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Topological Properties of the Approximate Subdifferential*

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The approximate subdifferential introduced by Mordukhovich has attracted much attention in recent works on nonsmooth optimization. Potential advantages over other concepts of subdifferentiability might be related to its nonconvexity. This is motivation to study some topological properties more in detail. As the main result, it is shown that any weakly compact subset of any Hilbert space may be obtained as the Kuratowski–Painlevé limit of approximate subdifferentials from a one-parametric family of Lipschitzian functions. Sharper characterizations are possible for strongly compact subsets. As a consequence, in any Hilbert space the approximate subdifferential of a suitable Lipschitzian function may be homeomorphic (both in the strong and weak topology) to the Cantor set. Further results relate the approximate subdifferential to specific topological types, to the one-dimensional case (which is extraordinary in some sense), and to the value function of a $\mathscr{C}^1$-optimization problem. © 1997 Academic Press

1. INTRODUCTION

The approximate subdifferential was first introduced in finite dimensions by Mordukhovich [21] via normal cones to epigraphs. An equivalent definition based on Dini subdifferentials was found by Ioffe [10]. The same authors gave several approaches for generalizing this concept to the infinite-dimensional case [19, 20, 11, 12]. Being nonconvex in general, the approximate subdifferential enjoys a minimality property among a family of ‘‘reasonable’’ subdifferentials (compare [10], Theorem 9). On the other hand, it offers a rich calculus. We merely point out the chain rules by Ioffe [12] or Jourani and Thibault [15] or the expression for the subdifferential

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of marginal functions provided by Mordukhovich [22], which all work without the need of convexification. As a consequence, many promising applications in nonsmooth optimization have been reported. For instance, Glover, Craven, and Flåm [5] and Glover and Craven [6] considered first order optimality conditions in Banach spaces. More general optimality conditions were established before by El Abdouni and Thibault [4]. Other papers are concerned with the metric regularity of feasible sets [14, 16, 17], [23]. It is interesting to note that, in finite dimensions, Mordukhovich [23] could give a complete characterization of the metric regularity of multivalued mappings, whereas in the Banach space setting Jourani and Thibault [16, 17] developed an approach via so-called compactly Lipschitzian mappings, thereby exploiting the chain rule mentioned above. Potential advantages of the approximate subdifferential might be related to the fact that it is not restricted to be a convex set. This motivates the question of which topological properties can be expected in general. It will turn out that, even for Lipschitzian functions, there is a rich variety of topological types that can occur.

Starting with the basic definitions, denote by X, X^* a Banach space with its dual and let $f\colon X \to \mathbb{R} \cup \{-\infty, \infty\}$ be an extended-valued function. First, recall the Dini subdifferential [9] of f at some point $x \in X$:

$$\partial^- f(x) = \begin{cases} \{x^* \in X^* | x^*(h) \leq d^- f(x; h) \ \forall h \in X\}, & |f(x)| < \infty, \\ \emptyset, & \text{else}, \end{cases}$$

where $d^- f(x; h) = \liminf_{u \to h,\, t \downarrow 0} t^{-1}(f(x + tu) - f(x))$ refers to the lower Dini directional derivative of f at x in direction h. For the following denote by $\mathscr{F}$ the collection of finite-dimensional subspaces of X and for any subset $S \subseteq X$ put

$$\begin{aligned} f_S(x) &:= f(x), \qquad \text{if } x \in S, \\ &= \infty, \qquad \text{else}. \end{aligned}$$

DEFINITION 1.1 (approximate subdifferential). For $z \in X$ define

$$\partial_a f(z) = \begin{cases} \bigcap\limits_{L \in \mathscr{F}} \limsup\limits_{\substack{x \to z \\ f(x) \to f(z)}} \partial^- f_{x+L}(x), & \text{if } |f(z)| < \infty, \\ \emptyset, & \text{else}. \end{cases}$$

In the definition, lim sup has to be understood in the Kuratowski–Painlevé sense with respect to the strong topology in X and the weak*-topology in X^*. More explicitly, for some multifunction $K\colon X \rightrightarrows X^*$ one has

$$\begin{aligned} x^* \in & \limsup_{\substack{x \to z \\ f(x) \to f(z)}} K(x) \\ & \Leftrightarrow \quad \text{there are nets } x_\alpha \to z,\ x_\alpha^* \rightharpoonup_{w^*} x^*\colon f(x_\alpha) \to f(z),\ x_\alpha^* \in K(x_\alpha). \end{aligned}$$

In a similar way the lim inf of such a multifunction is defined by

$$x^* \in \liminf_{\substack{x \to z \\ f(x) \to f(z)}} K(x) \quad \Leftrightarrow \quad \text{for all nets } x_\alpha \to z \text{ with } f(x_\alpha) \to f(z)$$

$$\text{there is a net } x_\alpha^* \rightharpoonup_{w^*} x^*,\ x_\alpha^* \in K(x_\alpha).$$

If both kinds of limits coincide (the second being always contained in the first) then one simply speaks of the limit lim of K at z.

Certainly, one always has the inclusion $\partial^- f(x) \subseteq \partial_a f(x)$. For locally Lipschitzian functions the main relation between the approximate and Clarke's subdifferential is $\partial_C f(x) = \operatorname{clco} \partial_a f(x)$, where clco refers to the convex, weak*-closure, and furthermore $\partial_a f(z)$ is weak*-compact. The following lemma (see Proposition 2.1 and Corollary 5.1.1 in [11]) gives expressions of $\partial_a f(z)$ which may be easier to handle than the original involved definition:

LEMMA 1.1. *One always has*

$$\partial_a f(z) = \Big\{ x^* \in X^* \mid \text{there are nets } x_\alpha \to z,\ x_\alpha^* \rightharpoonup_{w^*} x^*,\ L_\alpha \in \mathscr{F}:$$

$$f(x_\alpha) \to f(z),\ x_\alpha^* \in \partial^- f_{x_\alpha + L_\alpha}(x_\alpha),\ L_\alpha \text{ cofinal with } \mathscr{F} \Big\}$$

