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# OPTIMAL CONTROL OF THE SWEEPING PROCESS

G. Colombo[1] R. Henrion[2] N. D. Hoang[3] and B. S. Mordukhovich[3]

[1]Dipartimento di Matematica Pura e Applicata
Università di Padova, via Trieste 63, 35121 Padova, Italy (colombo@math.unipd.it)
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[2]Weierstrass Institute for Applied Analysis and Stochastics
10117 Berlin, Germany (henrion@wias-berlin.de)
Research of this author was partially supported by the DFG Research Center MATHEON in Berlin

[3]Department of Mathematics
Wayne State University, Detroit, MI 48202, USA (ndhoang@wayne.edu, boris@math.wayne.edu)
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**Abstract.** We formulate and study an optimal control problem for the sweeping (Moreau) process, where control functions enter the moving sweeping set. To the best of our knowledge, this is the first study in the literature devoted to optimal control of the sweeping process. We first establish an existence theorem of optimal solutions and then derive necessary optimality conditions for this optimal control problem of a new type, where the dynamics is governed by discontinuous differential inclusions with variable right-hand sides. Our approach to necessary optimality conditions is based on the method of discrete approximations and advanced tools of variational analysis and generalized differentiation. The final results obtained are given in terms of the initial data of the controlled sweeping process and are illustrated by nontrivial examples.

**Keywords.** sweeping process, optimal control, dissipative differential inclusions, variational analysis, generalized differentiation.

**AMS (MOS) subject classification:** 49J52, 49J53, 49K24, 49M235, 90C30

# 1 Introduction and Problem Formulation

The sweeping, or Moreau, process was introduced in the 1970s by Jean-Jacques Moreau as a tool for modeling elastoplastic mechanical systems (see, e.g., [14] with the references to Moreau's earlier work) and later has become an active research topic of its own interest; we refer the reader to [4, 10] and the bibliographies therein for more details and analysis. Originally the sweeping process was formulated as an evolution inclusion in a Hilbert space $H$. More precisely, let $I$ be a real interval, and let $t \rightrightarrows C(t)$ be a Lipschitzian set-valued mapping from $I$ into $H$ with closed and convex values. Then the *sweeping process* is described by the dissipative differential inclusion

$$\begin{cases} -\dot{x}(t) & \in N\big(x(t); C(t)\big) \quad \text{a.e.} \quad t \in I, \\ x(0) & = x_0 \in C(0), \end{cases} \tag{1.1}$$

where $N(\cdot;\Omega)$ stands as usual for the classical normal cone to a convex set $\Omega$; see (2.5) with $N(\bar{x};\Omega) = \emptyset$ for $\bar{x} \notin \Omega$. Moreau's motivation for this terminology came from the fact that $x(t)$ could be interpreted, especially if $C(t)$ has nonempty interior, as the evolution of $x_0$ according to the displacement of $C(\cdot)$. As written in [15], "the moving point $t \mapsto x(t)$ remains at rest as long as it happens to lie in its interior; when caught up with the boundary of the moving set, it can only proceed in an inward normal direction, as if pushed by its boundary, as to go on belonging to $C(t)$." The classical theory of the sweeping process establishes the existence and uniqueness of a Lipschitz continuous trajectory; see, e.g., [14, §5] and the references therein. It is worth mentioning that, from the theoretical viewpoint, the sweeping process was one of the important motivations for further developing *convex analysis* and the theory of *differential inclusions*.

Note that in mechanical applications (see, e.g., [14, §6.c]) the moving set may be taken as a translation of a certain convex subset of a fixed subspace of $H$. Furthermore, the well-known Skorokhod problem on the reflected Brownian motion in stochastic analysis can be treated in fact as a version of the sweeping process with a moving set that is a translation of a fixed convex set. Other particular versions of the sweeping process over polyhedral moving sets are studied in [8], where the reader can find more references and practical applications different from those mentioned above.

The mathematical theory of the sweeping process has been developed in the following main directions: perturbations of the dynamics allowing state dependent moving sets, weakening the time regularity, and/or dropping the convexity of the moving set; see more discussions and references in [4, 10]. To the best of our knowledge, in *all* the publications on the sweeping process the moving set $C(t)$ is *given.* On the other hand, it seems quite natural, from both viewpoints of the theory and applications, to consider *optimal control problems* for the sweeping process, where the moving set $C(t)$ is controlled by some constrained functions to be chosen in order to minimize a given cost. Our paper is devoted to a mathematical formulation and first analysis of this apparently new topic.

Let us provide a more rigorous formulation of the optimal control problem studied in this paper. For simplicity we consider here only the case when the moving set $C(t)$ is controlled by one functional parameter via an inequality constraint

$$C_u(t) := \{x \in \mathbb{R}^n \mid u(t,x) \le 0\}, \quad t \in [0,T], \tag{1.2}$$

where the control $u(t,x)$ belongs to a suitable class $\mathcal{U}$ of functions from $[0,T] \times \mathbb{R}^n$ to $\mathbb{R}$ that are convex in $x$; see below. When $u(\cdot)$ and hence $C(\cdot)$ are fixed, it is well known from [14] that there is a unique sweeping trajectory $x_u$ corresponding to $u$ via (1.1) with the moving set $C_u(\cdot)$ in (1.2). Given a terminal/Mayer cost function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ and a running cost $\ell\colon \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$, consider the following *optimal control problem*:

$$\text{minimize} \quad J[x] := \ \varphi\big(x(T)\big) + \int_0^T \ell\big(x(t), \dot{x}(t)\big)\,dt, \tag{1.3}$$

over trajectories $x = x_u$ of (1.1) with the moving set $C(t) = C_u(t)$ generated by controls $u$ belonging to the prescribed class $\mathcal{U}$.

