Journal of Convex Analysis
Volume 30 (2023), No. 3, 1025–1052

A Turnpike Property for Optimal Control Problems with Dynamic Probabilistic Constraints

Martin Gugat
Department of Data Science, Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany
martin.gugat@fau.de

Holger Heitsch, René Henrion
Weierstrass Institute, Berlin, Germany
holger.heitsch@wias-berlin.de, rene.henrion@wias-berlin.de

Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

Received: May 27, 2022
Accepted: January 15, 2023

We consider systems that are governed by linear time-discrete dynamics with an initial condition and a terminal condition for the expected values. We study optimal control problems where in the objective function a term of tracking type for the expected values and a control cost appear. In addition, the feasible states have to satisfy a conservative probabilistic constraint that requires that the probability that the trajectories remain in a given set F is greater than or equal to a given lower bound. An application are optimal control problems related to storage management systems with uncertain in- and output. We give sufficient conditions that imply that the optimal expected trajectories remain close to a certain state that can be characterized as the solution of an optimal control problem without prescribed initial- and terminal condition. In this way we contribute to the study of the turnpike phenomenon that is well-known in mathematical economics and make a step towards the extension of the turnpike theory to problems with probabilistic constraints.

Keywords: Probabilistic constraints, chance constraints, probabilistic robustness, here-and-now-decision, turnpike phenomenon, turnpike result, terminal constraint, probabilistic turnpike.

2010 Mathematics Subject Classification: 90C20, 90C31.

1. Introduction

The turnpike phenomenon for optimization problems has been discussed in mathematical economics by P. A. Samuelson already in [4]. Ever since, the turnpike phenomenon has been analyzed for optimal control problems of different types, see for example [3, 7, 10]. The turnpike phenomenon for infinite horizon optimal control is studied in [21]. Turnpike properties in the calculus of variations and optimal control are considered in [11, 12, 20]. For optimal control problems with partial differential equations see also [18] and the references therein.

In order to obtain decisions that are robust against uncertainties in the problem data, probabilistic constraints are a useful tool if information on the corresponding probability distribution is available (see [16]). Probabilistic constraints require that the probability to remain feasible is greater than or equal to a lower bound p that is

ISSN 0944-6532 / $ 2.50

prescribed as a problem parameter by the decision maker. They play a prominent role in risk averse water reservoir management under uncertain inflows (e.g.,[14, 16, 19]) but could equally well apply to gas reservoirs. Recently, probabilistic constraints (or Value-at-Risk constraints) have attracted increasing interest in optimal control or PDE constrained optimization (e.g., [5, 6, 8, 15].

Although the study of the turnpike phenomenon is an active area of current research, results on the turnpike property for optimization problems with probabilistic constraints are not yet available in the literature.

This paper investigates the turnpike property for discrete time optimal control problems with probabilistic constraints (chance constraints). For probabilistic constraints continuous in time (a special case of so-called *probust* constraints), we refer to [1, 9]. The underlying random distribution is supposed to be continuous. We consider a probabilistic constraint where it is required that the probability that the whole trajectory remains in a given convex set F is greater than or equal to a given parameter p.

It is the nature of these constraints that for a longer time horizon, they are harder to satisfy than for a short time horizon. Therefore in some cases if the probability threshold p is not adapted to the time horizon there is a maximal time horizon where the probabilistic constraint admits a nonempty feasible set. Hence also in our turnpike result for optimization problems with probabilistic constraints we consider a time dependent probability threshold p_T.

We present a turnpike result that states that the optimal expected trajectories approach a certain state (the *turnpike*, which is defined by the optimal trajectory of the problem with free initial and free terminal state) in the sense that there is an upper bound for the Euclidean distance between the trajectories of the expected values that is independent of the time horizon. Since probabilistic constraints are an excellent modeling tool for problems of optimal control and optimal design, also for this case, the turnpike structure of the generated trajectories is of interest.

This paper has the following structure. In Section 2 we introduce the time-discrete system, a quadratic objective function and define an optimization problem with a probabilistic constraint.

In Section 3 we show that the solutions of the relaxed problem without the probabilistic constraint have an exponential turnpike property. Moreover, we show a turnpike property for the problems where the probabilistic constraint is replaced by a probabilistic penalty term in the objective function. Finally we also discuss the problem with the probabilistic constraint.

In Section 4 numerical experiments are presented that illustrate the probabilistic turnpike phenomenon. At the end of the paper, some conclusions are discussed.

2. Optimal control of time-discrete systems

We consider a linear time-discrete system. The initial state $l_0 \in \mathbb{R}^n$ is given and for $t \in \{1, 2, 3, ...\}$ the evolution of the state $l_t \in \mathbb{R}^n$ is influenced by identically distributed random variables $\xi_t \in \mathbb{R}^n$ and governed by the linear recursion

$$l_t = A\, l_{t-1} + Bx_t + \xi_t \tag{1}$$

with linear operators A and B and control variables $x_t \in X = \mathbb{R}^n$.

Assume that
$$A^T = A, \tag{2}$$
that A is positive definite and that B is invertible. Expanding the recursion (1), the state vector l can be written as an affine linear mapping of control and random variables:
$$l(x, \xi) = Px + Q\xi + r. \tag{3}$$
As an example consider the linear recursion
$$l_t = l_{t-1} + x_t + \xi_t$$
for $t \in \{1, ..., T\}$ that models the water level in a reservoir for hydroelectricity generation. It can also be used as a model of gas storage. Gas storage is important for power generation in gas-fired power stations in the case of a lack of electricity that is generated from renewable energy. Also the storage of hydrogen can play an important role in a future hydrogen economy, see [2].

Let a closed convex set $F \subset \mathbb{R}^n$ and a desired state
$$l^{(\delta)} \in F \tag{4}$$
be given. We assume that for all $t \in \{1, 2, ..., T\}$ we have
$$\mathbb{E}\xi_t = E$$
and that
$$l^{(\delta)} = A\, l^{(\delta)} + Bx^{(\delta)} + E. \tag{5}$$
Let a weight $\gamma > 0$ be given. For $k \in \{1, ..., T\}$, we define the objective function J_T with a control cost and a tracking term that is stated in terms of expected values as
$$J_T(x) = \sum_{t=0}^{T} \|\mathbb{E}l_t - l^{(\delta)}\|^2 + \gamma \sum_{t=1}^{T} \|Bx_t - Bx^{(\delta)}\|^2. \tag{6}$$
Here, for $z \in \mathbb{R}^n$ we use the notation $\|z\| = \sqrt{\sum_{i=1}^n z_i^2}$. Define the probability
$$\varphi_T(x) = \mathbb{P}\left(l_t \in F \ \text{ for all } \ t \in \{1, ..., T\}\right)$$
in the sense that the initial state for $t = 0$ is l_0 and l_t is the corresponding random state generated with the control $x \in X^T$ by (1).

For a natural number T and $p_T \in (0, 1)$ we define the probabilistic constraint
$$\varphi_T(x) \geq p_T \tag{7}$$
and the optimization problem
$$\mathbf{P}(T,\, l_0): \qquad \min_{x \in X^T} J_T(x) \ \text{ subject to } \ \mathbb{E}l_T = l^{(\delta)} \ \text{ and } \ (7).$$
This is a problem where a here-and-now decision has to be taken based upon the information that is available at the time $t = 0$.

If the feasible set is nonempty, that is if p_T is sufficiently small, our assumptions imply that a solution of $\mathbf{P}(T,\, l_0)$ exists.

This can be seen as follows. Define the feasible set of $\mathbf{P}(T,\, l_0)$

$$\Upsilon_T = \{x \in X^T : \varphi_T(x) \geq p_T\}$$

that contains the feasible control vectors that generate the trajectories $(l_t)_{t=1}^T$ with the starting point l_0. Since the relation $l_t \in F$ for all $t \in \{1, \ldots, T\}$ can be represented as the inequality $h(x,\xi) \geq 0$ with $l_t(x,\xi)$ from (3) and with the continuous function

$$h(x,\xi) = -\max_{t \in \{1,\ldots,T\}} \operatorname{dist}\,(l_t(x,\xi), F), \tag{8}$$

it follows that φ_T is an upper semicontinuous function. Hence the feasible set Υ_T is closed.

Note that the objective function J_T is continuous. Assume that $x_S \in \Upsilon_T$ is a feasible control. Due to the growth of the objective function J_T the lower-level set

$$M_T = \{x \in X^T : J_T(x) \leq J_T(x_S)\}$$

is compact. Without changing the optimal control we can replace the feasible set Υ_T of $\mathbf{P}(T,\, l_0)$ by the set $\Upsilon_T \cap M_T$. Since this set is compact, the existence of an optimal control follows.

We finish this section with a statement on the log-concavity of the probability function φ_T. As observed above, one may use the function h in (8) for the representation $\varphi_T(x) = \mathbb{P}(h(x,\xi) \geq 0)$. Thanks to (3), the functions

$$\operatorname{dist}\,(l_t(x,\xi), F) = \operatorname{dist}\,(\cdot, F)(P_t x + Q_t \xi + r_t)$$

are convex as compositions of the convex (by convexity of F) distance function $\operatorname{dist}\,(\cdot, F)$ with an affine linear mapping. As a consequence, h is concave. Now, the following Lemma is a direct consequence of a classical result by Prékopa [16, Theorem 10.2.1]:

Lemma 2.1. *If ξ has a density f_ξ such that $\ln f_\xi$ is concave (e.g., Gaussian and many other prominent multivariate distributions), then $\ln \varphi_T$ is a concave function.*

3. Turnpike properties for the optimal controls and trajectories

In this section we consider decisions x that have to be taken before the ξ_t are observed, that is we are looking for a decision that is taken at the time $t = 0$ and yields a control that is optimal subject to uncertainty about the random perturbations ξ_t for all $t \in \{1, 2, ..., T\}$. This type of choice is often called a *here-and-now-decision.*

First we present an exponential turnpike property for the solution of $\mathbf{P}(T,\, l_0)$ for the case that the probabilistic constraint is not active. Our turnpike results in Theorem 3.1 states that for the problem where the probabilistic constraint is not active, in the optimal trajectories the distance between the expected state and the desired state decays exponentially fast with t. Next, we consider problems with a logarithmic penalty term for the probabilities and show that the optimal trajectories have a turnpike property in the sense that the optimal trajectories approach the optimal trajectories for the corresponding problem with free initial and terminal state. In Theorem 3.5 below we state this turnpike result.

Then we also discuss the problem with the probabilistic constraint. In order to show a turnpike result in this case we have to adapt the probability level p_T to the time-horizon.

In the sequel we assume that the feasible set of $\mathbf{P}(T,\, l_0)$ has non-empty interior. More precisely, we assume that there exists a control $\hat{z}^{(T)} \in X^T$ such that

$$\varphi_T(\hat{z}^{(T)}) > p_T \quad \text{and} \quad \mathbb{E}\hat{l}_T = l^{(\delta)}. \tag{9}$$

Here, $\hat{l}_T$ refers to the final state resulting from the control $\hat{z}^{(T)}$ and the dynamics (1). In the sequel we assume that the sequence $(p_T)_{T=1}^{\infty}$ is decreasing.

For $\lambda \in [0,\, 1]$ we introduce the problem with a probabilistic penalty term

$$\mathbf{Q}(T, l_0, \lambda): \qquad \min_{x \in X^T} \lambda\, J_T(x) - (1-\lambda) \ln\left(\varphi_T(x)\right) \quad \text{subject to} \quad \mathbb{E}l_T = l^{(\delta)}.$$

In problem $\mathbf{Q}(T, l_0, \lambda)$, the probabilistic constraint (7) is replaced by a penalty term in the objective function and the initial state l_0 is still prescribed.

The aim of our turnpike analysis is to provide insights on the behavior of the solutions of $\mathbf{P}(T,\, l_0)$ for different values of T, in particular for large time-horizons. Therefore it is important to keep in mind that each component of the optimal state l_t $(t \in \{1, ..., T\})$ for $\mathbf{P}(T,\, l_0)$ also depends on the time horizon T. This also holds for the optimal control x_t $(t \in \{1, ..., T\})$ for $\mathbf{P}(T,\, l_0)$. An emphasis on this dependence would require a notation like $l_t^{(T)}$, $x_t^{(T)}$ $(t \in \{1, ..., T\})$. However, since this would deteriorate the readibility of the paper, we have decided to use the more concise form l_t, where the dependence of T is not stated explicitly in each component.

3.1. An exponential turnpike result for the case that the probabilistic constraint is not active

We start with an exponential turnpike result for the case that the probabilistic constraint is not active. In this case, the optimal control solves a deterministic problem.

