbb{R} \cup \{-\infty, \infty\}$ and $(z, \bar{u}) \in \mathbb{R}^p \times U$ put

$$\partial_a^* f(z,\bar{u}) = \begin{cases} \limsup\limits_{\substack{(x,u)\to(z,\bar{u}) \\ f(x,u)\to f(z,\bar{u})}} \partial^- f_u(x) & \text{if } |f(z,\bar{u})| < \infty \\ \emptyset & \text{else} \end{cases}$$

where $f_u(x) = f(x, u)$.

The partial approximate subdifferential enjoys the same upper semicontinuity property as the original one, i.e.

$$\partial_a^* f(z,\bar{u}) = \limsup_{\substack{(x,u)\to(z,\bar{u}) \\ f(x,u)\to f(z,\bar{u})}} \partial_a^* f(x,u)$$

3 Results

Owing to its definition, topological properties of the approximate subdifferential are closely related to the local behaviour of the Dini subdifferential. The following conditions on ∂^- (the first of which is included only for the sake of completeness) will imply consecutively weaker characterizations of ∂_a:

(A1) $\partial^- f(z) = \limsup\limits_{\substack{x \to z \\ f(x) \to f(z)}} \partial^- f(x)$

(A2) $\liminf\limits_{\substack{x \to z \\ f(x) \to f(z) \\ \partial^- f(x) \neq \emptyset}} \partial^- f(x) \neq \emptyset$

(A3) $\partial^- f(z)$ is compact and there exists $\varepsilon > 0$ such that all x with $\|x - z\|$, $|f(x) - f(z)| < \varepsilon$ and $\partial^- f(x) \neq \emptyset$ fulfill $\partial^- f(x) \cap \partial^- f(z) \neq \emptyset$

Lemma 3.1:

1. **(A1)** $\Rightarrow \partial_a f(z)$ *is convex.*
2. **(A2)** $\Rightarrow \partial_a f(z)$ *is star-shaped.*
3. **(A3)** $\Rightarrow \partial_a f(z)$ *is connected.*
4. *If f is locally Lipschitzian then*
   (A3) $\Rightarrow$ **(A1)** *as well as*
   (A2) $\Rightarrow$ **(A1)** *or* $\partial_a f(z)$ *is a singleton*

Proof:

ad 1.: This follows immediately from definition 2.3 and the convexity of $\partial^- f(z)$.
ad 2.: First, choose

$$y^* \in \liminf_{\substack{x \to z \\ f(x) \to f(z) \\ \partial^- f(x) \neq \emptyset}} \partial^- f(x) \tag{1}$$

Now, select any $x^* \in \partial_a f(z)$. By definition 2.3, this means existence of sequences

$$x_n \to z,\ f(x_n) \to f(z),\ x_n^* \to x^*,\ x_n^* \in \partial^- f(x_n)$$

From (1) it follows, taking into account the restrictions below the 'liminf' sign, that there exists a sequence $y_n^* \to y^*$ with $y_n^* \in \partial^- f(x_n)$. For arbitrarily fixed $t \in [0, 1]$ one gets

$$w_n^* = tx_n^* + (1-t)y_n^* \in \partial^- f(x_n),\ w_n^* \to tx^* + (1-t)y^*$$

by convexity of $\partial^- f(x_n)$. Consequently, $[x^*, y^*] \subseteq \partial_a f(z)$ for the line segment joining x^* and y^*. In this way, y^* chosen in (1) is itself an element of $\partial_a f(z)$ and has the property, that $[x^*, y^*] \subseteq \partial^a f(z)$ for arbitrary $x^* \in \partial_a f(z)$. This means starshapedness of the approximate subdifferential.