It is important to emphasize that the formulated dynamic optimization problem is *not* an optimization problem over a differential inclusion of the type $\dot{x} \in F(t,x)$ well-studied in the framework of variational analysis and control theory under Lipschitzian requirements on $F(t,\cdot)$. Indeed, in our case the velocity set $F(t,x) = -N\big(x; C(t)\big)$ is *not fixed*, since the sweeping set $C(t) = C_u(t)$ is different for each control $u \in \mathcal{U}$. Thus in (1.3) we optimize in fact the shape of the set $F(t,x)$, which somehow relates this problem to dynamic *shape optimization*. Observe to this end that when $C(t)$ is fixed in (1.1), it does not make sense to formulate any optimization problem for the differential inclusion

$$\dot{x} \in F(t,x) := -N\big(x; C(t)\big), \quad t \in [0,T],$$

since the latter inclusion admits a *unique* solution for every initial point $x(0) = x_0 \in C(0)$.

In what follows we study the new dynamic optimization problem (1.1)–(1.3) and its specifications by using advanced tools of *variational analysis* and *generalized differentiation*. After presenting some background material in Section 2 we devote Section 3 to establishing verifiable conditions for the *existence* of optimal controls in the problem formulated above. The methods and results developed in this direction are based on one hand on the classical ideas of lower semicontinuity via convexity and coercivity, while on the other hand they employ advanced techniques of *variational convergence*.

The remaining larger part of the paper is mainly devoted to deriving *necessary optimality conditions* for problem (1.1)–(1.3), which is a much harder task than the existence of optimal controls due to the reasons mentioned above. Although the methods developed in the paper work in more general settings, for definiteness and simplicity we concentrate here on the case when the $C(t)$ in (1.3) is a *moving hyperplane*. Note that this particular case of the sweeping process occurs in many practical mechanical applications; see [10, 14]. In this framework the controlled sweeping dynamics in (1.1) is described by

$$\begin{cases} \dot{x}(t) \in -N\big(x(t);C(t)\big) \text{ a.e. } t \in [0,T], \ \ x(0) = x_0 \in C(0) \\ \text{with } \ C(t) := \big\{x \in \mathbb{R}^n \big| \ \langle u(t), x\rangle \le b(t)\big\} \ \text{ and} \\ \|u(t)\| = 1 \ \text{ a.e. } \ t \in [0,T], \end{cases} \tag{1.4}$$

where control actions $u\colon [0,T] \to \mathbb{R}^n$ and $b\colon [0,T] \to \mathbb{R}$ are Lipschitz continuous with Lipschitz constants $L_u \ge 0$ and $L_b \ge 0$, respectively, and where the corresponding trajectories $x\colon [0,T] \to \mathbb{R}^n$ are absolutely continuous. By using the normal cone construction in convex analysis (2.5), we can equivalently rewrite the differential inclusion in (1.4) as

$$\dot{x}(t) \in F\big(x(t), u(t), b(t)\big), \ \ x(0) = x_0,$$

where the velocity mapping $F\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^n$ is fixed being given by

$$F(x,u,b) := \begin{cases} \big\{z \big| \ \langle z, x - v\rangle \le 0 \ \ \forall v \ \text{ s.t. } \ \langle u, v\rangle \le b\big\} & \text{if } \ \langle u, x\rangle \le b, \\ \emptyset & \text{otherwise.} \end{cases} \tag{1.5}$$

Denoting further $y := (x, u, b) \in \mathbb{R}^{2n+1}$, we arrive at the differential inclusion

$$\dot{y}(t) \in G\big(y(t)\big) := F\big(x(t), u(t), b(t)\big) \times \mathbb{R}^n \times \mathbb{R} \tag{1.6}$$

with $F$ defined in (1.5). Observe that the mapping $G$ in (1.6) is *not Lipschitz* continuous and also does not satisfy similar properties of the Lipschitz type (sub-Lipschitz, Lipschitz-like, pseudo-Lipschitz, Aubin continuous) that have been used in optimization problems for differential inclusions; see [3, 7, 9, 13, 19] and the references therein. In fact, the mapping $G$ above is even *discontinuous*. On the other hand, $G$ satisfies (due to the dissipativity of $F$) the *one-sided Lipschitz* (OSL) property in the following sense: there is $L \in \mathbb{R}$ such that

$$\langle y_1 - y_2, z_1 - z_2\rangle \le L\|y_1 - y_2\|^2 \ \text{ for all } \ z_i \in G(y_i) \ \text{ and } \ y_i, \quad i = 1, 2. \tag{1.7}$$

To study the above optimal control problem, we develop the method of *discrete approximations* employed in [11, 13] to derive necessary optimality conditions for Lipschitzian differential inclusions. Note that the convergence of discrete approximations (but not optimality conditions) was established in [5] for OSL differential inclusions under some additional assumptions that are not satisfied in the framework of (1.6). In particular, the assumptions of [5] imply the continuity of $G$, which does not follow from (1.7).

In this paper we follow the discrete approximating scheme of [11] and, by taking into account a particular structure of the mapping $G$ in (1.6), justify the *strong convergence* (in the $W^{1,p}$-norm as $p \geq 1$) of optimal solutions for discrete approximations of the continuous-time problem under consideration to its given optimal trajectory. Then we derive necessary optimality conditions for the discrete-time problems by using appropriate constructions and techniques of generalized differentiation. The results derived for the discrete problems are expressed in terms of the *coderivative* of the mapping $F$ in (1.5), which is fully calculated in the paper via the initial data of the controlled sweeping process (1.4). It allows us to pass to the limit in the necessary optimality conditions for discrete approximations and establish in this way constructive necessary conditions for the original optimal control problem of the sweeping process. The results obtained are illustrated by nontrivial examples, where the derived optimality conditions allow us to explicitly determine optimal solutions.

The rest of the paper (after Section 3) is organized as follows. In Section 4 we justify the possibility of reducing the original unbounded differential inclusion in (1.4) to a *bounded* one under uniform Lipschitzian requirements on control functions; this is widely employed in the sequel. Section 5 deals with calculating the coderivative for a broad class of normal cone mappings that appear, in particular, in necessary optimality conditions for discrete approximations. These coderivative calculations are certainly of independent interest for general variational analysis as well as for other applications.