Theorem 3.1. *Assume that* (7) *is nonactive at a solution of* $\mathbf{P}(T, l_0)$*. Then, such solution is unique and has a discrete exponential turnpike structure in the sense that there exists a number* $z_\gamma \in (0,\, 1)$ *that is independent of* l_0 *and* T *such that for all* $t \in \{1, ..., T\}$ *we have the turnpike inequality*

$$\|\mathbb{E}l_t - l^{(\delta)}\|^2 \leq z_\gamma^t\, \|\mathbb{E}l_0 - l^{(\delta)}\|^2. \tag{10}$$

For all eigenvalues λ_k *of the matrix* A *define the polynomial*

$$p_k(\omega) = \omega^2 - \left[\frac{1}{\lambda_k}\left(1 + \frac{1}{\gamma}\right) + \lambda_k\right]\omega + 1. \tag{11}$$

Then we can choose $\quad z_\gamma = \max\limits_{k \in \{1,\ldots,n\}} \; \min\limits_{z \in \mathbb{C}: p_k(z)=0} |z|^2.$

For the optimal control $x \in X^T$ *of* $\mathbf{P}(T, l_0)$*, for all* $t \in \{1, ..., T\}$ *we have the turnpike inequality*

$$\|x_t - x^{(\delta)}\|^2 \leq \|B^{-1}\|^2\, (1 + \|A\|)^2\, z_\gamma^{t-1}\, \|\mathbb{E}l_0 - l^{(\delta)}\|^2 \tag{12}$$

where $\|B^{-1}\|$ *and* $\|A\|$ *denote the spectral matrix norms.*

Proof. For the proof we first observe that problem $\mathbf{Q}(T, l_0, 1)$ is identical to the relaxed problem

$$\mathbf{R}(T, l_0): \qquad \min_{x \in X^T} J_T(x) \text{ subject to } \mathbb{E} l_T = l^{(\delta)}$$

where the probabilistic constraint does not appear. Due to linearity, for the expected values, we have the recursion

$$\mathbb{E} l_t = A\, \mathbb{E} l_{t-1} + B\, x_t + E \quad (t \in \{1, \ldots, T\}). \tag{13}$$

Since the objective function J_T only depends on the expected values, this implies that in fact, we have a deterministic problem that we can solve. Equation (13) yields

$$B\, x_t = \mathbb{E} l_t - A\, \mathbb{E} l_{t-1} - E \quad (t \in \{1, \ldots, T\}).$$

This implies that we can write the objective function in terms of

$$\alpha_t := \mathbb{E} l_t - l^{(\delta)} \quad (t \in \{0, \ldots, T\}). \tag{14}$$

Then we have for $t \in \{1, \ldots, T\}$:

$$B x_t - B x^{(\delta)} = \mathbb{E}(l_t - l^{(\delta)}) - A\, (\mathbb{E} l_{t-1} - l^{(\delta)}) = \alpha_t - A\, \alpha_{t-1}. \tag{15}$$

Hence, the constrained problem $\mathbf{R}(T, l_0)$ is equivalent with the free minimization of the objective

$$\tilde{J}_T(\alpha) := \|\alpha_0\|^2 + \sum_{t=1}^{T} \left(\|\alpha_t\|^2 + \gamma \|\alpha_t - A\, \alpha_{t-1}\|^2 \right). \tag{16}$$

We note that for $\tilde{J}_T$ only $\alpha := (\alpha_1, \ldots, \alpha_{T-1})$ is variable, while $\alpha_0 = l_0 - l^{(\delta)}$ and $\alpha_T = 0$ (as a consequence of the terminal constraint in $\mathbf{R}(T, l_0)$) are constant. Recalling that $A = A^T$, differentiation yields for $t \in \{1, ..., T-1\}$

$$\begin{aligned} \nabla_{\alpha_t} \tilde{J}_T(\alpha) &= 2 \left[\alpha_t + \gamma \left(\alpha_t - A\, \alpha_{t-1} + A^2\, \alpha_t - A \alpha_{t+1} \right) \right] \\ &= 2 \left[-\gamma\, A\, \alpha_{t-1} + ((1+\gamma) I + \gamma A^2)\, \alpha_t - \gamma\, A \alpha_{t+1} \right]. \end{aligned}$$

Thus the necessary optimality condition implies the equation

$$A \alpha_{t+1} = \left(\left(1 + \frac{1}{\gamma}\right) I + A^2 \right) \alpha_t - A\, \alpha_{t-1}. \tag{17}$$

Note that due to convexity, (17) is also a sufficient condition for the optimality of a trajectory that minimizes (16).

Due to (2) there exists an orthonormal basis $v^{(1)}, ..., v^{(n)}$ of eigenvectors of the symmetric matrix A that correspond to the real eigenvalues λ_1,...,λ_n. Our aim is to express the optimal trajectories as a linear combination of the orthonormal basis vectors $v^{(k)}$ with $k \in \{1, ..., n\}$. In order to proceed, for $k \in \{1, ..., n\}$ define the polynomial

$$P_k(\omega) = \lambda_k\, \omega^2 - \left(1 + \frac{1}{\gamma} + \lambda_k^2\right) \omega + \lambda_k.$$

Let z_k denote a number such that $P_k(z_k) = 0$. For $t \in \{0, 1, 2, ...\}$ define the vector $\alpha_t^{(k)} = z_k^t\, v^{(k)} \in \mathbb{R}^n$.

Note that $P_k(z_k) = 0$ implies

$$\lambda_k\, z_k^{t+2} = \left(1 + \frac{1}{\gamma} + \lambda_k^2\right) z_k^{t+1} - \lambda_k\, z_k^t.$$

Hence we have $$\lambda_k\, \alpha_{t+2}^{(k)} = \left(1 + \frac{1}{\gamma} + \lambda_k^2\right) \alpha_{t+1}^{(k)} - \lambda_k\, \alpha_t^{(k)}.$$

Since for all $s \in \{0, 1, 2, ...\}$, the $\alpha_s^{(k)}$ are eigenvectors corresponding to the eigenvalue λ_k, this implies that the $\alpha_t^{(k)}$ satisfy (17). Since $\lambda_k \neq 0$, we can define the polynomial $p_k = \frac{1}{\lambda_k} P_k$ as in (11). With the roots of p_k we obtain an explicit representation of the optimal state. If one root is z_k, the other root is $\frac{1}{z_k}$. Note that since

$$\Delta = \left[\frac{1}{\lambda_k}\left(1 + \frac{1}{\gamma}\right) + \lambda_k\right]^2 - 4 > 0, \tag{18}$$

p_k has two different real roots. The initial state has the representation

$$l_0 = l^{(\delta)} + \sum_{k=1}^{n} \rho_k\, v^{(k)}$$

(where the coefficients ρ_1, ρ_2,... ρ_n are uniquely determined). We represent the optimal state as a linear combination of the $\alpha_t^{(k)}$ corresponding to the roots z_k and $\frac{1}{z_k}$. The initial condition and the terminal constraint $\mathbb{E} l_T = 0$ yield a system of $2n$ linear equations for the $2n$ coefficients. With suitable coefficients $(\hat{g}_k,\, \hat{h}_k)$ $(k \in \{1, ..., n\})$ for $t \in \{1, ..., T\}$ the optimal state is given by

$$\mathbb{E} l_t = l^{(\delta)} + \sum_{k=1}^{n} \rho_k \left(\hat{g}_k z_k^t v^{(k)} + \hat{h}_k z_k^{-t} v^{(k)}\right).$$

For $t = 0$ we obtain $l_0 - l^{(\delta)} = \sum_{k=1}^{n} \rho_k \left(\hat{g}_k + \hat{h}_k\right) v^{(k)}$. This yields $\hat{g}_k + \hat{h}_k = 1$ for all $k \in \{1, ..., n\}$. For $t = T$ we obtain the equation

$$\mathbb{E} l_T - l^{(\delta)} = 0 = \sum_{k=1}^{n} \rho_k \left(\hat{g}_k z_k^T v^{(k)} + \hat{h}_k z_k^{-T} v^{(k)}\right).$$

This yields $z_k^T\, \hat{g}_k + z_k^{-T}\, \hat{h}_k = 0$ for all $k \in \{1, ..., n\}$. Thus we obtain

$$\hat{g}_k = \frac{z_k^{-T}}{z_k^{-T} - z_k^T}, \quad \hat{h}_k = \frac{-z_k^T}{z_k^{-T} - z_k^T}$$

and $$\mathbb{E} l_t = l^{(\delta)} + \sum_{k=1}^{n} \rho_k\, \frac{z_k^{t-T} - z_k^{T-t}}{z_k^{-T} - z_k^T} v^{(k)}. \tag{19}$$

By our construction, this trajectory satisfies (17), hence it minimizes (16).

For the control that generates this trajectory we have $Bx_t = Bx^{(\delta)} + \alpha_t - A\alpha_{t-1}$. Since this control generates an optimal trajectory, this is an optimal control for $\mathbf{R}(T, l_0)$. Since the optimization problem $\mathbf{R}(T,\, l_0)$ has a strongly convex objective function and the constraints are linear, the solution is uniquely determined.

Now we show that for the problem without the probabilistic constraint, the expected values of the optimal state approach the desired state $l^{(\delta)}$ exponentially fast. In order to show this we introduce the notation

$$f_{k,t} = \frac{z_k^{t-T} - z_k^{T-t}}{z_k^{-T} - z_k^{T}}.$$

Then (19) implies
$$\|\alpha_t\|^2 = \sum_{k=1}^{n} (\rho_k)^2 \; |f_{k,t}|^2 \,. \tag{20}$$

Since we can assume without restriction that $|z_k| < 1$ we have the inequality

$$\left| \frac{1 - z_k^{2(T-t)}}{1 - z_k^{2T}} \right| \leq 1.$$

Hence the following inequality holds:

$$|f_{k,t}| = \left| \frac{z_k^t - z_k^{2T-t}}{1 - z_k^{2T}} \right| = |z_k|^t \; \left| \frac{1 - z_k^{2(T-t)}}{1 - z_k^{2T}} \right| \leq |z_k|^t.$$

Define $z_\gamma = \max_{k \in \{1,\ldots,n\}} |z_k|^2 < 1$. Then we have

$$\|\alpha_t\|^2 = \sum_{k=1}^{n} (\rho_k)^2 \, |f_{k,t}|^2 \leq \sum_{k=1}^{n} (\rho_k)^2 |z_k|^{2t} \leq \sum_{k=1}^{n} (\rho_k)^2 z_\gamma^t = z_\gamma^t \, \|\alpha_0\|^2. \tag{21}$$

Thus we obtain (10) for the relaxed problem $\mathbf{R}(T, l_0)$.

For the controls, (15) implies

$$x_t - x^{(\delta)} = B^{-1}\alpha_t - B^{-1} A \, \alpha_{t-1} \quad (t \in \{1, \ldots, T\}). \tag{22}$$

Hence (21) yields $\|x_t - x^{(\delta)}\| \leq \|B^{-1}\| \, z_\gamma^{t/2} \, \|\alpha_0\| + \|B^{-1}\| \, \|A\| \, z_\gamma^{(t-1)/2} \, \|\alpha_0\|$.

Hence (12) follows. This completes the proof. □

Note that the exponential decay implies that the optimal value $\nu(T, \, l_0)$ of the optimization problem $\mathbf{R}(T, \, l_0)$ is uniformly bounded with respect to T and $l_0 \in U$.

Define
$$\eta^* = \sup_{T \in \{1,2,3,\ldots\}, l_0 \in l^{(\delta)} + U} \nu(T, \, l_0) < \infty. \tag{23}$$

Remark 3.2. If the optimal state of the relaxed problem $\mathbf{R}(T, \, l_0)$ that is generated by the optimal control $x_T(l_0)$ satisfies the probabilistic constraint (7) (which is the case if $p_T \geq 0$ is sufficiently small), it is also the solution of $\mathbf{P}(T, \, l_0)$ and satisfies the exponential turnpike inequality (10).

In the next subsections, we investigate the role of the probabilistic constraint for the turnpike phenomenon. We start with the problem where the corresponding probability appears as a penalty term in the objective function.

3.2. Results with probabilistic penalty term

Now we present a turnpike result for the problem with the probabilistic constraint. Here the state that is approached in the interior of the time-interval (the 'turnpike') is defined as the solution of the corresponding problem with free terminal and free initial state, which obviously is independent of prescribed initial and terminal data.

First we state a result about the growth of $-\ln(\varphi_T(x))$.

Lemma 3.3. *We have*
$$\lim_{\|x\|\to\infty} -\ln(\varphi_T(x)) = \infty. \tag{24}$$

Proof. Since the set F is bounded, there exists a number $R_F \geq \|l_0\|$ such that $f \in F$ implies the inequality $\|f\| \leq R_F$. For all $s \in \{1, ..., T\}$ we have $\xi_s = l_s - Al_{s-1} - Bx_s$. This implies $\|\xi_s\| \geq \|Bx_s\| - \|l_s\| - \|A\|\,\|l_{s-1}\|$ where $\|A\|$ denotes the spectral norm of A. For all $s \in \{1, ..., T\}$ we have

$$\begin{aligned} \varphi_T(x) &= \mathbb{P}(l_t \in F \text{ for all } t \in \{1, ..., T\}) \\ &\leq \mathbb{P}(\|l_t\| \leq R_F \text{ for all } t \in \{1, ..., T\}) \\ &\leq \mathbb{P}(\|\xi_s\| \geq \|Bx_s\| - R_F\,(1 + \|A\|)). \end{aligned}$$

Let a sequence of controls $x^{(k)} \in X^T$ be given such that $\lim_{k\to\infty} \|x^{(k)}\| = \infty$. Then there exists an $s \in \{1, ..., T\}$ such that $\lim_{k\to\infty} \|Bx_s^{(k)}\| = \infty$.