ad 3.: First note, that it is sufficient to show that for each $x^* \in \partial_a f(z)$ there exists some $y^* \in \partial^- f(z)$ such that the line segment $[x^*, y^*]$ is contained in $\partial_a f(z)$. In fact, if there are given two arbitrary elements $x_a^*, x_b^* \in \partial_a f(z)$, then one could conclude that $[x_a^*, y_a^*], [x_b^*, y_b^*] \subseteq \partial_a f(z)$ for some $y_a^*, y_b^* \in \partial^- f(z)$. Now, convexity of $\partial^- f(z)$ implies $[y_a^*, y_b^*] \subseteq \partial^- f(z) \subseteq \partial_a f(z)$ (see property 1 stated above). Hence

$$[x_a^*, y_a^*] \cup [y_a^*, y_b^*] \cup [y_b^*, x_b^*] \subseteq \partial_a f(z)$$

and $\partial_a f(z)$ is connected.

In order to prove the mentioned fact, assume $x^* \in \partial_a f(z)$. By definition, there exist sequences $x_n \to z$, $x_n^* \to x^*$ such that $f(x_n) \to f(z)$ and $x_n^* \in \partial^- f(x_n)$. Owing to assumption **(A3)** there exists $y_n^* \in \partial^- f(x_n) \cap \partial^- f(z)$. Since $\partial^- f(z)$ is compact, one has $y_{n_k}^* \to y^* \in \partial^- f(z)$ for some subsequence. For arbitrary $t \in [0, 1]$ it holds

$$w_{n_k}^* = tx_{n_k}^* + (1-t)y_{n_k}^* \in \partial^- f(x_{n_k}),\ w_{n_k}^* \to tx^* + (1-t)y^* \ .$$

by convexity of $\partial^- f(x_{n_k})$. Therefore $[x^*, y^*] \subseteq \partial_a f(z)$.

ad 4.: Suppose that $\partial^- f$ is not uppersemicontinuous for some locally Lipschitzian f at z, i.e. **(A1)** is violated. Then there exists a point $x^* \in \partial_a f(z) \backslash \partial^- f(z)$. By convexity and closedness of $\partial^- f(z)$ (see property 1 stated in section 2) one even has $x^* \in \partial_a f(z) \backslash A$ for some proper superset $A \supset \partial^- f(z)$ which is closed and convex.

Again, there are sequences $x_n \to z$, $x_n^* \to x^*$ with $x_n^* \in \partial^- f(x_n) \subseteq \partial_c f(x_n)$ (see property 3 in section 2). According to [1] one may represent Clarke's subdifferential as

$$x_n^* \in \partial_c f(x_n) = co\{y^* \mid \exists y_{n_k} \to x_n \colon \nabla f(y_{n_k}) \to y^*\} \tag{2}$$

where ∇ refers to the gradient which is supposed to exist at y_{n_k}. Now, if $y^* \in A$ would hold for all y^* from (2) then convexity of A would imply $x_n^* \in A$. On the other hand, it follows $x_n^* \notin A$ for $n \geq n_0$ from the closedness of A. Thus, for each fixed number $n \geq n_0$ we may choose some element $y^* \notin A$ from (2). By closedness of A a point y_n may be found such that $\|y_n - x_n\| \leq 1/n$ and $\nabla f(y_n) \notin A$. As a result, there exists a sequence $\{y_n\}$ with

$$y_n \to z,\ f(y_n) \to f(z)\ \text{(continuity of } f\text{)},\ \partial^- f(y_n) = \{\nabla f(y_n)\} \neq \emptyset \tag{3}$$

Furthermore, $\partial^- f(y_n) \cap \partial^- f(z) = \emptyset$, which contradicts assumption **(A3)**.

For proving violation of **(A2)**, first assume $\partial^- f(z) \neq \emptyset$. Then

$$\liminf_{\substack{x \to z \\ f(x) \to f(z) \\ \partial^- f(x) \neq \emptyset}} \partial^- f(x) \subseteq \partial^- f(z) \subset A \tag{4}$$

The first inclusion is seen as follows: Let $w^* \in$ 'liminf' and consider the constant sequence $x_n \equiv z$. By definition of 'liminf' there has to be a sequence $w_n^* \to w^*$, $w_n^* \in \partial^- f(z)$, which means $w^* \in \partial^- f(z)$.