In Section 6 we construct well-posed discrete approximations of the controlled sweeping process and establish their strong convergence to optimal solutions of (1.3)–(1.4). Finally, Section 7 contains the derivation of necessary optimality conditions for discrete approximations and then for the original continuous-time problem by passing to the limit with the vanishing step of discretization with employing the coderivative calculations. Illustrative examples conclude the paper.

## 2 Tools of Generalized Differentiation

In this section we present some basic definitions and preliminaries on generalized differentiation in variational analysis, which are widely used in the formulations and proofs of the major results. We mainly follow the book [12] and also refer the reader to [2, 13, 16, 17] for related and additional materials. Our notation is standard in variational analysis; see, e.g., [12]. Recall that, for a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the collection

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{ y\in\mathbb{R}^m \Big|\ \exists \text{ sequences } x_k\to\bar{x},\ y_k\to y \text{ such that } y_k\in F(x_k) \text{ for all } k\in I\!N := \{1,2,\ldots\}\Big\} \tag{2.1}$$

is known as the *Kuratowski-Painlevé outer/upper limit* of $F$ at $\bar{x}$. We mention also that $I\!B$ stands for the closed unit ball in the space in question, that $B(x,r)$ denoted the closed ball centered at $x$ with radius $r>0$, and that the symbols "co" and "cone" signify the convex and conic hulls of a set, respectively.

Given a subset $\Omega\subset\mathbb{R}^n$ locally closed around $\bar{x}\in\Omega$, the *Bouligand-Severi tangent/contingent cone* to $\Omega$ at $\bar{x}$ is defined by

$$T(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{t\downarrow 0} \frac{\Omega-\bar{x}}{t}. \tag{2.2}$$

Then the *Fréchet/regular normal cone* to $\Omega$ at $\bar{x}$ can be equivalently defined by

$$\widehat{N}(\bar{x};\Omega) := T^*(\bar{x};\Omega) = \Big\{ v\in\mathbb{R}^n \Big|\ \limsup_{x\xrightarrow{\Omega}\bar{x}} \frac{\langle v, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le 0\Big\}, \tag{2.3}$$

where the notation $\Lambda^*$ stands for the dual/polar operation applied to a set $\Lambda\subset\mathbb{R}^n$, i.e., $\Lambda^* := \{v\in\mathbb{R}^n|\ \langle v,u\rangle\le 0 \text{ for all } u\in\Lambda\}$, and where the symbol "$x\xrightarrow{\Omega}\bar{x}$" means that $x\to\bar{x}$ with $x\in\Omega$. The equivalently defined limiting construction

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x\xrightarrow{\Omega}\bar{x}} \widehat{N}(x;\Omega) = \operatorname*{Lim\,sup}_{x\to\bar{x}} \big\{\operatorname{cone}\big[x-\Pi(x;\Omega)\big]\big\} \tag{2.4}$$

is known as the *Mordukhovich/limiting normal cone* to $\Omega$ at $\bar{x}$, where $\Pi(x;\Omega)$ stands for the Euclidean projection of $x$ onto $\Omega$. When the set $\Omega$ is convex, both normal cones (2.3) and (2.4) reduce to the normal cone of convex analysis

$$N(\bar{x};\Omega) = \widehat{N}(\bar{x};\Omega) = \big\{v\in\mathbb{R}^n \big|\ \langle v, x-\bar{x}\rangle \le 0 \text{ for all } x\in\Omega\big\} \tag{2.5}$$

while, in contrast to (2.3), the limiting normal cone (2.4) is generally nonconvex for simple nonconvex sets, e.g., in both cases of $\Omega := \{(x_1,x_2)\in\mathbb{R}^2|\ x_2 = |x_1|\}$ and $\Omega := \{(x_1,x_2)\in\mathbb{R}^2|\ x_2\ge -|x_1|\}$. Nevertheless, in contrast to (2.3), the limiting normal cone (2.4) and the corresponding subdifferential and coderivative constructions for extended-real-valued functions and set-valued mappings, respectively, satisfy comprehensive *calculus rules* based on the *variational/extremal principles* of variational analysis. Recall that the *subdifferential* of $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ at $\bar{x}$ with $\varphi(\bar{x}) < \infty$ generated by (2.4) is given by

$$\partial\varphi(\bar{x}) := \big\{x \in \mathbb{R}^n \big|\ (v,-1) \in N\big((\bar{x},\varphi(\bar{x})); \operatorname{epi}\varphi\big)\big\} \tag{2.6}$$

and the corresponding *coderivative* of $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is

$$D^*F(\bar{x},\bar{y})(u) := \big\{v \in \mathbb{R}^n \big|\ (v,-u) \in N\big((\bar{x},\bar{y}); \operatorname{gph} F\big)\big\}, \quad u \in \mathbb{R}^m, \tag{2.7}$$

where $\operatorname{epi}\varphi := \{(x,\mu) \in \mathbb{R}^{n+1} |\ \mu \ge \varphi(x)\}$ and $\operatorname{gph} F := \{(x,y) \in \mathbb{R}^{n+m} \big|\ y \in F(x)\}$. We refer the reader to [12, 16] for more details and equivalent representations of (2.6) and (2.7).

## 3 Existence of Optimal Controls

In this section we establish the existence of solutions to the optimal control problem (1.1)–(1.3) for the sweeping process under appropriate assumptions on the initial data. Let us first describe in more details the set $\mathcal{U}$ of feasible controls to (1.1)–(1.3).