For all $t \in \{1, ..., T\}$ we have
$$\lim_{k\to\infty} \mathbb{P}(\|\xi_t\| \geq k) = 0. \tag{25}$$

This yields $\lim\limits_{k\to\infty} \mathbb{P}(\|\xi_s\| \geq \|Bx_s^{(k)}\| - R_F\,(1+\|A\|)) = 0$ and assertion (24) follows. $\square$

Due to (25) there exists a number $k_{0,T} > 0$ such that for all $t \in \{1, ..., T\}$ we have the inequality $\mathbb{P}(\|\xi_t\| \geq k_{0,T}) < p_T$. Thus if for a control $x \in X^T$ and a natural number $s \in \{1, .., T\}$ we have

$$\|Bx_s\| \geq k_{0,T} + R_F\,(1 + \|A\|), \tag{26}$$

we also have $\varphi_T(x) < p_T$, and thus x is not feasible for $P(l_0, T)$.

By Lemma 3.3, for all $\lambda \in [0, 1]$ for the objective function of $\mathbf{Q}(T, l_0, \lambda)$ we have

$$\begin{aligned} &\lim_{\|x\|\to\infty} \inf_{\lambda\in[0,1]} \lambda\, J_T(x) - (1 - \lambda)\ln(\varphi_T(x)) \geq \\ &\qquad \lim_{\|x\|\to\infty} \min\{J_T(x),\, -\ln(\varphi_T(x))\} = \infty. \end{aligned} \tag{27}$$

Let $x_T(l_0)$ denote the optimal control for $\mathbf{Q}(T, l_0, 1)$ presented in Theorem 3.1 and define
$$C_{prob}(T) = -\ln \varphi_T(x_T(l_0)) \tag{28}$$
(where we set $C_{prob}(T) = \infty$ if $\varphi_T(x_T(l_0)) = 0$). We define the set

$$\aleph_T := \bigcup_{\lambda\in[0,\,1]} \aleph_T(\lambda),$$

where, for $\lambda \in [0, 1]$,

$$\aleph_T(\lambda) := \{x \in X^T : \lambda\, J_T(x) - (1-\lambda)\ln \varphi_T(x) \leq \lambda\, J_T(x_T(l_0)) + (1-\lambda)\, C_{prob}(T)\}.$$

Lemma 3.4. *Assume that $\varphi_T(x_T(l_0)) > 0$ and that ξ has a density f_ξ such that $\ln(f_\xi)$ is concave. Then, for each $\lambda \in [0,1]$, the sets $\aleph_T(\lambda)$ are nonempty, compact and convex. Moreover, the set $\aleph_T$ is nonempty and compact.*

Proof. For each $\lambda \in [0,1]$ the set $\aleph_T(\lambda)$ contains $x_T(l_0)$, hence is nonempty. Much more, $\aleph_T$ is nonempty. As a consequence of Lemma 2.1, the $\aleph_T(\lambda)$ are convex. They are also closed thanks to the upper semicontinuity of φ_T (see Section 2). The set $\aleph_T$ is bounded due to (27) and by our assumption that $\varphi_T(x_T(l_0)) > 0$. This implies that the sets $\aleph_T(\lambda)$ are bounded too, hence compact. It remains to verify the closedness of $\aleph_T$. To this aim, consider a sequence $\{x_n\} \subseteq \aleph_T$ with $x_n \to x^*$ for some x^*. Then, there exists some sequence $\{\lambda_n\} \subseteq [0,1]$ with $x_n \in \aleph_T(\lambda_n)$. Passing to a subsequence which we do not relabel, we may assume that $\lambda_n \to \lambda^* \in [0,1]$. Then, by upper semicontinuity of φ_T it follows that

$$\begin{aligned}\lambda\, J_T(x_T(l_0)) + (1-\lambda)\, C_{prob}(T) &\geq \liminf_n \, (\lambda_n\, J_T(x_n) - (1-\lambda_n) \ln \varphi_T(x_n)) \\ &= \lambda^* J_T(x^*) - (1-\lambda^*) \limsup_n \, \ln \varphi_T(x_n) \geq \lambda^* J_T(x^*) - (1-\lambda^*) \ln \varphi_T(x^*).\end{aligned}$$

Hence, $x^* \in \aleph_T(\lambda^*) \subseteq \aleph_T$, as was to be shown. □

In the next theorem we state that for a certain value of λ, problem $\mathbf{Q}(T,\, l_0,\, \lambda)$ is equivalent to $\mathbf{P}(T,\, l_0)$.

Theorem 3.5. *Let $T \in \mathbb{N}$ be arbitrarily given. Assume that $C_{prob}(T) < \infty$ for $C_{prob}(T)$ in (28). Let ξ have a density f_ξ such that $\ln(f_\xi)$ is concave (e.g., multivariate Gaussian). Then, for all $\lambda \in (0,\, 1]$, problem $\mathbf{Q}(T,\, l_0,\, \lambda)$ has a unique solution and there exists a number $\lambda^* \in (0,1]$ such that the solution of $\mathbf{Q}(T,\, l_0,\, \lambda^*)$ is equal to the solution of $\mathbf{P}(T,\, l_0)$.*

Proof. According to Lemma 2.1, our assumption on the density of ξ implies that $\ln \varphi_T$ is concave. Hence, for all $\lambda \in (0,\, 1]$, the objective function of problem $\mathbf{Q}(T,\, l_0,\, \lambda)$ is strongly convex. Since the optimal controls can be found in the nonempty, compact and convex set $\aleph_T(\lambda)$ (see Lemma 3.4), the existence of a unique solution of $\mathbf{Q}(T,\, l_0,\, \lambda)$ follows. By the concavity of $\ln \varphi_T$, problem $\mathbf{P}(T,\, l_0)$ is a convex optimization problem. Similar to the proof of Theorem 3.1 we can transform it to an optimization problem in terms of $\alpha := (\alpha_t)_{t=1}^{T-1}$ with $\alpha_t = \mathbb{E} l_t - l^{(\delta)}$ for $t = 0, \ldots, T$:

$$\text{minimize } \tilde{J}_T(\alpha) \quad \text{subject to} \quad -\ln \tilde{\varphi}_T(\alpha) \leq -\ln p_T. \tag{29}$$

Here, $\tilde{J}_T$ is defined in (16) and, using the linear transformation (15), $\tilde{\varphi}_T$ is defined as

$$\tilde{\varphi}_T(\alpha) := \varphi_T([B^{-1}(\alpha_t - A\alpha_{t-1} + Bx^{(\delta)})]_{t=1}^T) = \varphi_T(x), \tag{30}$$

where in (29) $\alpha_0 = l_0 - l^{(\delta)}$ and $\alpha_T = 0$ are constants in these problems. Observe that the concavity of $\ln \varphi_T$ implies that of $\ln \tilde{\varphi}_T$ by linearity of the inner mapping. Hence, (29) is a convex optimization problem too. Moreover, with $\hat{z}^{(T)}$ from (9), we may resolve (15) for α with $x := \hat{z}^{(T)}$ starting with $\alpha_0 := l_0 - l^{(\delta)}$ and ending – thanks to the endpoint condition in (9) – as required with

$$\begin{aligned}\alpha_T &= A\alpha_{T-1} + \mathbb{E}(l_t - l^{(\delta)}) - A\,(\mathbb{E} l_{t-1} - l^{(\delta)}) = A\alpha_{T-1} - A\,(\mathbb{E} l_{t-1} - l^{(\delta)}) \\ &= A\alpha_{T-1} - A\alpha_{T-1} = 0.\end{aligned}$$

Using the correspondence (30) between α and controls, this yields some $\hat{\alpha}^{(T)}$ with $\hat{\alpha}_0^{(T)} = l_0 - l^{(\delta)}$, $\hat{\alpha}_T^{(T)} = 0$ and $\tilde{\varphi}_T(\hat{\alpha}^{(T)}) = \varphi_T(\hat{z}^{(T)}) > p_T$. This means that $\hat{\alpha}_T$ is a Slater point for problem (29). Consequently, the necessary and sufficient conditions for a solution α of (29) amount to the existence of a multiplier $\mu \geq 0$ such that $\varphi_T(\alpha) \geq p_T$ and

$$0 \in \nabla \tilde{J}_T(\alpha) + \mu\, \partial \left(-\ln \left(\tilde{\varphi}_T(\alpha)\right)\right), \quad \mu(\tilde{\varphi}_T(\alpha) - p_T) = 0 \tag{31}$$

where ∂ denotes the subgradient of convex analysis. Note that the last equation in (31) represents the complementarity constraint associated with the inequality in (29).

In the following, denote by $x^{(\lambda)}$ the solution of $\mathbf{Q}(T,\, l_0,\ \lambda)$ (whose unique existence we have shown in the beginning of this proof). If $\varphi_T(x^{(1)}) \geq p_T$, then $x^{(1)}$ is a solution of $\mathbf{P}(T,\, l_0)$ as well and we may choose $\lambda^* = 1$ in the statement of the theorem. Therefore, we assume now that $\varphi_T(x^{(1)}) < p_T$. Assume for a moment, that there exists some $\lambda^* \in (0, 1)$ such that

$$\varphi_T(x^{(\lambda^*)}) = p_T. \tag{32}$$

Then, by definition, $x^{(\lambda^*)}$ solves $\mathbf{Q}(T,\, l_0,\ \lambda^*)$ and we show that it also solves $\mathbf{P}(T,\, l_0)$ as claimed in the Theorem. Indeed, like $\mathbf{P}(T,\, l_0)$ in (29), $\mathbf{Q}(T,\, l_0,\, \lambda^*)$ can be formulated as a (free) convex problem in terms of the variable α:

$$\text{minimize}\, \lambda^* \tilde{J}_T(\alpha) - (1 - \lambda^*) \ln \tilde{\varphi}_T(\alpha). \tag{33}$$

Denote by α^* the vector in correspondence with $x^{(\lambda^*)}$ via (15). Then, by (30) and (32),

$$\tilde{\varphi}_T(\alpha^*) = \varphi_T(x^{(\lambda^*)}) = p_T. \tag{34}$$

Moreover, since $x^{(\lambda^*)}$ is the solution of $\mathbf{Q}(T,\, l_0,\ \lambda^*)$, α^* is the solution of (33) which is equivalent with the condition

$$0 \in \partial\big(\lambda^* \tilde{J}_T(\alpha^*) + (1-\lambda^*)(-\ln \tilde{\varphi}_T(\alpha^*))\big) = \lambda^* \nabla \tilde{J}_T(\alpha^*) + (1-\lambda^*)\, \partial \left(-\ln \tilde{\varphi}_T(\alpha^*)\right). \tag{35}$$

Here, we have applied the sum rule for the convex subdifferential which is justified by, e.g., [17, Theorem 2.85] because $\tilde{J}_T$ is continuous and convex, $-\ln \tilde{\varphi}_T$ is convex and $-\ln \tilde{\varphi}_T(\alpha^*) < \infty$ as a consequence of (34) and our general assumption $p_T > 0$. Now, defining

$$\mu := (1 - \lambda^*)/\lambda^* > 0, \tag{36}$$

we get – thanks to $\lambda^* \in (0, 1)$ – that the inclusion inside (31) is satisfied for α^*. The same holds true for the equality (complementarity condition) as a consequence of (34). Hence, α^* satisfies the necessary and sufficient optimality conditions of problem (29) which entails that it is a solution of this problem. Translated to the original description in terms of the x-variables, this means that $x^{(\lambda^*)}$ is a solution of $\mathbf{P}(T,\, l_0)$ as was to be shown.