(*here "L_α cofinal with $\mathscr{F}$" means* $\forall L \in \mathscr{F}\ \forall \alpha\ \exists \alpha_0 \geq \alpha,\ L \subseteq L_{\alpha_0}$). *If X is a Banach space which is separable or admits an equivalent Gateaux-differentiable norm, then for each lower semicontinuous function $f\colon X \to \mathbb{R} \cup \{-\infty, \infty\}$ it holds that*

$$\partial_a f(z) = \limsup_{\substack{x \to z \\ f(x) \to f(z) \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f(x),$$

where $\partial_\varepsilon^- f(x)$ is defined as

$$\partial_\varepsilon^- f(x) = \begin{cases} \{x^* \in X^* \mid x^*(h) \leq d^- f(x; h) + \varepsilon \|h\|\ \forall h \in X\}, \\ \quad \text{if } |f(x)| < \infty, \\ \emptyset, \qquad \text{else}. \end{cases}$$

2. RESULTS

As one can already see from the simple example $f(x) = -|x|$, where $\partial_a f(0) = \{-1, 1\}$, the approximate subdifferential need not be convex, not even connected. In a first step for characterizing its topological properties, one could look for certain conditions yielding specific topological types. In the following lemma such conditions are imposed onto the local behavior of the Dini subdifferential, which is the main ingredient of the approximate subdifferential.

LEMMA 2.1. *For* $f\colon X \to \mathbb{R}$, $z \in X$ (*Banach space*) *consider the following conditions*:

(A1) $\partial^- f_{z+L}(z) = \limsup_{x \to z,\, f(x) \to f(z)} \partial^- f_{x+L}(x)\ \forall L \in \mathscr{F}$.

(A2) $\liminf_{x \to z,\, f(x) \to f(z)} \partial^- f(x) \neq \emptyset$.

(A3) $\partial^- f(z)$ *is weak*-compact and there exists* $\varepsilon > 0$ *such that the relation* $\partial^- f(x) \cap \partial^- f(z) \neq \emptyset$ *holds for all* x *with* $\|x - z\|$ *and* $|f(x) - f(z)| < \varepsilon$. *Then one has the following implications.*

1. (A1) $\Rightarrow$ $\partial_a f(z)$ *is convex.*
2. (A2) $\Rightarrow$ $\partial_a f(z)$ *is star-shaped.*
3. (A3) $\Rightarrow$ $\partial_a f(z)$ *is connected.*
4. *If* $\dim X < \infty$ *and* f *is locally Lipschitzian, then* (A3) $\Rightarrow$ (A1) *and* (A2) $\Rightarrow$ (A1) *or* $\partial_a f(z)$ *is a singleton.*

Proof. For verifying implication 1, note that the definition of $\partial_a f(z)$ together with (A1) means

$$\partial_a f(z) = \bigcap_{L \in \mathscr{F}} \partial^- f_{z+L}(z),$$

which is convex by convexity of $\partial^- f_{z+L}(z)$. Next, let (A2) be fulfilled. Then there exists some $y^* \in \liminf_{x \to z,\ f(x) \to f(z)} \partial^- f(x)$. In order to show star-shapedness of $\partial_a f(z)$ it is sufficient to prove that the line segments $[x^*, y^*]$ are contained in $\partial_a f(z)$ for all $x^* \in \partial_a f(z)$. Now, let any such x^* be given. By Lemma 1.1, there are nets $x_\alpha \to z$, $x^*_\alpha \rightharpoonup_{w^*} x^*$, and $L_\alpha \subseteq \mathscr{F}$ such that $f(x_\alpha) \to f(z)$, $x^*_\alpha \in \partial^- f_{x_\alpha + L_\alpha}(x_\alpha)$, and L_α is cofinal with $\mathscr{F}$. According to the definition of lim inf there exists a net $y^*_\alpha \rightharpoonup_{w^*} y^*$ with $y^*_\alpha \in \partial^- f(x_\alpha) \subseteq \partial^- f_{x_\alpha + L_\alpha}(x_\alpha)$. Then, by convexity of the Dini subdifferential, one gets

$$\partial^- f_{x_\alpha + L_\alpha}(x_\alpha) \ni t x^*_\alpha + (1 - t) y^*_\alpha \rightharpoonup_{w^*} t x^* + (1 - t) y^*$$

for all $t \in [0, 1]$, but this means $[x^*, y^*] \subseteq \partial_a f(z)$. For proving implication 3, it is sufficient to show that for each $x^* \in \partial_a f(z)$ there is some $y^* \in \partial^- f(z)$ such that $[x^*, y^*] \subseteq \partial_a f(z)$. Then, indeed, two arbitrary points $x^*_1, x^*_2 \in \partial_a f(z)$ may be joined by a path $[x^*_1, y^*_1] \cup [y^*_1, y^*_2] \cup [y^*_2, x^*_2]$ which is completely contained in $\partial_a f(z)$ because $[y^*_1, y^*_2] \subseteq \partial^- f(z) \subseteq \partial_a f(z)$ by convexity of the Dini subdifferential. This means (path-) connectedness of $\partial_a f(z)$. In order to verify the afore-mentioned assumption, let any $x^* \in \partial_a f(z)$ be given. Then the same kind of nets $x_\alpha, x^*_\alpha, L_\alpha$ as in the proof of implication 2 exist. Moreover, after passing to subnets if necessary, one may assume that L_α is not only cofinal with $\mathscr{F}$, but even residual, i.e., for all $L \in \mathscr{F}$ there exists an α_0 such that $L \subseteq L_\alpha$ for all $\alpha \geq \alpha_0$. Now (A3) means existence of a net $y_\alpha^* \in \partial^- f(x_\alpha) \cap \partial^- f(z) \subseteq \partial^- f_{x_\alpha + L_\alpha}(x_\alpha) \cap \partial^- f(z)$. By the assumed weak*-compactness of $\partial^- f(z)$ we have $y_{\alpha'}^* \rightharpoonup_{w^*} y^*$ for some subnet and some $y^* \in \partial^- f(z)$. Again convexity of the Dini subdifferential implies

$$\partial^- f_{x_{\alpha'} + L_{\alpha'}}(x_{\alpha'}) \ni t x_{\alpha'}^* + (1-t) y_{\alpha'}^* \rightharpoonup_{w^*} t x^* + (1-t) y^* \qquad \forall t \in [0,1].$$