Given a time interval $[0,T]$, a point $x_0 \in \mathbb{R}^n$, constants $L > 0$ and $\eta < 0$, and a real function $\psi, \psi_2\colon [0,\infty) \to \mathbb{R}$ satisfying the growth/coercivity conditions

$$\lim_{\rho\to\infty} \psi_i(\rho) = \infty, \quad i = 1,2, \tag{3.1}$$

we define the control set $\mathcal{U}$ by

$$\begin{aligned}
\mathcal{U} := \Big\{&u\colon [0,T]\times\mathbb{R}^n \to \mathbb{R} \text{ such that}\\
&(a)\quad u(\cdot,x) \text{ is } L\text{-Lipschitz for all } x \in \mathbb{R}^n,\\
&(b)\quad u(t,\cdot) \text{ is convex for all } t \in [0,T],\\
&(c)\quad \psi_1(\|x\|) \le u(t,x) \le \psi_2(\|x\|) \text{ for all } t \in [0,T] \text{ and } x \in \mathbb{R}^n,\\
&(d)\quad \text{there is a constant } M > 0 \text{ with the property that for all}\\
&\qquad\ t \in [0,T] \text{ there exists } x_t \in \mathbb{R}^n \text{ such that}\\
&\qquad\ u(t,x_t) \le \eta \text{ and } \|x_t - x_0\| \le M\Big\}.
\end{aligned}$$

Let us prove the compactness of the set of feasible controls in a suitable topology.

**Proposition 3.1 (compactness of feasible controls).** *The control set $\mathcal{U}$ defined above is compact in the topology of the uniform convergence on compact subsets of $[0,T]\times\mathbb{R}^n$.*

**Proof.** Let $\{K_m|\ m \in \mathbb{N}\}$ be a nested sequence of closed balls covering $\mathbb{R}^n$. By conditions (b) and (c) in the definition of the control set $\mathcal{U}$ for each $m \in I\!N$ there is constant $L_m \geq 0$ such that for *every* $u \in \mathcal{U}$ the function $x \to u(t,x)$ is $L_m$-Lipschitzian on $K_m$ whenever $t \in [0,T]$. By (a) and the above remark we can use the Arzelà-Ascoli theorem, which yields that for each $m$ a subsequence $\{u_k^m|\ k \in \mathbb{N}\}$ uniformly converges on $[0,T] \times K_m$. Employing the diagonal process, we find a subsequence $\{u_m\}$ uniformly converging on every compact subset of $[0,T] \times \mathbb{R}^n$ to some function $u$. Our intention is to show that $u \in \mathcal{U}$.

It is easy to observe that $u(\cdot,x)$ is Lipschitz continuous on $[0,T]$ with constant $L$ for all $x \in \mathbb{R}^n$, that $u(t,\cdot)$ is convex on $\mathbb{R}^n$ for all $t \in [0,T]$, and that $u(t,x) \geq \psi(\|x\|)$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$. Thus properties $(a)$, $(b)$, and $(c)$ in the definition of $\mathcal{U}$ are satisfied. To justify the remaining property (d), pick any $t \in [0,T]$ and $m \in \mathbb{N}$ and then select $x_t^m \in B(x_0,M)$ with $u_m(t,x_t^m) \leq \eta$. By taking a converging subsequence for each $t \in [0,T]$, we find a point $x_t$ such that $u(t,x_t) \leq \eta$. Therefore $u \in \mathcal{U}$, which justifies (d) and completes the proof of the proposition. △

The next proposition summarizes the main properties of the set $C_u$ in (1.2) defined for any feasible control $u \in \mathcal{U}$.

**Proposition 3.2 (properties of moving controlled sets).** *Let $u \in \mathcal{U}$, and let $C_u(t)$ be the corresponding moving set defined in* (1.2). *Then we have the following:*

**(C1)** *$C_u(t)$ is nonempty, compact, and convex for all $t \in [0,T]$.*

**(C2)** *There is $R > 0$ depending only on $\psi$ and such that*

$$C_u(t) \subset B(x_t,R) \ \text{ for all } \ t \in [0,T].$$

*The latter implies, whenever $u \in \mathcal{U}$, that*

$$C_u(t) \subset B(x_0,M+R) \ \text{ for all } \ t \in [0,T].$$

**(C3)** *For each $t \in [0,T]$ and each $x \in C_u(t)$ with $u(t,x) = 0$ the normal cone to $C_u(t)$ at $x$ is represented by*

$$N(x;C_u(t)) = \mathbb{R}^+ \partial_x u(t,x),$$

*where $\mathbb{R}^+ := [0,\infty)$, and where $\partial_x u(t,x)$ stands for the subdifferential of convex analysis of $u(t,\cdot)$ taken at the point $x \in C_u(t)$.*

**(C4)** *There is a constant $L' \geq 0$ depending only on $L,T,R,M,\eta$ and such that for all $u \in \mathcal{U}$ the set-valued mapping $t \rightrightarrows C_u(t)$ is Lipschitz continuous on $[0,T]$ with constant $L'$.*

**Proof**. Properties (C1) and (C2) are immediate consequences of conditions (b), (c), and (d) in the definition of $\mathcal{U}$ and of the growth assumption (3.1). Property (C3) follows from the convexity of $u(t,\cdot)$ due to, e.g., [16, Theorem 6.14]. It remains to justify property (C4).

To proceed, observe first that the set-valued mapping $t \rightrightarrows \text{epi}(u(t,\cdot))$ is Lipschitz continuous on $[0,T]$ with constant $L$ by property (a) in the definition of $\mathcal{U}$. Indeed, letting $x \in \mathbb{R}^n$ and $\xi \geq u(t,x)$, we get

$$d((x,\xi),\text{epi}(u(s,\cdot)) \leq |u(t,x)-u(s,x)| \leq L|t-s| \quad \text{for all} \quad t,s \in [0,T].$$