It remains to justify the existence of $\lambda^* \in (0, 1)$ with (32). Define

$$\lambda^* := \sup\{\lambda \in (0,\, 1] \mid \varphi_T(x^{(\lambda)}) \geq p_T\}.$$

We show first that $\lambda^* > 0$ which amounts to saying that there exists some $\lambda \in (0,\,1]$ with $\varphi_T(x^{(\lambda)}) \geq p_T$. Assume to the contrary that $\varphi_T(x^{(\lambda)}) < p_T$ for all $\lambda \in (0,\,1]$. Then, by optimality of $x^{(\lambda)}$ and by feasibility of $x^{(0)}$ for problem $\mathbf{Q}(T,\,l_0,\;\lambda)$, it follows that

$$\begin{aligned}\lambda J_T(x^{(\lambda)}) - (1-\lambda)\ln p_T &< \lambda J_T(x^{(\lambda)}) - (1-\lambda)\ln \varphi_T(x^{(\lambda)})\\ &\leq \lambda J_T(x^{(0)}) - (1-\lambda)\ln \varphi_T(x^{(0)})\end{aligned}$$

for all $\lambda \in (0,1]$. Since all $x^{(\lambda)}$ belong to the compact set $\aleph_T$ by Lemma 3.4 and since J_T is bounded on this set, we may pass to the limit $\lambda \downarrow 0$, and arrive at $\varphi_T(x^{(0)}) \leq p_T$. On the other hand, $x^{(0)}$ is the optimal solution of $\mathbf{Q}(T, l_0, 0)$ which amounts to maximizing φ_T under the endpoint constraint $\mathbb{E}l_T = l^{(\delta)}$. Hence, we obtain from (9) the contradiction $\varphi_T(x^{0)}) \geq \varphi_T(\hat{z}^{(T)}) > p_T$. Thus, $\lambda^* \in (0,1]$.

Next, we verify that $\varphi_T(x^{(\lambda^*)}) \geq p_T$. By definition of λ^*, there is a sequence $\lambda_k \uparrow \lambda^*$ with $\varphi_T(x^{(\lambda_k)}) \geq p_T$. Since the $x^{(\lambda_k)}$ belong to the compact set $\aleph_T$ (see Lemma 3.4), we may assume that $x^{(\lambda_k)} \to x^*$. Observe that, since all $x^{(\lambda_k)}$ as solutions of $\mathbf{Q}(T,\,l_0,\;\lambda_k)$ satisfy the endpoint condition $\mathbb{E}l_T = l^{(\delta)}$, the same holds true for x^*. Let x be arbitrary such that $\mathbb{E}l_T = l^{(\delta)}$. Then, since $x^{(\lambda_k)}$ is the solution of $\mathbf{Q}(T,\,l_0,\;\lambda_k)$ and the objective of that problem is lower semicontinuous, we obtain

$$\begin{aligned}\lambda^* J_T(x^*) - (1-\lambda^*)\varphi_T(x^*) &\leq \liminf_k \lambda_k J_T(x^{(T,\lambda_k)}) - (1-\lambda_k)\varphi_T(x^{(T,\lambda_k)})\\ \leq \liminf_k \lambda_k J_T(x) - (1-\lambda_k)\varphi_T(x) &= \lambda^* J_T(x) - (1-\lambda^*)\varphi_T(x).\end{aligned}$$

This means that x^* is the solution of $\mathbf{Q}(T,\,l_0,\;\lambda^*)$, i.e., $x^* = x^{(\lambda^*)}$. Now, the upper semicontinuity of φ_T yields the desired inequality

$$p_T \leq \limsup_k \varphi_T(x^{(\lambda_k)}) \leq \varphi_T(x^*) = \varphi_T(x^{(\lambda^*)}).$$

As a consequence, $\lambda^* < 1$ because $\varphi_T(x^{(1)}) < p_T$. Summarizing, we have that $\lambda^* \in (0,1)$ and $\varphi_T(x^{(\lambda^*)}) \geq p_T$.

In the last step we show that actually $\varphi_T(x^{(\lambda^*)}) = p_T$. For $k \in \mathbb{N}$ sufficiently large it holds that $\lambda^* + 1/k \leq 1$ and, hence, by definition of λ^*, for k large enough, $\varphi_T(x^{(\lambda^*+1/k)}) < p_T$. Then, by optimality of $x^{(\lambda^*+1/k)}$ and by feasibility of $x^{(\lambda^*)}$ for problem $\mathbf{Q}(T,\,l_0,\;\lambda^*+1/k)$, it follows that

$$\begin{aligned}&(\lambda^*+1/k)\,J_T(x^{(\lambda^*+1/k)}) - (1-\lambda^*-1/k)\ln p_T\\ &< (\lambda^*+1/k)\,J_T(x^{(\lambda^*+1/k)}) - (1-\lambda^*-1/k)\ln \varphi_T(x^{(\lambda^*+1/k)})\\ &\leq (\lambda^*+1/k)\,J_T(x^{(\lambda^*)}) - (1-\lambda^*-1/k)\ln \varphi_T(x^{(\lambda^*)})\end{aligned}$$

for all k sufficiently large. Repeating an argument, already used before in this proof, we may assume that $x^{(\lambda^*+1/k_l)} \to_l x^{(\lambda^*)}$ for a subsequence. Invoking now the continuity of J_T, we end up, after passing to the limits above, at $\varphi_T(x^{(\lambda^*)}) \leq p_T$ which finally yields the desired relation $\varphi_T(x^{(\lambda^*)}) = p_T$. □

3.3. A turnpike result for the case that the probabilistic constraint is active

In the sequel we denote by λ_T the multiplier λ^* from Theorem 3.5 associated with an arbitrary $T \in \mathbb{N}$. Accordingly we define the following sequence of problems with free initial state and free terminal state

$$\hat{\mathbf{Q}}(T): \qquad \min_{(\hat{l}_0,\, x)\in\mathbb{R}^n\times X^T} \lambda_T\, J_T(x) - (1-\lambda_T)\ln\left(\varphi_T(x)\right) \quad (T \in \mathbb{N}), \tag{37}$$

where now, in contrast to the previous problems, $\hat{l}_0$ is a variable initial state. In the following, we denote by $\hat{l}_t$ $(t = 0, \ldots, T)$ the random states generated by the optimal solution of $\hat{Q}(T)$. Next we state a probabilistic turnpike result:

Lemma 3.6. *Let the assumptions of Theorem* 3.5 *be valid for all* $T \in \mathbb{N}$. *Assume that there exists some constant* R *such that for all* $T \in \mathbb{N}$

$$\|\mathbb{E}\hat{l}_t\| \le R \quad \forall t \in \{0, \ldots, T\}. \tag{38}$$

Moreover, suppose that $\kappa > 0$ *for*

$$\kappa := \inf_{T\in\mathbb{N}} \mathbb{P}\left(\hat{l}_T - \mathbb{E}\hat{l}_T + l^{(\delta)} \in F \mid \hat{l}_t \in F \ \forall t \in \{1, ..., T-1\}\right). \tag{39}$$

Then, there exists $C_1 > 0$ *such that the random states* $(l_t)_{t=1}^T$ *generated by the optimal control of* $\mathbf{P}(T,\, l_0)$ *satisfy the estimate*

$$\sum_{t=0}^{T} \|\mathbb{E}l_t - \mathbb{E}\hat{l}_t\|^2 \le \frac{C_1}{\lambda_T} \quad \forall T \in \mathbb{N}. \tag{40}$$

Proof. Fix an arbitrary $T \in \mathbb{N}$. For $Z = (z_0, ..., z_T)^\top \in X^{T+1}$ define the function

$$H_1(Z) := \|z_0\|^2 + \sum_{t=1}^{T}\left(\|z_t\|^2 + \gamma\|z_t - A\, z_{t-1}\|^2\right). \tag{41}$$

Then H_1 is strongly convex in the sense that for all $s \in [0,1]$ and all $Z,\, Y \in X^{T+1}$ we have the inequality

$$H_1((1-s)Z + sY) \le (1-s)H_1(Z) + sH_1(Y) - s\,(1-s)\|Z - Y\|^2. \tag{42}$$

This can be seen as follows. For $H_2(Z) := \|z_0\|^2 + \sum_{t=1}^T \|z_t\|^2$ we have

$$H_2((1-s)Z + sY) = (1-s)H_2(Z) + sH_2(Y) - s\,(1-s)\|Z - Y\|^2.$$

Since H_1 is the sum of H_2 and a convex function, (42) follows. Define

$$H(\alpha) := \lambda_T\, H_1(\alpha) - (1-\lambda_T)\ln\left(\tilde{\varphi}_T(\alpha)\right) \quad (\alpha \in X^{T+1}),$$

where $\tilde{\varphi}$ is as in (30), but now with α_0, α_T being variables. Note that H is the objective function of $\hat{\mathbf{Q}}(T)$ when similarly as in the proof of Theorem 3.1, problem $\hat{\mathbf{Q}}(T)$ is restated as an optimization problem in terms of α as defined in (14). Due to (42) our assumptions imply that H is a strongly convex function in the sense that for all $s \in [0,1]$ and all $Z,\, Y \in X^{T+1}$ we have the inequality

$$H((1-s)Z + sY) \le (1-s)H(Z) + sH(Y) - \lambda_T\, s\,(1-s)\|Z - Y\|^2,$$

where we exploited (42), the concavity of $\ln\tilde{\varphi}$ according to Theorem 3.5 (see remark below (30)) and $\lambda_T \le 1$ by the same Theorem.

For all $s \in (0,\, 1]$ this is equivalent to the inequality

$$H(Y) \geq H(Z) + \frac{H((1-s)Z + sY) - H(Z)}{s} + \lambda_T\,(1-s)\|Z - Y\|^2.$$

If Z is a point such that $H(\tilde{Y}) \geq H(Z)$ for all $\tilde{Y} \in X^{T+1}$, this yields

$$H(Y) \geq H(Z) + \lambda_T \sup_{s\in(0,1]} (1-s)\|Z-Y\|^2 = H(Z) + \lambda_T\,\|Z - Y\|^2.$$

Following our previous reformulation of problems in the new variable α, we may restate $\hat{\mathbf{Q}}(T)$ as

$$\min_{\alpha \in X^{T+1}} H(\alpha)$$

and $\mathbf{Q}(T,\, l_0,\; \lambda_T)$ as the corresponding problem with fixed $\alpha_0 = l_0 - l^{(\delta)}$ and $\alpha_T = 0$ (see remarks below (30)). Let $\hat{v}(T)$ denote the optimal value of $\hat{\mathbf{Q}}(T)$ and $v(T,\, l_0,\; \lambda_T)$ the optimal value of $\mathbf{Q}(T,\, l_0,\; \lambda_T)$. Since problems $\mathbf{Q}(T,\, l_0,\; \lambda_T)$ and $\mathbf{P}(T,\, l_0)$ are equivalent by Theorem 3.5, we have that

$$v(T,\, l_0,\; \lambda_T) \geq \hat{v}(T) + \lambda_T \sum_{t=0}^{T} \|\alpha_t^* - \hat{\alpha}_t^*\|^2 \tag{43}$$

where $\alpha_t^* := \mathbb{E} l_t - l^{(\delta)}$ and $\hat{\alpha}_t^* := \mathbb{E}\hat{l}_t - l^{(\delta)}$ and $\mathbb{E}l_t$, $\mathbb{E}\hat{l}_t$ are the expected states generated by the optimal solutions of $\mathbf{Q}(T,\, l_0,\; \lambda_T)$ and $\hat{\mathbf{Q}}(T)$, respectively.

Since the matrices A, B are regular by our basic assumptions, there exists a control $\tilde{q} \in X^T$ that generates for the deterministic dynamics

$$\eta_t = A\eta_{t-1} + B\tilde{q}_t \tag{44}$$

the deterministic trajectory $(\eta_0, \eta_1, ..., \eta_T) = (l_0 - \hat{l}_0, 0, \ldots, 0, l^{(\delta)} - \mathbb{E}\hat{l}_T)$.
To be precise, we have $\tilde{q} = (-B^{-1}[A(l_0 - \hat{l}_0)], 0, \ldots, 0, B^{-1}[l^{(\delta)} - \mathbb{E}\hat{l}_T])$.

Starting with $l_0^u := l_0$ the control $u := \hat{x} + \tilde{q}$ with the (uncertain) dynamics

$$l_t^u := A\, l_{t-1}^u + Bu_t + \xi_t \quad (t = 1, \ldots, T)$$

generates the trajectory $(l_t^u)_{t=0}^T = (\hat{l}_t + \eta_t)_{t=0}^T$ which is equal to

$$l_0,\, \hat{l}_1, \ldots, \hat{l}_{T-1}, \hat{l}_T + l^{(\delta)} - \mathbb{E}\hat{l}_T. \tag{45}$$

Since $\mathbb{E}(\hat{l}_T + l^{(\delta)} - \mathbb{E}\hat{l}_T) = l^{(\delta)}$, the control u is feasible for $\mathbf{Q}(T,\, l_0, \lambda_T)$.