L_α being residual w.r.t. $\mathscr{F}$, the subnet $L_{\alpha'}$ is easily shown to be cofinal with $\mathscr{F}$, whence $[x^*, y^*] \subseteq \partial_a f(z)$, as desired. The last implication (4), was shown in [7]. ∎

By this lemma, conditions (A1), (A2), and (A3) imply successively weaker topological types. There are examples of lower semicontinuous functions in two variables such that (A2) is fulfilled but convexity is violated or (A3) is fulfilled but star-shapedness of the approximate subdifferential fails to hold (see [7]). In this sense, the indicated conditions are specific. On the other hand, the last statement of the lemma shows that in the (finite-dimensional) Lipschitzian case, (A2) and (A3) imply nothing but convexity. In this situation it seems hard to find reasonable conditions similar to Lemma 2.1 which are specific for a certain topological type. This motivates a more detailed study of the locally Lipschitzian case. Before proving the main result, we shall need the following lemma which facilitates the computation of $\partial_a f(z)$ according to Definition 1.1. The third assertion of the lemma could be deduced from more general results in [13], but to make the presentation self-contained we include its proof, which in the given specific setting is quite simple.

LEMMA 2.2. *Let X be a Hilbert space, $C \subseteq X$ a weakly compact subset, and $f\colon X \to \mathbb{R}$ defined by $f(z) = \min\{\langle z, x\rangle | x \in C\}$. Denote $E(z) = \{x \in C | \langle z, x\rangle = f(z)\}$. Then the following equalities hold*:

$$d^- f(z; h) = \min\{\langle x, h\rangle | x \in E(z)\}, \tag{1}$$

$$\#E(z) = 1 \quad \Rightarrow \quad E(z) = \partial^- f(z); \qquad \#E(z) \geq 2 \quad \Rightarrow \quad \partial^- f(z) = \emptyset, \tag{2}$$

$$\limsup_{\substack{z \to 0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f(z) = \limsup_{z \to 0} \partial^- f(z). \tag{3}$$

Proof. Since f is a concave function, its directional derivative exists and coincides with $d^- f(z; h)$. On the other hand, the value of the directional derivative computes from (1); see, e.g., [2, Proposition 4.4].

Concerning the first relation of (2) one concludes from $E(z) = \{x\}$ and from (1) that $d^-f(z;h) = \langle x, h\rangle \ \forall h \in X$; hence $\partial^- f(z) = \{x\}$. For the second relation, note that the nonemptiness of $\partial^- f(z)$ implies (by definition) $d^-f(z;h) \geq -d^-f(z;-h) \ \forall h \in X$. Now, choosing $x^1, x^2 \in E(z)$, $x^1 \neq x^2$, and assuming $\partial^- f(z) \neq \emptyset$ we obtain from (1) for arbitrary $h \in X$,

$$\begin{aligned}\min\{\langle x^1, h\rangle, \langle x^2, h\rangle\} &\geq \min_{x\in E(z)} \langle x, h\rangle = d^-f(z;h) \geq -d^-f(z;-h) \\ &= -\min_{x\in E(z)} \langle x, -h\rangle = \max_{x\in E(z)} \langle x, h\rangle \\ &\geq \max\{\langle x^1, h\rangle, \langle x^2, h\rangle\}.\end{aligned}$$

This inequality holding for all $h \in X$, one gets the contradiction $x^1 = x^2$.

Finally, for proving (3), first observe that the inclusion $\supseteq$ is trivially fulfilled because of $\partial_\varepsilon^- \supseteq \partial^-$. For the reverse inclusion let

$$x \in \limsup_{\substack{z\to 0\\ \varepsilon\downarrow 0}} \partial_\varepsilon^- f(z).$$

This means existence of nets $z_\lambda \to 0$, $x_\lambda \rightharpoonup_w x$, $\varepsilon_\lambda \to 0$, $x_\lambda \in \partial_{\varepsilon_\lambda}^- f(z_\lambda)$, and $\varepsilon_\lambda > 0$. Let U be a (strong) open neighborhood of 0 and V be a weak open neighborhood of x. Then V contains a base neighborhood of the type

$$W = \{y \in X \| \langle y - x, y^i\rangle| < \varepsilon,\ y^i \in X,\ i = 1, \ldots, n\}$$

and for $\lambda \geq \lambda_0$ one has $|\langle x_\lambda - x, y^i\rangle| < \varepsilon/2$ $(i = 1, \ldots, n)$. This implies $y \in W \subseteq V$ whenever $\|y - x_\lambda\| < \varepsilon/(2(1 + \max\|y^i\|))$. Summarizing, the existence of an index λ^* such that $z_{\lambda^*} \in U$ and $x_{\lambda^*} + B_{\varepsilon_{\lambda^*}} \subseteq V$ follows. The last relation yields $E(z_{\lambda^*}) \subseteq V$. Indeed, since $x_{\lambda^*} \in \partial_{\varepsilon_{\lambda^*}}^- f(z_{\lambda^*})$, one has for any $v \in E(z_{\lambda^*})$ and $h \in X$ that [compare the definition of $\partial_\varepsilon^- f$ and (1)]:

$$\langle x_{\lambda^*}, h\rangle \leq \min\{\langle v', h\rangle | v' \in E(z_{\lambda^*})\} + \varepsilon_{\lambda^*}\|h\| \leq \langle v, h\rangle + \varepsilon_{\lambda^*}\|h\|.$$

From this it follows that $\|x_{\lambda^*} - v\| \leq \varepsilon_{\lambda^*}$; hence $E(z_{\lambda^*}) \subseteq x_{\lambda^*} + B_{\varepsilon_{\lambda^*}} \subseteq V$.