Now, suppose that the 'liminf' contains some element w^*. Then, to the sequence $\{y_n\}$ constructed in (3), there must correspond a sequence $y_n^* \in \partial^- f(y_n)$ with $y_n^* \to w^*$. However, by (3), $y_n^* = \nabla f(y_n) \notin A$, which yields $w^* \notin A$ from the closedness of A and a contradiction to (4).

It remains to show that, in the case $\partial^- f(z) = \emptyset$ **(A2)** enforces $\partial_a f(z)$ to reduce to a singleton. If Clarke's subdifferential of f at z would contain at least two elements, then, again exploiting the representation (2) (now with x_n replaced by z), provides the existence of sequences $\{y_n^i\}$ $(i = 1, 2)$, with

$$y_n^i \to z,\ \partial^- f(y_n^i) = \{\nabla f(y_n^i)\},\ \nabla f(y_n^i) \to \alpha_i,\ \alpha_1 \neq \alpha_2$$

This again contradicts **(A2)**. Because of $\emptyset \neq \partial_a f(z) \subseteq \partial_c f(z)$ (see property 4. of section 2) the approximate subdifferential at z must reduce to a singleton. □

By Assertion 4 of lemma 3.1 it is confirmed that the value of assumptions **(A2)** and **(A3)** is restricted to functions not being locally Lipschitzian since, otherwise nothing else than convexity of the approximate subdifferential is implied (see assertion 1 of the lemma). On the other hand, for the non-Lipschitzian case, conditions **(A2)** and **(A3)** are not meaningless. Both of them are satisfied, for instance, in example 2.1 where $\partial_a f(z)$ is star-shaped but not convex. In the following example **(A2)** fails to hold but **(A3)** is fulfilled. As a consequence, $\partial_a f(z)$ is connected but it is not star-shaped.

Example 3.1: Consider the function

$$f(x, y) = \begin{cases} 1 & \text{if } xy > 0 \\ |x| & \text{else} \end{cases}$$

which is lower semicontinous in a neighbourhood of the origin. Computing the *approximate subdifferential gives*

$$\partial_a f(0,0) = [(-\infty,\infty) \times \{0\}] \cup [\{-1\} \times (-\infty,0)] \cup [\{1\} \times (0,\infty)]$$

Finally, condition **(A2)** may circumvent the compactness assumption in **(A3)** as in the example

$$f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ or } xy < 0 \\ 0 & \text{else} \end{cases}$$

Before stating the results on the generality of topological types for approximate subdifferentials we need the following simple lemmas.

Lemma 3.2: For a compact set $K \subseteq \mathbb{R}^q$ and a function $g(z) = \min_{x \in K} z^T x$ put

$$I(z) = \{y \in K \mid z^T y \leq z^T x \; \forall x \in K\}$$

Then it holds

$$I(z) = \{\bar{x}\} \Rightarrow \partial^- g(z) = \{\bar{x}\} \tag{5}$$

$$\# I(z) \geq 2 \Rightarrow \partial^- g(z) = \varnothing \tag{6}$$

Proof: Clearly, the assumption in (5) implies $d^- g(z;h) = \bar{x}^T h \; \forall h \in \mathbb{R}^q$ from which the result follows (see definition 2.2). Concerning (6), assume $\bar{x} \in \partial^- g(z)$. Given arbitrary $x^a, x^b \in I(z)$ and $h \in \mathbb{R}^q$ it follows (compare definition 2.2)

$$\begin{aligned} \min\{x^{aT}h, x^{bT}h\} &\geq \min_{x \in I(z)} x^T h = d^- g(z;h) \geq \bar{x}^T h \geq -d^- g(z;-h) \\ &= -\min_{x \in I(z)} \{-x^T h\} = \max_{x \in I(z)} x^T h \\ &\geq \max\{x^{aT}h, x^{bT}h\} \end{aligned}$$

Therefore, $x^{aT}h = x^{bT}h \; \forall h \in \mathbb{R}^q$, hence $x^a = x^b$ and $\# I(z) = 1$, which contradicts the assumption of (6). □