Due to the definition of the objective function of $\mathbf{Q}(T,\, l_0, \lambda_T)$ and $\hat{\mathbf{Q}}(T)$, our construction implies the inequality

$$v(T,\, l_0, \lambda_T) - \hat{v}(T) \leq \lambda_T\,(J_T(u) - J_T(\hat{x})) - (1-\lambda_T)\,(\ln\varphi_T(u) - \ln\varphi_T(\hat{x}))\,. \tag{46}$$

First we derive an upper bound for the deterministic part $J_T(u) - J_T(\hat{x})$. Given (16) and with $\alpha_t^u := \mathbb{E}l_t^u - l^{(\delta)}$ for $t = 0, \ldots, T$, we get that

$$\begin{aligned} J_T(u) - J_T(\hat{x}) &= \|\alpha_0^u\|^2 - \|\hat{\alpha}_0^*\|^2 + \textstyle\sum_{t=1}^T \|\alpha_t^u\|^2 - \|\hat{\alpha}_t^*\|^2 \\ &+\gamma\left(\|\alpha_t^u - A\alpha_{t-1}^u\|^2 - \|\hat{\alpha}_t^* - A\hat{\alpha}_{t-1}^*\|^2\right) \leq \|\alpha_0^u\|^2 - \|\hat{\alpha}_0^*\|^2 \\ &+\gamma\left(\|\alpha_1^u - A\alpha_0^u\|^2 - \|\hat{\alpha}_1^* - A\hat{\alpha}_0^*\|^2 + \|A\alpha_{T-1}^u\|^2 - \|\hat{\alpha}_T^* - A\hat{\alpha}_{T-1}^*\|^2\right), \end{aligned} \tag{47}$$

because, thanks to (45) one has that $\alpha_t^u = \hat{\alpha}_t^*$ for $t = 1, \dots, T-1$ and because of $\alpha_T^u = \mathbb{E}(\hat{l}_T + l^{(\delta)} - \mathbb{E}\hat{l}_T) - l^{(\delta)} = 0$. For the probabilistic part of the objective function we proceed in a similar way. By definition of φ_T, we may write in terms of conditional probabilities

$$\varphi_T(u) = \mathbb{P}(l_t^u \in F\,(t = 1, \dots, T-1)) \cdot \mathbb{P}(l_T^u \in F \mid l_t^u \in F\,(t = 1, \dots, T-1))$$
$$\varphi_T(\hat{x}) = \mathbb{P}(\hat{l}_t \in F\,(t = 1, \dots, T-1)) \cdot \mathbb{P}(\hat{l}_T \in F \mid \hat{l}_t \in F\,(t = 1, \dots, T-1)).$$

By (45), the first factors coincide. Since also the log of a probability is negative, we may conclude that

$$\ln\varphi_T(u) - \ln\varphi_T(\hat{x}) \geq \ln \mathbb{P}(\hat{l}_T + l^{(\delta)} - \mathbb{E}\hat{l}_T \in F \mid \hat{l}_t \in F\,(t = 1, \dots, T-1)).$$

Thus, we may continue (46) by using (47) as

$$\begin{gathered} v(T,\, l_0, \lambda_T) - \hat{v}(T) \leq \lambda_T \left(\|\alpha_0^u\|^2 - \|\hat{\alpha}_0^*\|^2\right) \\ +\lambda_T\gamma \left(\|\alpha_1^u - A\alpha_0^u\|^2 - \|\hat{\alpha}_1^* - A\hat{\alpha}_0^*\|^2 + \|A\alpha_{T-1}^u\|^2 - \|\hat{\alpha}_T^* - A\hat{\alpha}_{T-1}^*\|^2\right) \\ -(1-\lambda_T) \ln \mathbb{P}(\hat{l}_T + l^{(\delta)} - \mathbb{E}\hat{l}_T \in F \mid \hat{l}_t \in F\,(t = 1, \dots, T-1)) \\ \leq \lambda_T \left(\|\alpha_0^u\|^2 + \gamma\|\alpha_1^u - A\alpha_0^u\|^2 + \gamma\|A\alpha_{T-1}^u\|^2\right) - (1-\lambda_T)\,\ln\kappa, \end{gathered}$$

where we we exploited that $\kappa > 0$ by assumption. Observing that

$$\begin{aligned} \|\alpha_0^u\| &\leq \|l_0\| + \|l^{(\delta)}\| \\ \|\alpha_1^u - A\alpha_0^u\| &\leq R + \|l^{(\delta)}\| + \|A\|(\|l_0\| + \|l^{(\delta)}\|) \\ \|A\alpha_{T-1}^u\| &\leq \|A\|(R + \|l^{(\delta)}\|), \end{aligned}$$

we arrive at $v(T,\, l_0, \lambda_T) - \hat{v}(T) \leq C_1$, where C_1 is independent of T. With (43) the above inequality implies

$$\sum_{t=0}^{T} \|\alpha_t^* - \hat{\alpha}_t^*\|^2 \leq \frac{1}{\lambda_T}\left[v(T, l_0, \lambda_T) - \hat{v}(T)\right] \leq \frac{C_1}{\lambda_T}. \qquad \square$$

In Lemma 3.6, the quotient $\frac{C_1}{\lambda_T}$ on the right-hand side of (40) becomes arbitrarily large if $\lambda_T \in (0, 1]$ converges to zero. In the following we derive a strictly positive lower bound for the corresponding values of λ_T. To show that there is a strictly positive uniform lower bound for the multipliers for problem $\mathbf{P}(T,\, l_0)$ with respect to T, we have to introduce a normalization with respect to T in the probabilistic constraint, that is, we adapt the probability level to the time-horizon. For this purpose for a given parameter $\zeta \in (0, 1)$ and $T \in \mathbb{N}$ we define

$$p_T = \zeta^T\, p_{\max}(T) \tag{48}$$

where $p_{\max}(T)$ is the optimal value of the probability maximizing problem $\mathbf{Q}(T,\, l_0,\, 0)$. Since $p_{\max}(T)$ is decreasing with T, also p_T is decreasing as a function of T.

Lemma 3.7. *Assume that the probability levels p_T in problems $\mathbf{P}(T,\, l_0)$ are given by (48). Suppose, moreover, that there exists some $\tilde{R}$ such that for all $T \in \mathbb{N}$.*

$$\|\mathbb{E}l_t\| \leq \tilde{R} \quad \forall t \in \{0, \dots, T\}, \tag{49}$$

Then, there is some $C_2 > 0$ such that $\lambda_T \geq C_2$ for all $T \in \mathbb{N}$.

Proof. The probabilistic constraint (7) with the time-dependent probability level is

$$\varphi_T(x) \geq p_T.$$

Under the assumptions of Theorem 3.5, define the convex function

$$g_T(x) = \ln(p_T) - \ln(\varphi_T(x)).$$

Then the probabilistic constraint (7) is equivalent to the convex constraint

$$g_T(x) \leq 0. \tag{50}$$

Let $x_S(T)$ be a solution of $\mathbf{Q}(T,\, l_0,\; 0)$ (i.e. a control that yields the maximum probability $\varphi_T(\cdot)$). Then for all $T \in \{2, 3, 4, ...\}$ we have the Slater condition (9) for problem $\mathbf{P}(T,\, l_0)$:

$$\varphi_T(x_S(T)) = p_{\max}(T) > \zeta^T\, p_{\max}(T) = p_T. \tag{51}$$

Define the affine subspace

$$\tilde{X}_T = \{x \in X^T \mid \mathbb{E} l_T = l^{(\delta)} \text{ under } l_0 = 0 \text{ and the dynamics (1)}\}.$$

Then, we can write the dual problem for $\mathbf{P}(T,\, l_0)$ as

$$\mathbf{D}(T,\, l_0): \qquad \max_{\mu \geq 0} \inf_{x \in \tilde{X}^T} L_T(x,\, \mu)$$

with the Lagrangian $L_T(x,\, \mu) = J_T(x) + \mu\, g_T(x)$. Let $\mu_T(l_0)$ denote the multiplier that corresponds to the optimal control $x_T(l_0)$ of problem $\mathbf{P}(T,\, l_0)$. Let $\beta(T, l_0)$ denote the optimal value of $\mathbf{P}(T, l_0)$. Due to the Slater condition (51) we have strong duality, which means that the optimal value of $\mathbf{P}(T, l_0)$ is equal to the optimal value of $\mathbf{D}(T,\, l_0)$, that is

$$\beta(T,\, l_0) = \inf_{x \in \tilde{X}^T} L_T(x,\, \mu_T(l_0)).$$

This yields the inequality $\beta(T,\, l_0) \leq L_T(x_S(T),\, \mu_T(l_0))$, which implies in turn

$$\mu_T(l_0) \leq \frac{\beta(T,\, l_0) - J_T(x_S(T))}{g_T(x_S(T))} = \frac{J_T(x_S(T)) - \beta(T,\, l_0)}{|g_T(x_S(T))|} = \frac{J_T(x_S(T)) - \beta(T,\, l_0)}{T|\ln(\zeta)|}. \tag{52}$$

Due to the recursion (13) for the expected values, we have

$$(x_S(T))_t = B^{-1}\left(\mathbb{E} l_t - A\mathbb{E} l_{t-1} - E\right).$$

This implies with (49)

$$\sup_{T \in \mathbb{N}} \max_{t \in \{1,\ldots,T\}} \|(x_S^{(T)})_t\| < \infty. \tag{53}$$

Note that by (6) J_T attains only values greater than or equal to zero. This yields $\beta(T,\, l_0) \geq 0$. Hence we obtain

$$\sup_{T \in \mathbb{N}} \frac{J_T(x_S(T)) - \beta(T,\, l_0)}{T} \leq \sup_{T \in \mathbb{N}} \frac{J_T(x_S(T))}{T}.$$

The objective function J_T is defined as the sum of $T+1$ terms each of which can be bounded by some common constant $\hat{M}$ thanks to (49). Hence (53) yields

$$\sup_{T\in\mathbb{N}} \frac{J_T(x_S(T))}{T} \le \sup_{T\in\mathbb{N}} \frac{(T+1)\,\hat{M}}{T} \le 2\,\hat{M} < \infty.$$

Due to (52) we have
$$\sup_{T\in\mathbb{N}} \mu_T(l_0) \le \frac{2\,\hat{M}}{|\ln(\zeta)|} < \infty.$$

Due to the relation (36), we have that $\mu_T = (1-\lambda_T)/\lambda_T$, whence

$$\lambda_T \ge \frac{|\ln(\zeta)|}{|\ln(\zeta)| + 2\,\hat{M}} =: C_2 \quad \forall T\in\mathbb{N}.$$ □

Now, we are in a position to formulate our main result on the probabilistic turnpike property of the expected states for the optimal control of problem $\mathbf{P}(T,\, l_0)$:

Theorem 3.8. *Under the assumptions of Lemma* 3.6 *and Lemma* 3.7*, the expected states* $(\mathbb{E}l_t)_{t=1}^{T}$ *generated by the optimal controls of the sequence of problems* $\mathbf{P}(T,\, l_0)$ *for* $T\in\mathbb{N}$ *have a turnpike structure near the expected states* $(\mathbb{E}\hat{l}_t)_{t=1}^{T}$ *generated by the optimal solutions of the sequence of problems* $\hat{Q}(T)$ *in the sense that there exists a constant* C *such that*

$$\sum_{t=0}^{T} \|\mathbb{E}l_t - \mathbb{E}\hat{l}_t\|^2 \le C \quad \forall T\in\mathbb{N}. \tag{54}$$

For the optimal control $x\in X^T$ *of* $\mathbf{P}(T,l_0)$ *and* $\hat{x}$ *of* $\hat{Q}(T)$ *we have the turnpike inequality*

$$\sum_{t=1}^{T} \|x_t - \hat{x}_t\|^2 \le 2\,C\,\|B^{-1}\|^2\,\left(1+\|A\|^2\right) \quad \forall T\in\mathbb{N}, \tag{55}$$

where $\|B^{-1}\|$ *and* $\|A\|$ *denote the spectral norms. Also here the control values* x_t *and* $\hat{x}_t$ *(*$t\in\{1,...,T\}$*) depend on the problem parameter* T *of* $\mathbf{P}(T,l_0)$*.*

Proof. Combine Lemmas 3.6 and 3.7 and put $C := C_1/C_2$. For the controls, (15) implies

$$x_t - x^{(\delta)} = B^{-1}\alpha_t^* - B^{-1}A\,\alpha_{t-1}^*,\;\; \hat{x}_t - x^{(\delta)} = B^{-1}\hat{\alpha}_t^* - B^{-1}A\,\hat{\alpha}_{t-1}^* \quad (t\in\{1,\dots,T\}).$$

Hence we have

$$x_t - \hat{x}_t = B^{-1}(\alpha_t^* - \hat{\alpha}_t^*) - B^{-1}A(\alpha_{t-1}^* - \hat{\alpha}_{t-1}^*).$$

Hence due to the definition of α_t^* and $\hat{\alpha}_t^*$ (54) yields

$$\sum_{t=1}^{T} \|x_t-\hat{x}_t\|^2 \le \sum_{t=1}^{T}\Big(\|B^{-1}\|\,\|\mathbb{E}l_t - \mathbb{E}\hat{l}_t\| + \|B^{-1}\|\,\|A\|\,\|\mathbb{E}l_{t-1} - \mathbb{E}\hat{l}_{t-1}\|\Big)^2$$

$$\le 2\|B^{-1}\|^2\,\left(1+\|A\|^2\right)\sum_{t=0}^{T}\|\mathbb{E}l_t - \mathbb{E}\hat{l}_t\|^2 \le 2\|B^{-1}\|^2\,\left(1+\|A\|^2\right)C \quad \forall T\in\mathbb{N}.$$

Then (55) follows. □

3.4. Discussion of assumptions

The assumptions of Theorem 3.8 (via those of Lemmas 3.6 and 3.7) concern, on the one hand, the uniform boundedness of expected states generated by the optimal solutions of the original problem and the problem with free initial and terminal state, and on the other hand the sequence of conditional probabilities in (39). While the former ones are intuitively clear, the latter one is purely technical. Conversely, the former ones are hard to ensure by reasonable conditions on the initial data whereas we will show in the following how (39) can be guaranteed in a standard setting. We note that the uniform boundedness of the respective optimal expected states might be verified empirically as in the numerical results of Fig. 4.5, where both trajectories stay within the desired region no matter how large the time horizon has been chosen.