Since $C \setminus V$ is weakly compact, this last inclusion together with continuity of f implies existence of some (strong) open neighborhood of U' of z_{λ^*} such that $E(y) \subseteq V \ \forall y \in U'$. As a concave function f is Gateaux-differentiable on a dense subset of X [1]. Now, $U \cap U'$ is a nonempty set which consequently must contain some point y such that f possesses some Gateaux derivative y^* at y. Then, by (2),

$$\partial^- f(y) = \{y^*\} = E(y) \subseteq V.$$

Summarizing, to each pair U, V of (strong, weak) open neighborhoods of $0, x$ we can assign $y_{[U,V]} \in U$, $y^*_{[U,V]} \in V$ such that $y^*_{[U,V]} \in \partial^- f(y_{[U,V]})$. In this way one obtains converging nets $y_\mu \to 0$, $y^*_\mu \rightharpoonup_w x$, and $y^*_\mu \in \partial^- f(y_\mu)$ ($\mu = [U, V]$); hence

$$x \in \limsup_{z \to 0} \partial^- f(z)$$

as desired. ∎

Combining (3) in Lemma 2.2 with Lemma 1.1 one gets

COROLLARY 2.1. *Under the assumption of Lemma* 2.2 *the approximate subdifferential of f* (*defined in the lemma*) *computes as*

$$\partial_a f(0) = \limsup_{z \to 0} \partial^- f(z).$$

As was stated in the introductory section, the approximate subdifferential of a locally Lipschitzian function defined on a Banach space is weak*-compact. In turn, the following theorem shows that in a Hilbert space setting each weakly compact set may be approximated by subdifferentials of Lipschitzian functions.

THEOREM 2.1. *For any weakly compact subset K of any Hilbert space* $(\mathscr{H}, \langle\ \rangle)$ *there exists a one-parametric family* $f_u\colon \mathbb{R} \times \mathscr{H} \to \mathbb{R}$ *of Lipschitzian functions with some common modulus L, such that* $\lim_{u \downarrow 0} \partial_a f_u(0,0) = \{0\} \times K$ *holds in the sense of Kuratowski–Painlevé convergence.*

Proof. Let c be a constant with $\|x\|^2 < c\ \forall x \in K$ (which exists by weak compactness of K) and fix a parameter u with $0 < u \leq 1/c$. In $\mathbb{R} \times \mathscr{H}$ (which is a Hilbert space by canonical extension of the inner product) consider the ellipsoidal surface

$$E_u = \left\{(t, x) \in \mathbb{R} \times \mathscr{H} \mid u\|x\|^2 + (t-1)^2 = 1\right\}$$

as well as the subset

$$K_u = \left\{(t, x) \in \mathbb{R} \times \mathscr{H} \mid x \in K, t = 1 - \sqrt{1 - u\|x\|^2}\right\}.$$

Note that K_u is correctly defined and $K_u \subseteq E_u$. Furthermore, the function $f_u\colon \mathbb{R} \times X \to \mathbb{R}$ with

$$f_u(\alpha, z) = \min\{\langle(\alpha, z), (t, x)\rangle \mid (t, x) \in \operatorname{clco} K_u\}$$

is correctly defined as well (by weak compactness of clco K_u). Since

$$K_u \subseteq [0,1] \times B(0; \sqrt{c}\,) \subseteq B\big((0,0); \sqrt{c+1}\,\big),$$

one has clco $K_u \subseteq B((0,0); \sqrt{c+1}\,)$; hence $\sqrt{c+1}$ is a modulus of Lipschitz continuity for f_u (not depending on u).

Next we show the validity of the following relations which are essential for proving the assertion of the theorem:

$$\operatorname{pr}\big(\partial_a f_u(0,0) | \{0\} \times \mathscr{H}\big) = \{0\} \times K, \tag{4}$$

$$\operatorname{pr}\big(\partial_a f_u(0,0) | \mathbb{R} \times \{0\}\big) \subseteq \Big[0, 1 - \sqrt{1-uc}\,\Big] \times \{0\}. \tag{5}$$

Here, $\operatorname{pr}(\cdot|M)$ denotes projection onto a closed subspace of M of $\mathbb{R} \times \mathscr{H}$.

To show first the inclusion $\{0\} \times K \subseteq \operatorname{pr}(\partial_a f_u(0,0) | \{0\} \times \mathscr{H})$ it is sufficient to verify the relation $K_u \subseteq \partial_a f_u(0,0)$ because of $\operatorname{pr}(K_u | \{0\} \times \mathscr{H}) = \{0\} \times K$. So let $(t,x) \in K_u$. For $n \in \mathbb{N}$ put $\alpha_n = (1-t)/n$ and $z_n = -(u/n)x$. Then, the Kuhn–Tucker conditions (recall that for $u > 0$, E_u is a regular surface) yield that the linear function $\langle(\alpha_n, z_n), \cdot\,\rangle$ restricted to E_u attains its minimum at the unique point (t,x). Since $(t,x) \in K_u \subseteq E_u$ and by a convexity argument one has

$$\begin{aligned}\{(t,x)\} &= \arg\min\{\langle(\alpha_n, z_n),(t',x')\rangle | (t',x') \in K_u\} \\ &= \arg\min\{\langle(\alpha_n, z_n),(t',x')\rangle | (t',x') \in \operatorname{clco} K_u\}.\end{aligned}$$

Therefore, with the notation introduced in Lemma 2.2, $E(\alpha_n, z_n) = \{(t,x)\}$ and (2) implies $\partial^- f_u(\alpha_n, z_n) = \{(t,x)\}$. Together with $(\alpha_n, z_n) \to (0,0)$ and using the trivial sequence $(t_n, x_n) \equiv (t,x)$, Corollary 2.1 provides

$$(t,x) \in \limsup_{(\alpha,z)\to(0,0)} \partial^- f_u(\alpha, z) = \partial_a f_u(0,0).$$