Fix now $x \in C_u(t)$ and $t,s \in [0,T]$, and let $(\bar{x},u(s,\bar{x}))$ be a unique projection of $(x,u(t,x))$ onto the convex set $\text{epi}(u(s,\cdot))$. Observe that

$$\|\bar{x}-x\| + |u(s,\bar{x})-u(t,x)| \leq \sqrt{2}L|s-t|,$$

which gives in turn that

$$\|\bar{x}-x\| \leq \sqrt{2}L|s-t|. \tag{3.2}$$

Recalling that $u(t,x) \leq 0$, the latter yields

$$u(s,\bar{x}) \leq \sqrt{2}L|s-t|. \tag{3.3}$$

If $u(s,\bar{x}) \leq 0$, then $\bar{x} \in C_u(s)$ and we are done. Otherwise, employing (C2) and property (d) from the definition of $\mathcal{U}$ gives us

$$\frac{u(s,\bar{x})-u(s,x_s)}{\|\bar{x}-x_s\|} \geq \frac{-\eta}{\|\bar{x}-x_s\|} \geq \frac{-\eta}{\sqrt{2}LT+R+2M} \tag{3.4}$$

due to $\|\bar{x}-x_s\| \leq \|\bar{x}-x_t\| + \|x_t - x_s\|$. By the continuity of $u(s,\cdot)$ there is a point $\bar{x}_s$ in the segment joining $\bar{x}$ and $x_s$ such that $u(s,\bar{x}_s)=0$, i.e., $\bar{x}_s \in C_u(s)$. The convexity of $u(s,\cdot)$ and property (3.4) imply the estimates

$$\frac{u(s,\bar{x})-u(s,\bar{x}_s)}{\|\bar{x}-\bar{x}_s\|} \geq \frac{u(s,\bar{x})-u(s,x_s)}{\|\bar{x}-x_s\|} \geq \frac{-\eta}{\sqrt{2}LT+R+2M}.$$

On the other hand, it follows from (3.3) that

$$u(s,\bar{x})-u(s,\bar{x}_s) = u(s,\bar{x}) \leq \sqrt{2}L|s-t|,$$

which allows us to arrive at the final estimate

$$\|\bar{x}-\bar{x}_s\| \leq \frac{(\sqrt{2}LT+R+2M)\sqrt{2}L}{-\eta}|t-s|.$$

Combining the latter with (3.2) concludes the proof of the proposition. △

Proposition 3.2 together with the classical theory of the sweeping process (see, e.g., [10]) implies that for every $u \in \mathcal{U}$ the Cauchy problem

$$\begin{cases} \dot{x}(t) & \in -N\big(x(t);C_u(t)\big), \\ x(0) & = x_0 \end{cases} \tag{3.5}$$

admits a unique solution $x_u\colon [0,T] \to \mathbb{R}^n$, which is Lipschitz continuous with constant $L'$.

Now we are ready to establish the existence of optimal controls in problem (1.1)–(1.3) under consideration, which is the main result of this section.

**Theorem 3.3 (existence of optimal controls for the sweeping process).** *In addition to properties* (a)–(d) *of the feasible control set* $\mathcal{U}$*, suppose that both terminal cost* $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *and running cost* $\ell\colon \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ *in* (1.3) *are proper and lower semicontinuous extended-real-valued functions, that* $\ell$ *is bounded from below on bounded sets, and that* $\ell(x,\cdot)$ *is convex for all* $x \in \mathbb{R}^n$*. Then the optimal control problem* (1.1)–(1.3) *admits a solution.*

**Proof.** Having in hand the constants $M$ and $R$ from the definition of $\mathcal{U}$ and from Proposition 3.2, respectively, denote $Q := B(x_0, M+R)$ and consider a minimizing sequence $\{u_m |\ m \in \mathbb{N}\} \subset \mathcal{U}$ for problem (1.1)–(1.3). By construction we have that the corresponding trajectories $x_m$ of (3.5) take values in $Q$ for all $m \in \mathbb{N}$ and $t \in [0,T]$. The compactness of $\mathcal{U}$ from Proposition 3.1 allows us to select a subsequence of $\{u_m\}$ (without relabeling), which uniformly on $[0,T] \times Q$ converge to some $\bar{u} \in \mathcal{U}$. Moreover, it follows from the above and the Newton-Leibniz formula that the corresponding trajectories $x_{u_m} := x_m$ of of (3.5) converges weakly in $W^{1,1}(0,T;\mathbb{R}^n)$ to some $\bar{x}$. Employing the standard semicontinuity results (see, e.g., [1, Theorem 13.1.1] and its proof), we conclude that

$$J[x] \le \liminf_{m\to\infty} J[x_m]$$

for the Bolza functional (1.3) under the assumptions made. To prove that $\bar{x}$ is a global minimizer for (1.1)–(1.3), it remains to show that $\bar{x}$ is a solution to the Cauchy problem (3.5) generated by the control $u = \bar{u}$.

To proceed, fix $t \in [0,T]$ such that the derivatives $\dot{x}_m(t)$ as $m \in I\!N$ and $\dot{x}(t)$ exist and a sequence of convex combinations of $\dot{x}_m(t)$ converges to $\dot{x}(t)$; the latter is possible due the classical Mazur theorem on weak closure and due to the fact that the strong convergence of a sequence in $L^1(0,T;\mathbb{R}^n)$ implies the a.e. convergence of a subsequence. It follows from [16, Theorem 7.17] that the control sequence $u_m(t,\cdot)$ converges epigraphically to $\bar{u}(t,\cdot)$ on $Q$. By Attouch's theorem (see, e.g., [16, Theorem 12.35]), the subdifferentials $\partial_x u_m(t,\cdot)$ converge graphically to $\partial_x \bar{u}(t,\cdot)$ on $Q$. Finally, [16, Theorem 5.37] ensures that $\dot{\bar{x}}(t) \in -N(\bar{x}(t); C_{\bar{u}}(t))$, which thus concludes the proof of this theorem. △

# 4 Reduction to Bounded Differential Inclusion

An underlying feature of the sweeping differential inclusion (1.1) and its specification in (1.4) is the intrinsic *unboundedness* of the right-hand side. However, known results in the theory and applications of differential inclusions deal with either bounded ones or with special Lipschitzian kinds of unboundedness, which is not the case of the set $C(t)$ and the differential inclusion in (1.4). On the other hand, it is proved by Thibault [18] that the unbounded differential inclusion of the sweeping process (1.1) can be reduced to a bounded one, by replacing the normal cone in (1.1) by the scaled subdifferential of the distance function, provided that the moving set $C(t)$ is Lipschitz continuous (or, more generally, absolutely continuous with respect to the Hausdorff distance). Neither of these assumptions holds for the set $C(t)$ in (1.4).