Now, we address the verification of (39). We start with a technical preparation. Fix an arbitrary $T \in \mathbb{N}$. We consider the time-discrete dynamic system of the random states $\hat{l}_0$, $\hat{l} = (\hat{l}_1, \ldots, \hat{l}_T)^\top$ that are generated by the optimal solution $\hat{x}$ of the free terminal state problem $\hat{\mathbf{Q}}(T)$ in (35). Together with the recursion in (1) we obtain

$$\hat{l}_t = A\hat{l}_{t-1} + B\hat{x}_t + \xi_t, \qquad t = 1, \ldots, T.$$

Let be $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_T)^\top$ and $\xi = (\xi_1, \ldots, \xi_T)^\top$. Defininе the lower triangular block matrix $\Delta \in \mathbb{R}^{nT\times nT}$ and the matrices $\bar{A} \in \mathbb{R}^{nT\times n}$, $\bar{B} \in \mathbb{R}^{nT\times nT}$ such that

$$\Delta := \begin{pmatrix} I & & & 0 \\ A & I & & \\ \vdots & & \ddots & \\ A^{T-1} & \cdots & A & I \end{pmatrix}, \qquad \bar{A} := \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^T \end{pmatrix}, \qquad \bar{B} := \begin{pmatrix} B & & 0 \\ & \ddots & \\ 0 & & B \end{pmatrix}.$$

Then the evolution of the states can be represented in closed form by

$$\hat{l} = \Delta\xi + \bar{A}\hat{l}_0 + \Delta\bar{B}\hat{x}. \tag{56}$$

Assuming that $\xi_t \sim \mathcal{N}(E, \Sigma)$ for $t = 1, \ldots, T$ are independently and identically distributed Gaussian random variables, and by defining

$$\bar{E} := \begin{pmatrix} E \\ \vdots \\ E \end{pmatrix} \quad \text{and} \quad \bar{\Sigma} := \begin{pmatrix} \Sigma & & 0 \\ & \ddots & \\ 0 & & \Sigma \end{pmatrix},$$

we obtain that

$$\xi \sim \mathcal{N}(\bar{E}, \bar{\Sigma}) \quad \text{and} \quad \hat{l} \sim \mathcal{N}(\Delta\bar{E} + \bar{A}\hat{l}_0 + \Delta\bar{B}\hat{x}, \Delta\bar{\Sigma}\Delta^\top).$$

Now, we are able to represent the joint density function for $\hat{l}$. Setting

$$\bar{\mu} := \Delta\bar{E} + \bar{A}\hat{l}_0 + \Delta\bar{B}\hat{x},$$

the density function reads

$$f_{\hat{l}}(z) = \frac{1}{\sqrt{(2\pi)^{nT}\det(\Sigma)^T}} \mathrm{e}^{-\frac{1}{2}(z-\bar{\mu})^\top(\Delta\bar{\Sigma}\Delta^\top)^{-1}(z-\bar{\mu})}. \tag{57}$$

Both, Δ and $\bar{\Sigma}$ involve special structures that allow for a simple representation of their inverses. It is easy to see that the inverse of Δ is of the form

$$\Delta^{-1} = \begin{pmatrix} I & & & 0 \\ -A & I & & \\ & \ddots & \ddots & \\ 0 & & -A & I \end{pmatrix}.$$

We assume that Σ has the Cholesky factorization $\Sigma = LL^\top$. Now let $\bar{L} \in \mathbb{R}^{nT\times nT}$ be defined as

$$\bar{L} := \begin{pmatrix} L & & 0 \\ & \ddots & \\ 0 & & L \end{pmatrix} \quad \text{with} \quad \bar{L}^{-1} = \begin{pmatrix} L^{-1} & & 0 \\ & \ddots & \\ 0 & & L^{-1} \end{pmatrix}.$$

Therefore, due to $\bar{\Sigma}^{-1} = (\bar{L}\bar{L}^\top)^{-1} = \bar{L}^{-\top}\bar{L}^{-1}$, we obtain for any $z \in \mathbb{R}^{nT}$ that

$$z^\top(\Delta\bar{\Sigma}\Delta^\top)^{-1}z \;=\; z^\top\Delta^{-\top}\bar{\Sigma}^{-1}\Delta^{-1}z \;=\; (\bar{L}^{-1}\Delta^{-1}z)^\top\bar{L}^{-1}\Delta^{-1}z. \tag{58}$$

The matrix $\bar{L}^{-1}\Delta^{-1}$ is of the form

$$M_T := \bar{L}^{-1}\Delta^{-1} = \begin{pmatrix} L^{-1} & & & 0 \\ -L^{-1}A & L^{-1} & & \\ & \ddots & \ddots & \\ 0 & & -L^{-1}A & L^{-1} \end{pmatrix}. \tag{59}$$

Applying (58) and (59), (57) reads

$$f_{\hat{l}}(z) = \frac{1}{\sqrt{(2\pi)^{nT}\det(\Sigma)^T}}\,\mathrm{e}^{-\frac{1}{2}(M_T(z-\bar{\mu}))^\top M_T(z-\bar{\mu})}. \tag{60}$$

Proposition 3.9. *Let $(\hat{l}_0, \hat{x})$ be the solution of the free initial state problem $\hat{\mathbf{Q}}(T)$, where $T \in \mathbb{N}$ is arbitrarily fixed, and denote by $\hat{l} = (\hat{l}_1, \ldots, \hat{l}_T)^\top$ the random states generated by this solution. Under the assumption that $\xi_t \sim \mathcal{N}(E, \Sigma)$, $t = 1, 2, \ldots$ are i.i.d. Gaussian random variables and under assumption* (38) *of Lemma* 3.6*, it holds that*

$$\mathbb{P}\left(\hat{l}_T - \mathbb{E}\hat{l}_T + l^{(\delta)} \in F \,\middle|\, \hat{l}_t \in F \;\forall t \in \{1, \ldots, T-1\}\right) \geq C,$$

where $C > 0$ is a constant independent of T.

Proof. To determine the formulated conditional probability for a given $T \in \mathbb{N}$ we want to apply the joint density function (60) of $\hat{l}$. To do this we consider the random variable

$$\eta := (\hat{l}_1, \ldots, \hat{l}_{T-1}, \hat{l}_T - \mathbb{E}\hat{l}_T)^\top$$

that is obtained from $\hat{l}$ by shifting the Tth component by the constant $\mathbb{E}\hat{l}_T$ to zero mean. Note that the covariance matrix does not change by this shift. In the following argumentation, for any vector $z = (z_1, \dots, z_T)^\top \in \mathbb{R}^{nT}$ we will denote by $z' = (z_1, \dots, z_{T-1})^\top \in \mathbb{R}^{n(T-1)}$ its first $T-1$ entries (each of dimension n). Having

$$M_T = \begin{pmatrix} & & & & 0 \\ & & M_{T-1} & & \vdots \\ & & & & 0 \\ 0 & \cdots & 0 & -L^{-1}A & L^{-1} \end{pmatrix},$$

the joint density function of η can be derived from the density of $\hat{l}$ given in (60). Since $\mathbb{E}\eta = (\bar{\mu}_1, \dots, \bar{\mu}_{T-1}, 0)$, we obtain

$$f_\eta(z) = \frac{1}{\sqrt{(2\pi)^{nT}\det(\Sigma)^T}}\, \mathrm{e}^{-\frac{1}{2}(\|M_{T-1}(z'-\bar{\mu}')\|^2 + \|L^{-1}z_T - L^{-1}A(z_{T-1}-\bar{\mu}_{T-1})\|^2)}, \tag{61}$$

where $\|\cdot\|$ denotes the respective Euclidean norm. Moreover, the density function of the reduced vector $\eta' = (\hat{l}_1, \dots, \hat{l}_{T-1})$ is obtained by

$$f_{\eta'}(z') = \frac{1}{\sqrt{(2\pi)^{n(T-1)}\det(\Sigma)^{(T-1)}}}\, \mathrm{e}^{-\frac{1}{2}\|M_{T-1}(z'-\bar{\mu}')\|^2}. \tag{62}$$

For the wanted conditional probability we have now (with $F^{T-1} := \underbrace{F \times \cdots \times F}_{T-1 \text{ times}}$)

$$\begin{aligned} p_T^{\mathrm{cond}} &:= \mathbb{P}\left(\hat{l}_T - \mathbb{E}\hat{l}_T + l^{(\delta)} \in F \,\middle|\, \hat{l}_t \in F \;\forall t \in \{1, \dots, T-1\}\right) \\ &= \mathbb{P}\left(\eta_T \in F - l^{(\delta)} \,\middle|\, \eta' \in F^{T-1}\right). \end{aligned}$$

The latter conditional expression can be represented in terms of the above densities (61) and (62). More precisely, with

$$\frac{\sqrt{(2\pi)^{n(T-1)}\det(\Sigma)^{(T-1)}}}{\sqrt{(2\pi)^{nT}\det(\Sigma)^T}} = \frac{1}{\sqrt{((2\pi)^n\det(\Sigma))}} =: \theta$$

it turns out that

$$p_T^{\mathrm{cond}} = \theta\, \frac{\int_{x\in F^{T-1}} \int_{y \in F - l^{(\delta)}} \mathrm{e}^{-\frac{1}{2}(\|M_{T-1}(x-\bar{\mu}')\|^2 + \|L^{-1}y - L^{-1}A(x_{T-1}-\bar{\mu}_{T-1})\|^2)} dy dx}{\int_{x\in F^{T-1}} \mathrm{e}^{-\frac{1}{2}(\|M_{T-1}(x-\bar{\mu}')\|^2} dx}. \tag{63}$$

Since F and therefore $F - l^{(\delta)}$ are compact and because $\bar{\mu}_{T-1} = \mathbb{E}\hat{l}_{T-1}$ and therefore $\bar{\mu}_{T-1} \le R$ due to assumption (38) (independently of T), for any $(z, y) \in F \times (F - l^{(\delta)})$ we can uniformly estimate

$$\begin{aligned} \|L^{-1}y - L^{-1}A(z - \bar{\mu}_{T-1})\| &\le \|L^{-1}\|\|y\| + \|L^{-1}A\|(\|z\| + \|\bar{\mu}_{T-1}\|) \\ &\le \hat{C}\|L^{-1}\| + \hat{C}\|L^{-1}A\| + R\|L^{-1}A\|. \end{aligned} \tag{64}$$

Here, $\hat{C}$ is a constant such that the unit ball $B_{\hat{C}}(0)$ contains both F and $F - l^{(\delta)}$. Applying inequality (64) to (63) we can bound the conditional probability p_T^{cond}

uniformly from below. Let λ denote the Lebesgue measure. Then we observe from (63) that

$$\begin{aligned} p_T^{\text{cond}} &\geq \theta \frac{\lambda(F - l^{(\delta)})\mathrm{e}^{-\frac{1}{2}(\hat{C}\|L^{-1}\|+\hat{C}\|L^{-1}A\|+R\|L^{-1}A\|)^2} \int_{x\in F^{T-1}} \mathrm{e}^{-\frac{1}{2}(\|M_{T-1}(x-\bar{\mu}')\|^2} dx}{\int_{x\in F^{T-1}} \mathrm{e}^{-\frac{1}{2}(\|M_{T-1}(x-\bar{\mu}')\|^2} dx} \\ &= \theta\lambda(F - l^{(\delta)})\mathrm{e}^{-\frac{1}{2}(\hat{C}\|L^{-1}\|+\hat{C}\|L^{-1}A\|+R\|L^{-1}A\|)^2} =: C > 0. \end{aligned}$$

Since the constants $\hat{C}$, θ and R from Lemma 3.6 do not depend on T, so C does not either, which completes the proof. □

In the numerical experiments presented in the next section the turnpike property stated in Theorem 3.8 is clearly visible and in particular for sufficiently large time horizons close to the middle of the time inteval the two expected trajectories almost coincide.