Before proving the reverse inclusion of (4) and (5) we show first the relation

$$(t^*, x^*) \in \partial^- f_u(\alpha, z) \quad \Rightarrow \quad x^* \in K, \qquad t^* \in \Big[0, 1 - \sqrt{1-uc}\,\Big]. \tag{6}$$

Indeed, from (2) in Lemma 2.2 one has

$$\{(t^*, x^*)\} = \arg\min\{\langle(\alpha, z),(t',x')\rangle | (t',x') \in \operatorname{clco} K_u\} \tag{7}$$

and by convexity of clco K_u there exists a sequence $\{(t_n, x_n)\} \subseteq K_u$ such that $\langle(\alpha,z),(t_n,x_n)\rangle \to_n \langle(\alpha,z),(t^*,x^*)\rangle$. Weak compactness of clco K_u then yields the existence of a weakly convergent subsequence $(t_{n_l}, x_{n_l}) \rightharpoonup_l$

$(\tau, y) \in \operatorname{clco} K_u$. Consequently, $\langle(\alpha, z),(\tau, y)\rangle = \langle(\alpha, z),(t^*, x^*)\rangle$; hence $(\tau, y) = (t^*, x^*)$ because of (7). It follows that $x_{n_l} \rightharpoonup_l x^*$ with $x_{n_l} \in K$ [since $(t_{n_l}, x_{n_l}) \in K_u$ and by definition of K_u]. Weak compactness of K yields $x^* \in K$, as desired. On the other hand, from the definition of K_u it follows that $0 \leq t_{n_l} \leq 1 - \sqrt{1 - uc}$, which proves the second implication of (6).

Now, let

$$(t, x) \in \partial_a f_u(0,0) = \limsup_{(\alpha, z) \to (0,0)} \partial^- f_u(\alpha, z)$$

(compare Corollary 2.1), i.e., there exist nets $(\alpha_\lambda, z_\lambda) \to (0,0)$ and $(t_\lambda, x_\lambda) \rightharpoonup (t, x)$ with $(t_\lambda, x_\lambda) \in \partial^- f_u(\alpha_\lambda, z_\lambda)$. Equation (6) yields $x_\lambda \in K$ and $t_\lambda \in [0, 1 - \sqrt{1 - uc}\,]$, whence $x \in K$ by weak compactness of K and $t \in [0, 1 - \sqrt{1 - uc}\,]$. Thus, (4) and (5) are completely proved.

In order to verify the assertion of the theorem it suffices to show the relation

$$\limsup_{u \downarrow 0} \partial_a f_u(0,0) \subseteq \{0\} \times K \subseteq \liminf_{u \downarrow 0} \partial_a f_u(0,0), \tag{8}$$

which directly implies equality between all three sets. Considering first some element $(t, x) \in \limsup_{u \downarrow 0} \partial_a f_u(0,0)$ one has converging nets $u_\lambda \to 0$, $(t_\lambda, x_\lambda) \rightharpoonup (t, x)$, and $(t_\lambda, x_\lambda) \in \partial_a f_{u_\lambda}(0,0)$. Then (4) implies $x_\lambda \in K$ which ensures $x \in K$ because of weak compactness of K. Moreover, (5) provides $t_\lambda \in [0, 1 - \sqrt{1 - u_\lambda c}\,]$. Therefore $t_\lambda \to 0$ and $t = 0$, as was to be proved.

Concerning the second inclusion let $x \in K$. We have to show that $(0, x) \in \liminf_{u \downarrow 0} \partial_a f_u(0,0)$. To this aim consider an arbitrary net $u_\lambda \to 0$. Put $x_\lambda \equiv x$ and choose—by virtue of (4) and (5)—some t_λ with $(t_\lambda, x) \in \partial_a f_{u_\lambda}(0,0)$ and $0 \leq t_\lambda \leq 1 - \sqrt{1 - u_\lambda c}$. Obviously, $(t_\lambda, x_\lambda) \rightharpoonup (0, x)$, as desired. ∎

It is clear that a similar approximation result as in Theorem 2.1 does not hold, for instance, for Clarke's subdifferential because of the restriction to convexity. This will become even clearer from the following topological derivations of the theorem.

LEMMA 2.3. *For any strongly compact subset K of any Hilbert space $\mathscr{H}$ there exists a Lipschitzian function f: $\mathbb{R} \times \mathscr{H} \to \mathbb{R}$ such that $\partial_a f(0,0)$ is homeomorphic to K in the strong topology. If moreover K is contained in some finite-dimensional subspace of $\mathscr{H}$, then $\partial_a f(0,0)$ is homeomorphic to K in the weak topology too.*

Proof. Fix any admissible parameter u in Theorem 2.1 (see beginning of the proof) and consider the corresponding set K_u. Via the functions ϕ: $K \to K_u$ and ψ: $K_u \to K$ defined by $\phi(x) = (1 - \sqrt{1 - u\|x\|^2}, x)$ and

$\psi(t,x) = x$ we see that ϕ defines a bijection from K to K_u with inverse $\phi^{-1} = \psi$. Moreover, ϕ is strongly continuous while ψ is both strongly and weakly continuous. Hence ϕ defines a homeomorphism between K and K_u in the strong topology. In particular K_u inherits strong compactness from K. Now we show that $\partial_a f(0,0) = K_u$. Define the Lipschitzian function f: $\mathbb{R} \times \mathscr{H} \to \mathbb{R}$ by $f(\alpha, z) = \min\{\langle(\alpha, z),(t,x)\rangle | (t,x) \in K_u\}$ which is justified by strong (hence weak) compactness of K_u. Repeating the lines of argumentation in the proof of Theorem 2.1 (but restricting considerations to K_u itself instead of its convex closure) one derives the inclusion $K_u \subseteq \partial_a f(0,0)$. The reverse inclusion would follows from the relation $\partial^- f(\alpha, z) \subseteq K_u \;\; \forall(\alpha, z)$ due to weak compactness of K_u and to the definition of the approximate subdifferential. To verify the mentioned relation, note that

$$(t^*, x^*) \in \partial^- f(\alpha, z)$$
$$\Rightarrow \{(t^*, x^*)\} = \arg\min\{\langle(\alpha, z),(t', x')\rangle | (t', x') \in K_u\}$$

as a consequence of (2) in Lemma 2.2. In particular, $(t^*, x^*) \in K_u$, as was to be shown. Concerning the second part of the lemma, note that in this case the function ϕ, introduced in the beginning of this proof, becomes weakly continuous as was also true for its inverse ψ. Therefore ϕ is a homeomorphism between K and K_u in the weak topology too. ∎