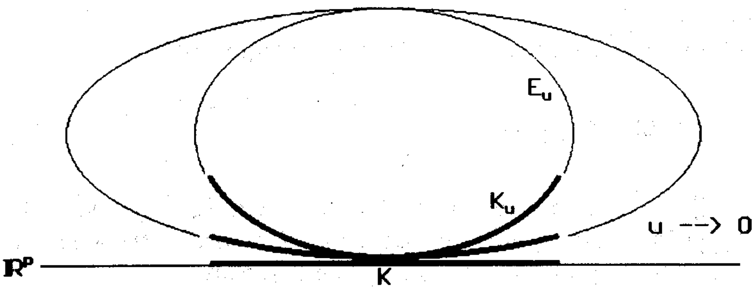

Fig. 1.

Lemma 3.3: For some fixed number $u \in \mathbb{R}$, $u \neq 0$ *consider the following ellipsoid (compare fig.* 1):

$$E_u = \{(x, t) \in \mathbb{R}^p \times \mathbb{R} \,|\, |u| \, \|x\|^2 + (t - 1)^2 = 1\} \tag{7}$$

Then for arbitrary $(z, \alpha) \in E_u$ *and* $\lambda < 0$, *the linear function* $h: \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$, $h(x, t) = \lambda |u| z^T x + \lambda(\alpha - 1)t$ *attains its minimum over* E_u *exactly at the point* $(x, t) = (z, \alpha)$.

The assertion of the last lemma is easily verified.

The following theorem indicates that any predefined compact subset of a finite dimensional space may occur as the *partial* approximate subdifferential (see definition 2.4) of some locally Lipschitzian function.

Theorem 3.1: For each compact subset $K \subseteq \mathbb{R}^p$ *there exists a locally Lipschitzian function* $f: \mathbb{R}^{p+2} \to \mathbb{R}$ *such that* $\partial_a^* f(0_{p+1}, 0) = K \times \{0\}$

Proof: Let C be some positive number fulfilling $\|x\|^2 < C \; \forall x \in K$. Then, for $|u| \leq 1/C$ the set

$$K_u = \{(x, t) \in \mathbb{R}^p \times \mathbb{R} \,|\, x \in K, t = 1 - \sqrt{1 - |u| \, \|x\|^2}\} \tag{8}$$

is well defined. Clearly, $K_0 = K \times \{0\}$ and, for $u \neq 0$, $K_u \subseteq E_u$ with respect to the parameter-dependent ellipsoid in (7). An illustration is given in figure 1.

The assertion of the theorem will be shown to hold for the following locally Lipschitzian function $f: \mathbb{R}^p \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$f(z, \alpha, u) = \begin{cases} \min\{z^T x + \alpha t \,|\, (x, t) \in K_u\} & \text{for } |u| \leq 1/C \\ \min\{z^T x + \alpha t \,|\, (x, t) \in K_{1/C}\} & \text{for } |u| \geq 1/C \end{cases} \tag{9}$$

1) $\partial_a^* f(0_p, 0, 0) \subseteq K \times \{0\}$
Let $(x^*, t^*) \in \partial_a^* f(0_p, 0, 0)$. By definition 2.4 there exist sequences

$$(x_n, t_n, u_n) \to (0_p, 0, 0), (x_n^*, t_n^*) \to (x^*, t^*)$$

with $(x_n^*, t_n^*) \in \partial^- f_{u_n}(x_n, t_n)$. Lemma 3.2 applied to the function $g = f_{u_n}$ and to the compact set K_{u_n} yields – having in mind that the set I defined in the lemma is always nonempty – $\{(x_n^*, t_n^*)\} = I(x_n, t_n) \subseteq K_{u_n}$. Consequently,