4. Numerical experiments for the probabilistic turnpike

In this section we present numerical experiments for the probabilistic turnpike for here-and-now decisions. For studying the turnpike phenomenon in a probabilistic setup, as an instance for a time-discrete system (1), we consider the linear recursion

$$l_t = l_{t-1} + x_t + \xi_t \tag{65}$$

for $t \in \{1, \ldots, T\}$. Here, equation (65) models the state level in a reservoir problem, for example the water level for hydroelectricity generation. For any time step $t \in \{1, \ldots, T\}$ the scalar state variable $l_t \in \mathbb{R}$ denotes the water level in the reservoir, the control variable $x_t \in \mathbb{R}$ is the amount of water to be filled or released at t, and $\xi_t \in \mathbb{R}$ is some random water inflow to the reservoir. We assume that the inflows ξ_t describe a sequence of identically distributed Gaussian random numbers with $\mathbb{E}\xi_t = E$ for $t = 1, \ldots, T$.

Instead of computing policies for optimal water releases for power generation, in our numerical tests we are rather interested in turning a given water level l_0 back to a desired level $l^{(\delta)} \in F := [a, b]$ in a cost optimal way. According to (5) we have

$$x^{(\delta)} = -E$$

and define the objective function of the optimal control problem by

$$J_T(x) = \sum_{t=0}^{T} (\mathbb{E}(l_t) - l^{(\delta)})^2 + \gamma \sum_{t=1}^{T} (x_t - x^{(\delta)})^2,$$

where γ is some non-negative weighting factor concerning the control cost. Introducing the probabilistic constraint

$$\varphi_T(x, l_0) = \mathbb{P}(l_t \in [a, b] \text{ for all } t \in \{1, \ldots, T\}),$$

finally, the optimization problem $\mathbf{P}(T, l_0)$ introduced in Section 2 reads

$$\min_{x\in\mathbb{R}^T} J_T(x) \quad \text{subject to} \quad \mathbb{E}l_T = l^{(\delta)} \text{ and } \varphi_T(x, l_0) \geq p \tag{66}$$

for a given and fixed probability level $p \in [0, 1]$.

4.1. Turnpike study for a short time horizon

In a first test series we want to study numerical examples, where we consider a short fixed time horizon T, varying initial water levels l_0 compared to different desired levels $l^{(\delta)}$ and a given fixed confidence interval $[a, b]$. In particular, we solve (66) numerically with the following data:

Confidence interval:	$[a, b]$	$=$	$[11, 25]$
Initial level:	l_0	$=$	5, 13
Desired level:	$l^{(\delta)}$	$=$	16, 20
Time horizon:	T	$=$	10
Control cost factor:	γ	$=$	5
Distribution of inflow t:	ξ_t	$\sim$	$\mathcal{N}(E, 1);\ E = -1$

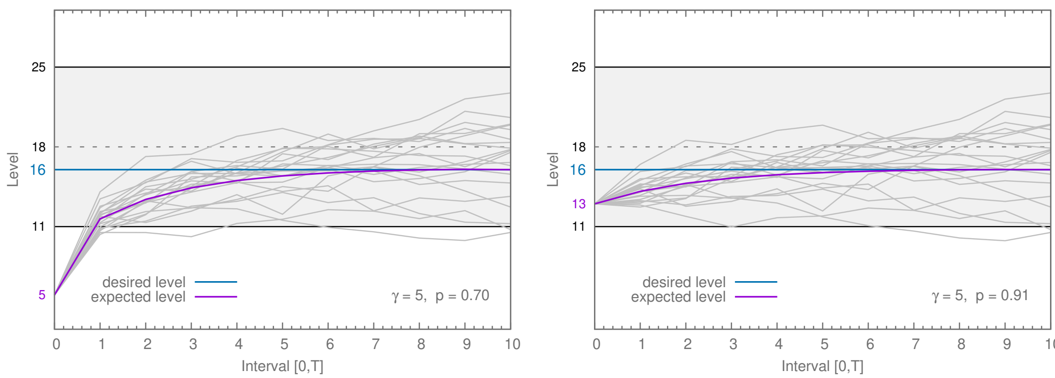

Figure 4.1: Solution of the turnpike problem for two examples with different initial level $l_0 = 5$ (left) and $l_0 = 13$. The level trajectories are computed for a fixed desired level $l^{(\delta)} = 16$ as well as probability levels $p = 0.70$ (left) and $p = 0.91$.

Computed solutions of the optimal control problem (66) for the first two numerical examples are shown in Fig. 4.1. The expected level trajectories of the reservoir for the given data are displayed for two different situations, where the initial level is located outside and inside the confidence interval, respectively. Beside the expected level, the figure also shows realizations of the level curves realizing the computed optimal control for randomly selected inflow scenarios (light gray) for the given time horizon. The expected level (shown by purple lines) are observed for probability levels $p = 0.70$ and $p = 0.91$, respectively. Clearly, in both example, by the optimal solution the system is controlled towards the desired level. However, if the initial level as in the first example is located outside the confidence interval, we observe a jump of the expected level into the confidence interval in the first time step in order to satisfy the probabilistic constraint. Afterwards, similar to the second example, the system is smoothly turned to the desired level, which is a consequence to the chosen parameter $\gamma > 0$. By this setting, due to the control cost within the objective function, abrupt rises of the reservoir levels will be avoided.

Next, we want to study the behavior of the reservoir levels when increasing the probability level inside the probabilistic constraint. With the same setup as in the two examples before we just change the probabilities to $p = 0.91$ and $p = 0.93$, respectively.

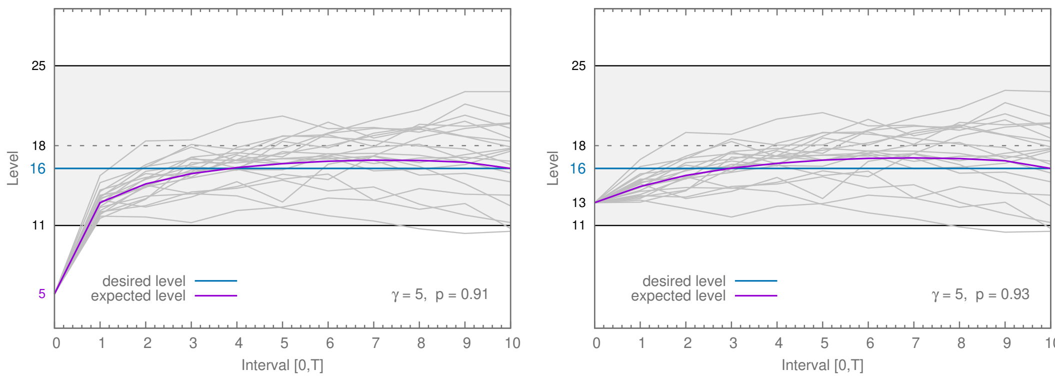

Figure 4.2: Solution of the turnpike problem for two examples with different initial level $l_0 = 5$ (left) and $l_0 = 13$, but, with increased probability levels $p = 0.91$ (left) and $p = 0.93$. The level trajectories are computed for the fixed desired level $l^{(\delta)} = 16$.

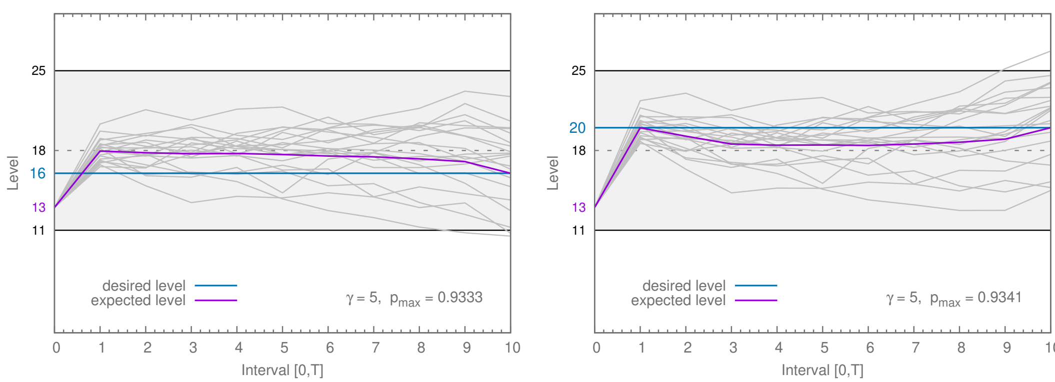

Figure 4.3: Solution of the turnpike problem for $p_{\max}$. Displayed are level trajectories for different desired levels $l^{(\delta)} = 16$ (left) and $l^{(\delta)} = 20$.

The results are shown in Fig. 4.2. The new observation is the following: When increasing the probability the expected reservoir level will be forced to leave the desired state in order to increase the probability that the state curves remain within the confidential bounds. As consequence, within intermediate time steps the expected state of the system exceeds the desired level and turns toward the center line of the confidence interval. However, as required by the constraints, at the end of the time horizon in both examples the expected value of the reservoir level turns back and reaches the desired level again.

As typical for optimization problems with probabilistic constraints there exists a maximum probability level $p_{\max}$ such that the feasibility set becomes empty for higher probability p, i.e. for $p_{\max} < p \leq 1$. The previous results are obtained for probability levels below the maximum probability. Now, we want to look at the turnpike behavior when reaching $p_{\max}$. If $p = p_{\max}$, the reservoir problem solution approaches the level state that maximizes the probability p in one step. This is shown in Fig. 4.3 for two instances, where we compare two different desired levels. In both cases it turns out that the expected reservoir levels almost ignore the desired level, because they are forced towards the center of confidence in order to match the maximum probability. They both turn to the desired level only to the end of the time horizon that is due to the terminal condition.

4.2. The turnpike property for increasing time horizons

Finally, we want to illustrate the probabilistic turnpike property for a increasing time horizon. The turnpike result from Section 3 describes the turnpike behavior of the system state with increasing time horizon T. In order to show this specific probabilistic turnpike phenomenon by the numerical example we want to setup the time horizon sequentially by $T = 40 / 60 / 80 / 100$. In addition, we adjust the standard deviation of the random vectors ξ_t and we want to allow correlations between different time steps.

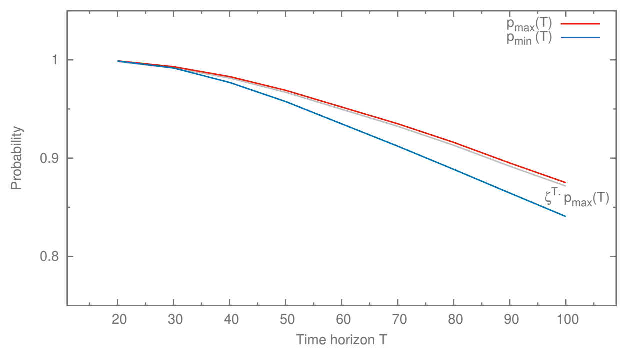

Figure 4.4: Bounding maximal and minimal probability curves $p_{\max}(\cdot)$ and $p_{\min}(\cdot)$ as function of the time horizon such that the level problem (66) is feasible and such that the probabilistic constraint is active. The graphic also shows a suitable time dependent choice of probabilities $p(T)$, where $p(T) = \zeta^T \cdot p_{\max}(T)$ with constant $\zeta = 0.99996$.

In particular, we assume an inflow process $\xi = (\xi_1, \dots, \xi_T)$ that follows a multivariate Gaussian distribution with tridiagonal covariance matrix of the form

$$\xi \sim \mathcal{N}(-I_T, \Sigma_T) \quad \text{and} \quad \Sigma_T = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & & 0 \\ \sigma_2 & \ddots & \ddots & \ddots & \\ \sigma_3 & \ddots & \ddots & \ddots & \sigma_3 \\ & \ddots & \ddots & \ddots & \sigma_2 \\ 0 & & \sigma_3 & \sigma_2 & \sigma_1 \end{bmatrix}$$

with $\sigma_1 = 0.05$, $\sigma_2 = 0.03$, $\sigma_3 = 0.015$, and where I_T denotes the T-dimensional identity matrix. All other fixed problem data of the level problem (66) follow the general setup above with initial level $l_0 = 13$ and desired level $l^{(\delta)} = 20$.

When increasing the number of time steps, the probability that the system remains within some given bounds drops down. This is due to the increasing variance of perturbations caused by the random inflow process. More precisely, the probability $p_{\max}(T)$ as function of the time horizon is strictly monotonic decreasing. On the other hand, for small enough probability levels the probabilistic constraint becomes inactive.