This lemma extends a corresponding result in [7] (Theorem 3.2) for finite-dimensional $\mathscr{H}$. It allows the following topological characterization of the approximate subdifferential:

COROLLARY 2.2. *For any Hilbert space $\mathscr{H}$ there exists a Lipschitzian function f: $\mathbb{R} \times \mathscr{H} \to \mathbb{R}$ such that $\partial_a f(0,0)$ is homeomorphic to the Cantor set in the weak as well as the strong topology. In particular, $\partial_a f(0,0)$ is totally disconnected in both topologies.*

Finally we note that in [7] it was proved that for each compact subset $K \subseteq \mathbb{R}^n$ there exists a locally Lipschitzian function f: $\mathbb{R}^{n+2} \to \mathbb{R}$ such that the so-called partial approximate subdifferential (introduced by Jourani and Thibault [14] in the context of functions depending on a parameter) of f coincides (at the origin) with $\{0\} \times K \subseteq \mathbb{R}^{n+1}$. Here, any given compact subset may be even realized itself (up to imbedding into a space with one extra dimension).

Revisiting Lemma 2.3 and Corollary 2.2 one notices that for the definition of the desired Lipschitzian function one additional dimension is needed. In particular, the construction (according to Corollary 2.2) of an approximate subdifferential being totally disconnected requires the domain of the Lipschitzian function f to have at least dimension 2. This suggests a question about possible topological types of the approximate subdifferential in the one-dimensional case. Theorem 2.2 below indicates, that for real continuous functions $f: \mathbb{R} \to \mathbb{R}$ the approximate subdifferential is either an interval or the disjoint union of two intervals, which is obviously a topological restriction.

For the proof of this one-dimensional characterization two auxiliary results are needed. To this aim, for functions $f: \mathbb{R} \to \mathbb{R}$ we introduce the following notations (with $z \in \mathbb{R}$):

$$d^l(z) = \liminf_{\substack{t \downarrow 0 \\ u \to -1}} t^{-1}\big(f(z+tu) - f(z)\big);$$

$$d^r(z) = \liminf_{\substack{t \downarrow 0 \\ u \to 1}} t^{-1}\big(f(z+tu) - f(z)\big),$$

where improper values $\pm\infty$ are allowed. Obviously,

$$\partial^- f(z) = \left[-d^l(z), d^r(z)\right] \tag{9}$$

holds, with the interval to be interpreted appropriately for improper values.

PROPOSITION 2.1. *Let $f: \mathbb{R} \to \mathbb{R}$ be continuous, and $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$. Then*

$$d^r(t_1) < -d^l(t_2) \quad \Rightarrow \quad \forall c \in \left(d^r(t_1), -d^l(t_2)\right) \exists \bar{t} \in (t_1, t_2): \\ c \in \left[-d^l(\bar{t}), d^r(\bar{t})\right] = \partial^- f(\bar{t}), \tag{10}$$

$$d^r(t_1) > -d^l(t_2) \quad \Rightarrow \quad \forall c \in \left(-d^l(t_2), d^r(t_1)\right) \exists \bar{t} \in (t_1, t_2): \\ c \in \left[d^r(\bar{t}), -d^l(\bar{t})\right]. \tag{11}$$

Proof. Concerning (10) let $c \in (d^r(t_1), -d^l(t_2))$. Then the function $x \mapsto f(x) - cx$, restricted to $x \in [t_1, t_2]$, attains its minimum at a point of the open interval (t_1, t_2). To see this, assume that t_1 is a minimizer. It follows that

$$f(t+t_1) - f(t_1) \geq ct \qquad \forall t \in [0, t_2 - t_1].$$

This however yields

$$\liminf_{\substack{t \downarrow 0 \\ u \to 1}} t^{-1}\big(f(t_1+ut) - f(t_1)\big) \geq c,$$

leading to the contradiction $c \leq d^r(t_1)$. In a similar way t_2 may be excluded as a minimizer via the contradiction $c \geq -d^l(t_2)$. As a consequence, there exists $\bar{t} \in (t_1, t_2)$ such that $f(x) - cx \geq f(\bar{t}) - c\bar{t}$ $\forall x \in [t_1, t_2]$. Choose $\delta > 0$ with $(\bar{t} - \delta, \bar{t} + \delta) \subseteq (t_1, t_2)$ to get

$$f(t+\bar{t}) - f(\bar{t}) \geq ct \qquad \forall t \in (-\delta, \delta),$$

which immediately implies (10). For (11) consider the maximum (rather than minimum) of $f(x) - cx$ and repeat the same arguments. ∎

LEMMA 2.4. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be continuous and* $\partial^- f(z) \neq \emptyset$ *for some* $z \in \mathbb{R}$. *Then* $\partial_a f(z)$ *is a closed (possibly unbounded) interval.*