In this section we show, by exploiting a special structure of the control sweeping process in (1.4), that the differential inclusion therein can be equivalently reduced to the bounded one of Thibault's type provided that feasible controls in (1.4) are *uniformly Lipschitzian*, i.e., their Lipschitz constants are are uniformly bounded.

**Theorem 4.1 (uniform boundedness).** *Assume that the Lipschitz constants of all the $b$-controls in* (1.4) *are uniformly bounded by some number $L_b \in (0,\infty)$ and that Lipschitz constants of all the $u$-controls therein are uniformly bounded by some $L_u \in (0,\infty)$. Then there is a number $M > 0$ depending only on $\|x_0\|$, $L_b$, and $L_u$ such that*

$$\|\dot{x}(t)\| \le M \quad \text{a.e. } t \in [0,T] \tag{4.1}$$

*for all feasible trajectories of* (1.4)*, and hence*

$$\|x(t)\| \le \|x_0\| + MT \quad \textit{whenever} \quad t \in [0,T]. \tag{4.2}$$

**Proof.** Using the notation

$$K(t) := \big\{x \in \mathbb{R}^n \big|\ \langle u(t), x(t)\rangle \le 0\big\} \quad \text{and} \quad v(t) := b(t)u(t),$$

we can write the moving set $C(t)$ in the form

$$C(t) = K(t) + v(t),$$

Suppose without loss of generality that $b(0) = \langle u(0), x_0\rangle$ and get the estimate

$$\|b\|_\infty \le \|x_0\| + L_b T.$$

Hence the function $v$ is Lipschitz continuous with Lipschitz constant

$$L := L_b + (\|x_0\| + L_b T)L_u.$$

Denoting by $\pi_C(x)$ a unique projection of $x$ into the closed and convex set $C$, consider the following *discretization algorithm*:

- For $m \in \mathbb{N}$ and $t_m^i := iT/m$ as $0 \le i \le m$, we first set

$$x_m^1 := \pi_{C(t_m^1)}(x_0) \ \text{ and } \ x_m(t) := x_0 + \frac{m}{T}t\Big(x_m^1 - x_0\Big), \quad t \in [0, t_m^1].$$

- Then set inductively

$$x_m^i := \pi_{C(t_m^i)}(x_m^{i-1}) \ \text{ and } \ x_m(t) := x_m^{i-1} + \frac{m}{T}\Big(t - t_m^{i-1}\Big)\Big(x_m^i - x_m^{i-1}\Big),$$

for $t \in [t_m^{i-1}, t_m^i]$.

Denoting further $y_m^i := \pi_{\{K(t_m^i)+v(t_m^{i-1})\}}(x_m^{i-1})$ for $m \in \mathbb{N}$ and $1 \le i \le m$, observe that

$$\begin{aligned}\|y_m^i - x_m^{i-1}\| \quad &\le \quad \|x_m^{i-1} - v(t_m^{i-1})\|\frac{L_u T}{m} = \|y^{i-1} - v(t_m^{i-2})\|\frac{L_u T}{m} \\ &\qquad \text{(since the projection into a subspace decreases the norm)} \\ &\le \quad \|x_m^{i-2} - v(t_m^{i-2})\|\frac{L_u T}{m} \le \ldots \le \|x_0 - v(0)\|\frac{L_u T}{m}\end{aligned}$$

and thus the estimates

$$\|x_m^i - y_m^i\| \le \|v(t_m^i) - v(t_m^{i-1})\| \le L\frac{T}{m}, \quad \|x_m^i - x_m^{i-1}\| \le \frac{LT}{m} + \|x_0 - v(0)\|\frac{L_u T}{m}.$$

The latter ensures that each $x_m(\cdot)$ is Lipschitz continuous with Lipschitz constant

$$M := L + \|x_0 - v(0)\|L_u. \tag{4.3}$$

Since $x_m$ converges uniformly to a unique solution of (1.4) by standard results on the catching up algorithm (see, e.g., [15]), we conclude that for every moving half space $C(t)$ the solution of (1.4) is Lipschitz continuous with the Lipschitz constant given by (4.3). △

The reduction obtained above is crucial in justifying the convergence of discrete approximations and deriving necessary optimality conditions for the optimal control problem of the sweeping process given in Section 6 and Section 7, respectively. The next proposition important in what follows shows how such a truncation affects the coderivatives (2.7) of set-valued mappings that appear in the derivation of necessary optimality conditions.

**Proposition 4.2 (coderivatives under truncation).** *Let* $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *be a set-valued mapping of closed graph, and let* $F_1\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *be defined by*

$$F_1(x) := F(x) \cap I\!B, \quad x \in \mathbb{R}^n. \tag{4.4}$$

*Take any pairs* $(\bar{x}, \bar{y}) \in \operatorname{gph} F_1$ *and* $v \in D^*F_1(\bar{x}, \bar{y})(u)$ *and assume that* $0 \notin D^*F(\bar{x}, \bar{y})(\bar{y})$ *if* $\|\bar{y}\| = 1$. *Then there is a number* $t \geq 0$ *such that*

$$v \in D^*F(\bar{x}, \bar{y})(u + t\bar{y}). \tag{4.5}$$

**Proof.** By the above definitions we have

$$(v, -u) \in N\big((\bar{x}, \bar{y}); \operatorname{gph} F_1\big) \;\text{ and }\; \operatorname{gph} F_1 = \operatorname{gph} F \times (\mathbb{R}^n \times I\!B).$$

It is well known that the normal cone to the closed unit ball $I\!B \subset \mathbb{R}^m$ at $\bar{y} \in I\!B$ with $\|\bar{y}\| = 1$ is computed by

$$N(\bar{y}; I\!B) = \big\{ty^* \in \mathbb{R}^m \big|\, t \geq 0,\ \|y^*\| = 1,\ \langle y^*, \bar{y}\rangle = 1\big\}.$$