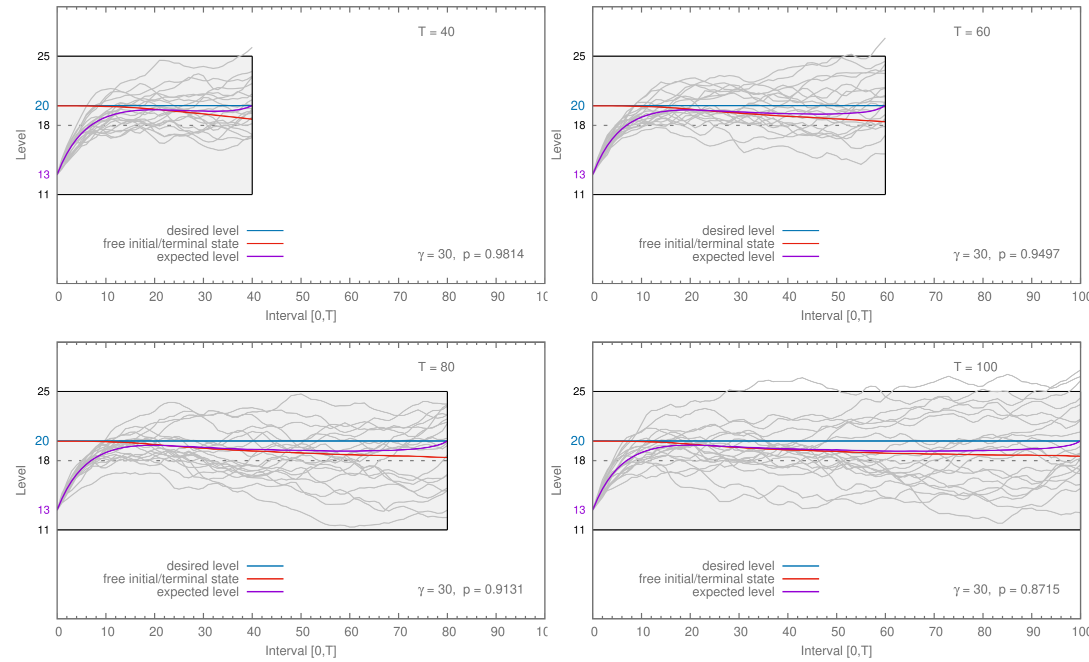

Figure 4.5: Probabilistic turnpike property for increasing time horizons $T = 40$, $60, 80, 100$. Shown here are the expected state trajectories compared to the corresponding free initial and terminal state curves.

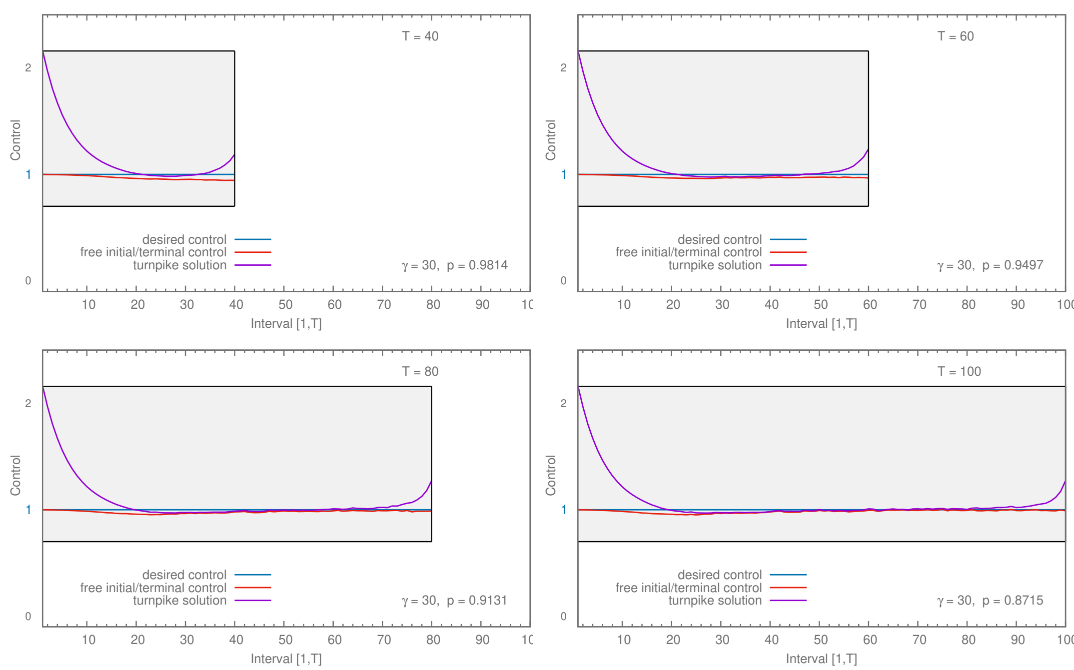

Figure 4.6: Probabilistic turnpike property disclosed by the optimal controls. Shown here are the computed optimal controls of the turnpike problems compared to the computed corresponding free initial and terminal state controls.

In consequence, the probability level p in (66) should be chosen between the upper bound $p_{\max}(T)$ and some lower bound $p_{\min}(T)$. Otherwise, the problem (66) is getting infeasible or it turns to a pure deterministic turnpike problem. Fig. 4.4 shows the bounding maximal and minimal probability curves $p_{\max}(\cdot)$ and $p_{\min}(\cdot)$ as function of the time horizon for the numerical example. In order to get a suitable probability levels for (66) we setup the probability p as function of $p_{\max}$. In the following we apply p according to the definition

$$p(T) := \zeta^T \cdot p_{\max}(T)\,,$$

where $\zeta \in (0,1)$ is some constant number that is chosen sufficient close to 1. In the numerical example ζ is assigned to $\zeta = 0.99996$ (cf. Fig. 4.4). The related probabilistic turnpike property for an increasing time horizon is studied in Fig. 4.5 and Fig. 4.6. The numerical results are shown for time horizons $T = 40\,/\,60\,/\,80\,/\,100$ for both the expected states (Fig. 4.5) and the optimal controls (Fig. 4.6). On the one hand we compare the expected state of the system, observed when applying the optimal control as solution of (66), with the expected state according to the optimal solution of the free initial state and free terminal state problem $\hat{\mathbf{Q}}(T)$ defined in (37). On the other hand we show the turnpike phenomenon as stated in Theorem 3.8 for the optimal controls themselves.

The pictures in Fig. 4.5 reveal that at the beginning of the time horizon the expected system state (when applying the problem with bounding conditions) turns from the defined initial state smoothly towards the expected state of the problem with free initial and free terminal state. Before reaching the end of the time horizon the expected level leaves the free initial/terminal state solution in order to match the deterministic terminal state condition, i.e. the expected system state of the bounded problem terminates with the desired level. The effect becomes more evident with a prolongation of the time horizon. The longer the time horizon the more intermediate time steps can be observed, where the computed expected level according to (66) is close to the expected level of the corresponding free initial/terminal solution. According to Fig. 4.6 a similar observation can be made for the optimal controls computed for the example problems on the different time scales. The numerical results confirm empirically the turnpike properties (54) and (55) stated in Theorem 3.8.

5. Conclusion

Motivated by the application of probabilistic constraints in dynamic optimal planning problems for the operation of gas networks, we have studied the turnpike property for time-discrete systems with an additive random perturbation. We have considered optimal control problems where the quadratic objective functional is stated in terms of expected values and a probabilistic constraint is prescribed. We have shown that under suitable assumptions we obtain a turnpike structure for the expected optimal state also for problems with probabilistic constraints. We have shown that for large time horizons the optimal expected trajectories approach the optimal expected trajectories of the problem with free initial and free terminal states in the majority of time steps.

In our analysis we consider problem $\mathbf{Q}(T,\, l_0,\, \lambda)$ where the probability that a given inequality constraint is satisfied appears in a penalty term in the objective functional with a certain weight that is parameterized by $\lambda \in [0, 1]$. We show that there exists a parameter λ for which the solution of the problem $\mathbf{P}(T,\, l_0)$ with a probabilistic constraint and given terminal state and prescribed expected terminal state coincides with the solution of problem $\mathbf{Q}(T,\, l_0,\, \lambda)$. This yields a turnpike result of a new type, where the turnpike trajectory is obtained as the solution of an auxiliary problem that depends on the parameter λ and thus indirectly (via the value of the parameter λ) also on the given initial state. It is important to emphasize that for the problem with the probabilistic contraint, this specific parameter is not known a priori and is not indpendent of the initial state. Since the parameter λ varies in a bounded set, this new type of turnpike result yields a family of limit trajectories that is parameterized by λ, whereas in the classical turnpike results only a single turnpike trajectory appears.

There are some open questions left, in particular about the verification of our assumptions in terms of the problem data, in particular the underlying probability distributions. We have considered a special finite-dimensional setting with affine linear dynamics. In the applications, in contrast to our setting the dynamics are often nonlinear, in fact often given by partial differential equations. In this infinite-dimensional setting specific probabilistic box-constraints are required for the feasible states. How the results can be generalized to this setting is a topic for future research. Such an analysis could be based upon the recent paper [13].

Acknowledgements. This work was supported by DFG in the framework of the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, Projects C03, C05 (author M.G.) and B04 (author H.H.). Author R.H. acknowledges support by the FMJH Program Gaspard Monge in optimization and operations research including support to this program by EDF.

References

[1] D. Adelhütte, D. Aßmann, T. Gonzlez Grandon, M. Gugat, H. Heitsch, F. Liers, R. Henrion, S. Nitsche, R. Schultz, M. Stingl, D. Wintergerst: *Joint model of probabilistic/robust (probust) constraints with application to gas network optimization*, Vietnam J. Math. 49 (2021) 1097–1130.

[2] J. Andersson, S. Grönkvist: *Large-scale storage of hydrogen*, Int. J. Hydrogen Energy 44/2 (2019) 11901–11919.

[3] T. Damm, L. Grüne, M. Stieler, K. Worthmann: *An exponential turnpike theorem for dissipative discrete time optimal control problems*, SIAM J. Control Optim. 52/3 (2014) 1935–1957.

[4] R. Dorfman, P. A. Samuelson, R. M. Solow: *Linear Programming and Economic Analysis*, McGraw-Hill, New York (1958).

[5] M. H. Farshbaf-Shaker, M. Gugat, H. Heitsch, R. Henrion: *Optimal Neumann boundary control of a vibrating string with uncertain initial data and probabilistic terminal constraints*, SIAM J. Control Optim. 58/4 (2020) 2288–2311.

[6] M. H. Farshbaf-Shaker, R. Henrion, D. Hömberg: *Properties of chance constraints in infinite dimensions with an application to PDE constrained optimization*, Set-Valued Var. Analysis 26 (2018) 821–841.

[7] T. Faulwasser, M. Korda, C. N. Jones, D. Bonvin: *On turnpike and dissipativity properties of continuous-time optimal control problems*, Automatica 81 (2017) 297–304.

[8] A. Geletu, A. Hoffmann, P. Schmidt, P. Li: *Chance constrained optimization of elliptic PDE systems*, ESAIM, Control Optim. Calc. Var. 26 (2020), 70 p.

[9] T. González Grandón, H. Heitsch, R. Henrion: *A joint model of probabilistic/robust constraints for gas transport management in stationary networks*, Comput. Manage. Sci. 14/3 (2017) 443–460.

[10] L. Grüne, R. Guglielmi: *Turnpike properties and strict dissipativity for discrete time linear quadratic optimal control problems*, SIAM J. Control Optim. 56/2 (2018) 1282–1302.

[11] L. Grüne, M. Schaller, A. Schiela: *Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations*, J. Diff. Equations 268/12 (2020) 7311–7341.

[12] L. Grüne, M. Schaller, A. Schiela: *Abstract nonlinear sensitivity and turnpike analysis and an application to semilinear parabolic PDEs*, ESAIM, Control Optim. Calc. Var. 27 (2021), art. no. 56, 28 p.

[13] M. Gugat: *On the turnpike property with interior decay for optimal control problems*, Math. Control Signals Systems 33 (2021) 237–258.

[14] H. A. Loiaciga: *On the use of chance constraints in reservoir design and operation modeling*, Water Resources Manage. 24 (1988) 1969–1975.

[15] J. Martínez-Frutos, F. Periago Esparza: *Optimal Control of PDEs under Uncertainty*, Springer, Berlin (2018).

[16] A. Prékopa: *Stochastic Programming*, Kluwer Academic, Dordrecht (1995).

[17] A. Ruszczyński: *Nonlinear Optimization*, Princeton University Press, Princeton (2006).

[18] E. Trélat, C. Zhang, E. Zuazua: *Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces*, SIAM J. Control Optim. 56/2 (2018) 1222–1252.

[19] W. van Ackooij, R. Zorgati, R. Henrion, A. Möller: *Joint chance constrained programming for hydro reservoir management*, Optim. Engineering 15 (2014) 509–531.

[20] A. J. Zaslavski: *Turnpike Properties in the Calculus of Variations and Optimal Control*, Nonconvex Optimization and its Applications 80, Springer, New York (2006).

[21] A. J. Zaslavski: *Turnpike Phenomenon and Infinite Horizon Optimal Control*, Springer Optimization and its Applications 99, Springer, Cham (2014).