Proof. By assumption, there is some $e \in \partial^- f(z) \subseteq \partial_a f(z)$. It will be sufficient to show that $a \in \partial_a f(z)$, $a < e$ implies $(a, e) \subseteq \partial_a f(z)$ (the proof is running along similar lines for $a > e$). The general definition, Definition 1.1, of the approximate subdifferential reduces in the present constellation (finite dimensions, continuous function) to $\partial_a f(z) = \limsup_{x \to z} \partial^- f(x)$, where the lim sup may be generated by sequences instead of nets. Accordingly, there exist sequences

$$t_k \to 0, \qquad a_k \to a, \qquad a_k \in \partial^- f(z + t_k). \tag{12}$$

Let $c \in (a, e)$ be arbitrarily given. We have to show that $c \in \partial_a f(z)$. For $k \geq k_0$ one has $c > a_k$ and we may suppose without loss of generality that $c \notin \partial^- f(z + t_k) \cup \partial^- f(z)$ [since otherwise the assertion $c \in \partial_a f(z)$ follows immediately]. Combining (12) with (9) one gets

$$-d^l(z + t_k) \leq a_k \leq d^r(z + t_k) < c < -d^l(z) \leq e \leq d^r(z) \qquad \forall k \geq k_0. \tag{13}$$

As an obvious consequence of this relation we have $t_k \neq 0\ \forall k$. One of the following two cases must hold true:

Case 1. There is a negative subsequence $t_{k_p} < 0$. Application of (10) to the relation $d^r(z + t_{k_p}) < c < -d^l(z)$ [see (13)] yields existence of $\tau_p \in (t_{k_p}, 0)$ with $c \in \partial^- f(z + \tau_p)$. Then $c \in \partial_a f(z)$ due to $\tau_p \to 0$.

Case 2. There is a positive subsequence $t_{k_p} > 0$. Fix an arbitrary $\bar{c} \in (c, e)$ and apply (11) to the relation $d^r(z) > \bar{c} > -d^l(z + t_{k_p})$ [compare (13)]. Hence, there exists $\tau_p \in (0, t_{k_p})$ with $\bar{c} \leq -d^l(z + \tau_p)$. Next choose an index $q(p)$ to fulfill $t_{k_{q(p)}} < \tau_p$. Application of (10) to the relation $d^r(z + t_{k_{q(p)}}) < c < \bar{c} \leq -d^l(z + \tau_p)$ finally provides $c \in \partial^- f(z + \mu_p)$ for some $\mu_p \in (t_{k_{q(p)}}, \tau_p)$. Now $c \in \partial_a f(z)$ due to $\mu_p \to 0$. ∎

According to Lemma 2.4 nonemptiness of the Dini subdifferential implies convexity of the approximate subdifferential in one dimension. Unfortunately, a generalization of this fact to the multidimensional case is not possible even in the Lipschitzian case. This can be seen from the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \max\{0, \min\{x, y\}\}$. Here $\partial^- f(0,0) = \{0, 0\} \neq \emptyset$, while $\partial_a f(0,0) = [(0,0),(0,1)] \cup [(0,0),(1,0)]$ (with brackets referring to line segments), which is obviously not convex.

THEOREM 2.2. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be continuous. Then* $\partial_a f(z)$ *contains at most two connected components (for all* $z \in \mathbb{R}$).

Proof. Negating the assertion means the existence of elements $a < b < c$ with $a, b, c \in \partial_a f(z)$ and which are contained in three pairwise different connected components. By definition there are sequences

$$t_k, \tau_k, \mu_k \to 0, \qquad a_k \to a, \qquad b_k \to b, \qquad c_k \to c,$$

$$a_k \in \partial^- f(z + t_k), \qquad b_k \in \partial^- f(z + \tau_k), \qquad c_i \in \partial^- f(z + \mu_k). \quad (14)$$

As a consequence of Lemma 2.4 one has $t_k, \tau_k, \mu_k \neq 0\ \forall k$; otherwise a_k (or b_k or c_k) $\in \partial^- f(z)$ for some k, so $\partial_a f(z)$ would be an interval in contradiction to our assumption of at least three connected components. Passing to appropriate subsequences one may assume, without loss of generality, that each of the sequences $\{t_k\}, \{\tau_k\}, \{\mu_k\}$ in (14) has a constant (definite) sign. For two of them this sign must be equal, so we have

$$t_k < 0, \tau_k < 0\ \forall k \quad \text{or} \quad t_k > 0, \tau_k > 0\ \forall k \qquad (15)$$

(the subsequent proof being identical for the remaining possible pairs $\{t_k\}, \{\mu_k\}$, and $\{\tau_k\}, \{\mu_k\}$). We are done if we can show that $(a, b) \subseteq \partial_a f(z)$, because then $[a, b] \subseteq \partial_a f(z)$ in contradiction to the assumption that a and b come from different connected components of $\partial_a f(z)$. So choose any $e \in (a, b)$. For some k_0 it holds that $a_k < e < b_k$ and $e \notin \partial^- f(z + t_k) \cup \partial^- f(z + \tau_k)\ \forall k \geq k_0$ (negating the second relation would immediately provide the desired result $e \in \partial_a f(z)$). Consequently one gets [compare (9)]

$$-d^l(z + t_k) \leq a_k \leq d^r(z + t_k) < e < -d^l(z + \tau_k) \leq b_k \leq d^r(z + \tau_k). \qquad (16)$$

From this chain it is evident that $t_k \neq t_k\ \forall k \geq k_0$, so one of the following two cases holds true:

Case 1. There exists a subsequence such that $t_{k_p} < \tau_{k_p}\ \forall p$. Then, (10) applied to the relation $d^r(z + t_{k_p}) < e < -d^l(z + \tau_{k_p})$ yields the existence of some $\alpha_p \in (t_{k_p}, \tau_{k_p})$ such that $e \in \partial^- f(z + \alpha_p)$. It follows $e \in \partial_a f(z)$ since $\alpha_p \to 0$.