$$x_n^* \in K \text{ and } t_n^* = 1 - \sqrt{1 - |u_n| \, \|x_n^*\|^2} \to 0$$

(see (8)). But this means $x^* \in K$ and $t^* = 0$.
2) $K \times \{0\} \subseteq \partial_a^* f(0_p, 0, 0)$
Let $x^* \in K$ and, for $n > C$, define $t_n^* = 1 - \sqrt{1 - \|x^*\|^2/n}$ to get $(x^*, t_n^*) \in K_{1/n}$. Put $x_n = -x^*/n^2$, $t_n = -(t_n^* - 1)/n$. Then, for fixed parameter $u = 1/n$, the partial function f_u of (9) evaluated at (x_n, t_n) reads as

$$f_{1/n}(x_n, t_n) = f(x_n, t_n, 1/n) = \min\{-x^T x^*/n^2 - t(t_n^* - 1)/n | (x, t) \in K_{1/n}\}$$

Application of lemma 3.3 with $u = 1/n$, $z = x^*$, $\alpha = t_n^*$, $\lambda = -1/n$ provides that the linear function

$$h(x, t) = -x^T x^*/n^2 - t(t_n^* - 1)/n$$

attains its minimum over $E_{1/n}$ exactly at the point (x^*, t_n^*). Owing to $(x^*, t_n^*) \in K_{1/n} \subseteq E_{1/n}$ it follows, that the same linear function attains its minimum over $K_{1/n}$ exactly at the mentioned point, or in other words: the set of active indices of $f_{1/n}$ at (x_n, t_n) is $I(x_n, t_n) = \{(x^*, t_n^*)\}$. Via (5) one arrives at $\partial^- f_{1/n}(x_n, t_n) = \{(x^*, t_n^*)\}$. By the corresponding definitions it holds $(x_n, t_n) \to (0_p, 0)$, $(x^*, t_n^*) \to (x^*, 0)$ and therefore $(x^*, 0) \in \partial_a^* f(0_p, 0, 0)$. □

For the usual, nonparametric approximate subdifferential, a little bit weaker, but topologically equivalent result may be derived by similar arguments.

Theorem 3.2: For each compact subset $K \subset \mathbb{R}^p$ and each number $\varepsilon > 0$ there exists a Lipschitzian function $\bar{f}: \mathbb{R}^{p+1} \to \mathbb{R}$ such that K is homeomorphic with $\partial_a \bar{f}(0)$ and dist $(K \times \{0\}, \partial_a \bar{f}(0)) < \varepsilon$.

Proof: First, we verify the equality

$$\partial_a f_u(0_p, 0) = K_u \quad 0 < u < 1/C \tag{10}$$

where f, K_u, C refer to theorem 3.1.

1) $\partial_a f_u(0_p, 0) \subseteq K_u$
Let $(x^*, t^*) \in \partial_a f_u(0_p, 0)$. By definition 2.3 there exist sequences

$$(x_n, t_n) \to (0_p, 0), (x_n^*, t_n^*) \to (x^*, t^*)$$

with $(x_n^*, t_n^*) \in \partial^- f_u(x_n, t_n)$. The same reasoning as in theorem 3.1 with u_n replaced by u yields $(x_n^*, t_n^*) \in K_u$, hence $(x^*, t^*) \in K_u$.
2) $K_u \subseteq \partial_a f_u(0_p, 0)$
For $(x^*, t^*) \in K_u$ define $x_n = -ux^*/n$, $t_n = -(t^* - 1)/n$. Along the same lines as in theorem 3.1 one arrives at $\partial^- f_u(x_n, t_n) = \{(x^*, t^*)\}$. Furthermore, $(x_n, t_n) \to (0_p, 0)$, hence $(x^*, t^*) \in \partial_a f_u(0_p, 0)$.

Now, for each $u \in (0, 1/C)$ the mapping $\phi: K \to K_u$, defined by

$$\phi(x) = (x, 1 - \sqrt{1 - |u| \, \|x\|^2})$$

is a homeomorphism between K and K_u. In fact, with $\psi: K_u \to K$, $\psi(y, t) = y$ one has $\psi \circ \phi = id_K$, $\phi \circ \psi = id_{K_u}$. This means that ϕ is bijective and both, ϕ and $\phi^{-1} = \psi$ are continuous. Summarizing, one may take $\bar{f} = f_u$ for any $u \in (0, 1/C)$ to prove the first assertion of the theorem via (10). Note that any such f_u is even (globally) Lipschitzian (see (9)).