On the other hand, we have for the Euclidean norm under consideration that

$$\|y^* - \bar{y}\|^2 = \langle y^*, y^*\rangle + \langle \bar{y}, \bar{y}\rangle - 2\langle y^*, \bar{y}\rangle = 0,$$

i.e., $y^* = \bar{y}$, and we can identify $N(\bar{y}; I\!B) = \{t\bar{y}|\, t \geq 0\}$. Thus

$$N\big((\bar{x}, \bar{y}); \mathbb{R}^n \times I\!B\big) = \begin{cases} \{(0,0)\} \text{ if } \|\bar{y}\| < 1. \\ \{(0, t\bar{y})|\, t \geq 0\} \text{ if } \|\bar{y}\| = 1. \end{cases}$$

By the assumption of $0 \notin D^*F(\bar{x}, \bar{y})(\bar{y})$ for $\|\bar{y}\| = 1$ we have

$$N\big((\bar{x}, \bar{y}); \operatorname{gph} F\big) \cap \big[-N\big((\bar{x}, \bar{y}); \mathbb{R}^n \times I\!B\big)\big] = \{(0,0)\}.$$

Applying the intersection rule for limiting normals (see, e.g., [12, Corollary 3.5]) gives us

$$N\big((\bar{x}, \bar{y}); \operatorname{gph} F_1\big) = N\big((\bar{x}, \bar{y}); \operatorname{gph} F\big) + N\big((\bar{x}, \bar{y}); \mathbb{R}^n \times I\!B),$$

and so there is $t \geq 0$ such that $(v, -u) - (0, t\bar{y}) \in N((\bar{x}, \bar{y}), \operatorname{gph} F)$, which amounts to (4.5) and completes the proof the proposition.

# 5 Calculating Coderivatives of Normal Cone Mappings

This section in entirely devoted to computing the coderivative (2.7) of the normal cone multifunction $F\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^n$ defined by

$$F(x, u, b) := -N\big(x; C(u, b)\big) \;\text{ with }\; C(u, b) := \big\{y \in \mathbb{R}^n \big|\, \langle u, y\rangle \leq b\big\}. \tag{5.1}$$

Note that coderivatives of normal cone mappings accumulate some *second-order* information about the processes under consideration and play a significant role in many aspects of variational analysis, optimization, and control. We refer the reader to [6] and bibliographies therein for calculations of coderivatives of normal cone mappings generated by convex polyhedra as well as for more discussions, and applications. To the best of our knowledge, nothing has been done for calculating coderivatives of more involved mappings (5.1), which is provided in this paper. Our motivation for this issue comes from applications to optimal control of the sweeping process while the results obtained are certainly of their own interest with potential applications to other variational topics.

As mentioned in Section 1, by definition of the normal cone in convex analysis the mapping $F$ in (5.1) can be represented in the explicit form (1.5). Furthermore, denoting

$$\mathbb{R}\{u\} := \big\{\lambda u \big|\ \lambda \in \mathbb{R}\big\},\ \ \mathbb{R}_+\{u\} := \big\{\lambda u \big|\ \lambda \geq 0\big\},\ \text{and}\ \mathbb{R}_-\{u\} := \big\{\lambda u \big|\ \lambda \leq 0\big\},$$

we have the following representation of (5.1):

$$F(x,u,b) = \begin{cases} \mathbb{R}_-\{u\} \ \text{ if } \ \langle u,x\rangle = b, \\ 0 \ \text{ if } \ \langle u,x\rangle < b, \\ \emptyset \ \text{ if } \ \langle u,x\rangle > b \end{cases} \tag{5.2}$$

for every $b \in \mathbb{R}$ and every $0 \neq u \in \mathbb{R}^n$. It follows from (5.2) that $\operatorname{gph} F = \Lambda_1 \cup \Lambda_2$ around any point $(x,u,b,z) \in \operatorname{gph} F$ with $u \neq 0$, where

$$\Lambda_1 := \big\{(x,u,b,z)\big|\ \langle u,x\rangle \leq b,\ z = 0\big\} \ \text{ and} \tag{5.3}$$

$$\Lambda_2 := \big\{(x,u,b,z)\big|\ \langle u,x\rangle = b,\ \exists\, \lambda \leq 0 \ \text{ with } \ z = \lambda u\big\}. \tag{5.4}$$

It is obvious that the set $\Lambda_1$ is closed. To check the closedness for $\Lambda_2$ in (5.4), take sequences $(u_k, z_k) \to (u,z)$ and $\lambda_k \leq 0$ with $z_k = \lambda_k u_k$ as $k \to \infty$ and show that $z = \lambda u$ for some $\lambda \leq 0$. Indeed, by $u \neq 0$ we get that $u^{(i)} \neq 0$ for at least one fixed component $i \in \{1,\ldots,n\}$. Thus $u_k^{(i)} \neq 0$ for all $k \in I\!N$ sufficiently large, and so $0 \geq \lambda_k = z_k^{(i)}/u_k^{(i)} \to z^{(i)}/u^{(i)} \leq 0$. Then it follows from $z_k = \lambda_k u_k$ as $k \to \infty$ that $z = \lambda u$ with $\lambda := z^{(i)}/u^{(i)}$.

Now we proceed, step by step based on the definitions in Section 2, with computing the coderivative of $F$ in terms of the initial data of the sweeping process. Let us start with computing the *contingent cone* (2.2) to the set $\Lambda_1$ and then to $\Lambda_2$. For convenience in this section we use notation $T_\Lambda(\cdot)$ instead of $T(\cdot;\Lambda)$, etc.