Case 2. There exists a subsequence such that $t_{k_p} > \tau_{k_p} \forall p$. First we consider the situation $t_k < 0$, $\tau_k < 0\ \forall k$ [see (15)]. Fix an arbitrary $\bar{e} \in (a, e)$. Again we may assume that $a_{k_p} < \bar{e}$. Then, (11) applied to the relation $d^r(z + \tau_{k_p}) > \bar{e} > -d^l(z + t_{k_p})$ yields the existence of some $\beta_p \in (\tau_{k_p}, t_{k_p})$ such that $\bar{e} \geq d^r(z + \beta_p)$. Obviously, $\beta_p < 0$ and $\beta_p \to 0$. In particular we may select an index $q(p)$ with $\tau_{k_{q(p)}} > \beta_p$. Then, (10) applied to the relation $d^r(z + \beta_p) \leq \bar{e} < e < -d^l(z + \tau_{k_{q(p)}})$ yields the existence of some $\gamma_p \in (\beta_p, \tau_{k_{q(p)}})$ such that $e \in \partial^- f(z + \gamma_p)$. Since $\gamma_p \to 0$ we arrive at $e \in \partial_a f(z)$. The second situation of (15) is treated in a similar manner, but now by choosing $\bar{e} \in (e, b)$, deriving the relation $\bar{e} \leq -d^l(z + \beta_p)$, and considering a sequence $t_{k_{q(p)}} < \beta_p$. ∎

We note that, in the case of locally Lipschitzian functions, the result of Theorem 2.2 can also be derived from [3]. From an example in the same reference one can see that on any countable subset of the line, which in particular could be dense, the approximate subdifferential of some suitably defined Lipschitz function is disconnected. The following proposition demonstrates that such behavior is possible even for a very nice nonsmooth function like the value function of a parametric optimization problem of class $\mathscr{C}^1$.

PROPOSITION 2.2. *There exists a function* $f \in \mathscr{C}^1(\mathbb{R}^2, \mathbb{R})$, *such that for the corresponding minimum function* $g(z) = \min\{f(x,z) | x \in [0,1]\}$ *the approximate subdifferential* $\partial_a g(z)$ *is disconnected* (*nonconvex in particular*) *on a dense set of points* $z \in \mathbb{R}$.

Proof. Denote by Z the set of all $z \in \mathbb{R}$ representable as $z = \pm i 2^{-j}$ $(i,j \in \mathbb{N})$. Then Z is a dense subset of $\mathbb{R}$ and it is possible to construct a $\mathscr{C}^1$-function $f: \mathbb{R}^2 \to \mathbb{R}$ with the properties:

$$\forall z \in Z, \qquad E(z) = \{x^1(z), x^2(z)\} \qquad \frac{\partial f}{\partial z}(x^1(z), z) \neq \frac{\partial f}{\partial z}(x^2(z), z), \tag{17}$$

where $E(z) = \{x \in [0,1] \| f(y,z) \geq f(x,z)\ \forall y \in [0,1]\}$. The details of this construction are described in [8] in the context of maximum functions with a dense subset of nondifferentiability. Now consider the minimum function

$$g(z) = \min\{f(x,z) | x \in [0,1]\},$$

which has the one-sided directional derivative

$$dg(z;h) = \min\left\{\frac{\partial f}{\partial z}(x,z) \cdot h | x \in E(z)\right\} \qquad \forall z, h \in \mathbb{R}.$$

Therefore, the lower Dini directional derivative $d^- g(z;h)$ (see the Introduction) coincides with $dg(z;h)$ and the Dini subdifferential simply becomes the "interval" $[\phi^l(z), \phi^r(z)]$, where

$$\phi^l(z) = \max\left\{\frac{\partial f}{\partial z}(x,z) | x \in E(z)\right\},$$

$$\phi^r(z) = \min\left\{\frac{\partial f}{\partial z}(x,z) | x \in E(z)\right\}.$$

Since $\phi^l(z) \geq \phi^r(z)$ it holds that $\partial^- g(z) = \{\phi^l(z)\} \cap \{\phi^r(z)\}$ for all $z \in \mathbb{R}$. Now, for some fixed $\bar{z} \in Z$ we have by (17) (without loss of generality),

$$E(\bar{z}) = \{\bar{x}^1, \bar{x}^2\}, \qquad \alpha_1 > \alpha_2, \qquad \text{where } \alpha_i = \frac{\partial f}{\partial z}(\bar{x}^i, \bar{z}) \ (i = 1, 2).$$

It follows that $\partial^- g(\bar{z}) = \emptyset$. Moreover, continuity of the function $\partial f/\partial z$ implies that

$$\forall \varepsilon > 0\ \exists \delta > 0, \qquad E(z) \subseteq \begin{cases} \bar{x}^1 + \varepsilon[-1,1], & \text{if } z \in (\bar{z} - \delta, \bar{z}), \\ \bar{x}^2 + \varepsilon[-1,1], & \text{if } z \in (\bar{z}, \bar{z} + \delta). \end{cases}$$

Therefore, $\lim_{z \uparrow \bar{z}} \phi^l(z) = \phi^l(\bar{z}) = \alpha_1$ and $\lim_{z \downarrow \bar{z}} \phi^r(z) = \phi^r(\bar{z}) = \alpha_2$. Summarizing we arrive at

$$\partial_a g(\bar{z}) = \limsup_{z \to \bar{z}} \{\phi^l(z)\} \cap \{\phi^r(z)\} \subseteq \limsup_{z \to \bar{z}} \{\phi^l(z), \phi^r(z)\} = \{\alpha_1, \alpha_2\}.$$

On the other hand, g being locally Lipschitzian, there are sequences $z_n \downarrow \bar{z}$ and $z_n \uparrow \bar{z}$, such that $g'(z_n)$ exists; hence $\partial^- g(z_n) = \{g'(z_n)\} = \{\phi^l(z_n)\} = \{\phi^r(z_n)\}$. It results that

$$\lim_{z_n \uparrow \bar{z}} \partial^- g(z_n) = \lim_{z_n \uparrow \bar{z}} \{\phi^l(z_n)\} = \{\alpha_1\},$$

$$\lim_{z_n \downarrow \bar{z}} \partial^- g(z_n) = \lim_{z_n \downarrow \bar{z}} \{\phi^r(z_n)\} = \{\alpha_2\},$$

whence $\partial_a g(\bar{z}) = \{\alpha_1, \alpha_2\}$. Since $\bar{z} \in Z$ was arbitrarily chosen, the approximate subdifferential is disconnected on a dense subset of $\mathbb{R}$. ∎

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