Finally, put $\bar{f} = f_u$ for any $u \in (0, \beta)$ with

$$\beta = \min\{1/C, [1 - (1 - \varepsilon)^2]/C\}$$

to get $dist(K \times \{0\}, K_u) < \varepsilon$ by definition of K_u and by (10) □

If relation (10) would hold also in the case $u = 0$ then, since $K \times \{0\} = K_0$, each compact set could be exactly realised as an approximate subdifferential of some Lipschitzian function. The crucial point is, that uniqueness of the minimum of the linear function f_u over K_u is enforced by embedding K_u into an ellipsoid for $u \in (0, 1/C)$. For $u \to 0$, this ellipsoid degenerates to $[\mathbb{R}^p \times \{0\}] \cup [\mathbb{R}^p \times \{2\}]$ and uniqueness of the minimum is lost in general. Therefore one cannot establish the inclusion $K_0 \subseteq \partial_a f_0(0_p, 0)$ with the ideas used here. Recalling, that

$$K \times \{0\} = K_0 = \limsup_{u \to 0} K_u$$

holds in the example, this may serve as well as an illustration for the relation

$$\limsup_{u\to 0} \partial_a f_u(z) \neq \partial_a f_0(z)$$

i.e., upper semicontinuity of the approximate subdifferential with respect to an exterior parameter fails to hold. This lack is cricumvented by the partial approximate subdifferential.

4 Conclusions

By theorem 3.2 each topological type of a compact set in $\mathbb{R}^p$ may occur as the approximate subdifferential of Lipschitzian functions. The fact that this subdifferential appears in quite a variety of shapes might be helpful when comparing it to other subdifferentials. Another consequence of the theorem is, that the Clarke subdifferential of any locally Lipschitzian function may be approximated arbitrarily close by the approximate subdifferential of simple min-type functions as in (9).

It is possible to extend the ideas used here to the infinite dimensional case in an appropriate setting, but care has to be taken of the generalized definition of the approximate subdifferential (see [6]). It remains an open question, whether each compact set itself (not just an homeomorphic image thereof) can be obtained as the approximate subdifferential of a locally Lipschitzian functions.

References

[1] Clarke FH (1983) Optimization and nonsmooth analysis. Wiley Interscience New York
[2] Glover BM, Craven BD (1994) A Fritz John optimality condition using the approximate subdifferential. Journal of Optimization Theory and Applications 82:253–265
[3] Glover BM, Craven BD, Flåm SD (1993) A generalized karush-kuhn-tucker optimality condition without constraint qualifications using the approximate subdifferential. Numerical Functional Analysis and Optimization 14:333–353
[4] Ioffe AD (1984) Calculus of dini subdifferentials of functions and contingent coderivatives of set-valued maps. Nonlinear Analysis, Theory, Methods and Applications, 8:517–539
[5] Ioffe AD (1984) Approximate subdifferentials and applications. I: The finite dimensional theory. Transactions of the American Mathematical Society 281:389–416
[6] Ioffe, AD (1986) Approximate subdifferentials and applications II: Functions on locally convex spaces. Mathematika 33:111–128

[7] Ioffe AD (1989) Approximate subdifferentials and applications. 3: The metric theory. Mathematika 36:1–38
[8] Jourani A, Thibault L (1990) Approximate subdifferential and metric regularity: The finite-dimensional case. Mathematical Programming 47:203–218
[9] Jourani A, Thibault L, (1993) Approximations and metric regularity in mathematical programming in banach space. Mathematics of Operations Research 18:390–401
[10] Mordukhovich BS (1976) The maximum principle in the problem of time-optimal control with nonsmooth constraints. Journal of Applied Mathematics Mechanics 40:960–969