**Proposition 5.1 (calculating the contingent cone to $\Lambda_1$).** *Let*

$$(\bar{x},\bar{u},\bar{b},\bar{z}) \in \Lambda_1$$

*with* $\bar{u} \neq 0$ *in* (5.3). *Then we have the expression*

$$T_{\Lambda_1}(\bar{x},\bar{u},\bar{b},\bar{z}) = \begin{cases} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \{0\} & \text{if } \langle \bar{u},\bar{x}\rangle < \bar{b}, \\ \big\{(h,p,l,0\}\big|\ \langle \bar{u},h\rangle + \langle \bar{x},p\rangle \le l\big\} & \text{if } \langle \bar{u},\bar{x}\rangle = \bar{b}. \end{cases} \tag{5.5}$$

**Proof.** The case of $\langle \bar{u},\bar{x}\rangle < \bar{b}$ is obvious. In the other case of $\langle \bar{u},\bar{x}\rangle = \bar{b}$ it is easy to see that $T_{\Lambda_1}(\bar{x},\bar{u},\bar{b},\bar{z}) = T_\Theta(\bar{x},\bar{u},\bar{b}) \times \{0\}$, where

$$\Theta := \{(x,u,b)|\ \langle u,x\rangle \le b\}.$$

Since the gradient of the function $\langle u,x\rangle - b$ is not zero at $(\bar{x},\bar{u},\bar{b})$, we get

$$T_\Theta(\bar{x},\bar{u},\bar{b}) = \big\{(h,p,l\}\big|\ \langle \bar{u},h\rangle + \langle \bar{x},p\rangle \le l\big\}$$

and thus arrive at the claimed formula (5.5). △

**Proposition 5.2 (calculating the contingent cone to $\Lambda_2$).** *Let*

$$(\bar{x},\bar{u},\bar{b},\bar{z}) \in \Lambda_2$$

*with* $\bar{u} \neq 0$ *in* (5.4). *Then we have*

$$T_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z}) =$$

$$\begin{cases} \big\{(h,p,l,m\}\big|\ \langle \bar{u},h\rangle + \langle \bar{x},p\rangle = l,\ \lambda p - m \in \mathbb{R}\{\bar{u}\}\big\} & \text{if } \bar{z} \neq 0, \\ \big\{(h,p,l,m\}\big|\ \langle \bar{u},h\rangle + \langle \bar{x},p\rangle = l,\ m \in \mathbb{R}_-\{\bar{u}\}\big\} & \text{if } \bar{z} = 0, \end{cases} \tag{5.6}$$

*where the multiplier* $\lambda < 0$ *is uniquely defined by* $\bar{z} = \lambda\bar{u}$.

**Proof.** It follows from definition (2.2) of the contingent cone that

$$T_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z}) = \big\{(h,p,l,m\}\big|\ \exists\, t_k \downarrow 0,\ \exists\,(h_k,p_k,l_k,m_k) \to (h,p,l,m) \text{ s.t.}$$
$$(\bar{x}+t_kh_k, \bar{u}+t_kp_k, \bar{b}+t_kl_k, \bar{z}+t_km_k) \in \Lambda_2\big\} \text{ for all } k \in I\!N,$$

which by the definition of $\Lambda_2$ in (5.4) amounts to

$$T_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z}) = \big\{(h,p,l,m\}\big|\ \exists\, t_k \downarrow 0,\ \exists\,(h_k,p_k,l_k,m_k) \to (h,p,l,m),$$
$$\exists\,\lambda_k \le 0 \text{ s.t. } \langle \bar{u}+t_kp_k, \bar{x}+t_kh_k\rangle = \bar{b}+t_kl_k,\ \bar{z}+t_km_k = \lambda_k(\bar{u}+t_kp_k)\big\}. \tag{5.7}$$

Dividing by $t_k$ and passing to the limit in (5.7) as $k \to \infty$, we get $\langle \bar{u},h\rangle + \langle \bar{x},p\rangle = l$. Observing further that

$$\bar{u}+t_kp_k \to \bar{u} \neq 0, \quad \bar{z}+t_km_k \to \bar{z},$$

it follows similarly to the above argument that $\lambda_k \to \lambda \le 0$. In particular, $\bar{z} = \lambda\bar{u}$ and thus $\lambda < 0$ if $\bar{z} \neq 0$. It also follows from (5.7) that

$$t_k^{-1}(\lambda - \lambda_k)\bar{u} = \lambda_kp_k - m_k \text{ for all } k \in I\!N.$$

Since $\bar{u} \neq 0$ and $\lambda_k p_k - m_k \to \lambda p - m$, we get that $t_k^{-1}(\lambda - \lambda_k) \to \gamma$ for some $\gamma \in \mathbb{R}$. Thus $\gamma \bar{u} = \lambda p - m$. If $\bar{z} = 0$, then $\lambda = 0$ and therefore $t_k^{-1}(-\lambda_k) \to \gamma \geq 0$ by $\lambda_k \leq 0$). Summarizing all the above, we arrive at the inclusion "$\subset$" in (5.6).

Next let us justify the reverse inclusion in (5.6) considering first the case of $\bar{z} \neq 0$. In this case $\bar{z} = \lambda \bar{u}$ with some $\lambda < 0$. Take $h, p, l, m$ such that $\lambda p - m \in \mathbb{R}\{\bar{u}\}$ and

$$\langle \bar{u}, h \rangle + \langle \bar{x}, p \rangle = l. \tag{5.8}$$

In particular, we get $\lambda p - m = \gamma \bar{u}$ for some $\gamma \in \mathbb{R}$. Putting

$$\begin{aligned} t_k &:= k^{-1}, \\ \lambda_k &:= \lambda - k^{-1}\gamma, \\ h_k &:= (\lambda/\lambda_k)h, \\ p_k &:= (\lambda/\lambda_k)p, \\ l_k &:= (\lambda/\lambda_k)l + k^{-1}\langle p_k, h_k \rangle, \\ m_k &:= m \;\text{ for all }\; k \in I\!N, \end{aligned}$$

observe that $t_k \downarrow 0$, $\lambda_k \to \lambda$ (hence $\lambda_k < 0$ for sufficiently large $k \in I\!N$) and that $(h_k, p_k, l_k, m_k) \to (h, p, l, m)$ as $k \to \infty$. It follows furthermore that

$$\bar{z} + t_k m_k = \lambda \bar{u} + k^{-1} m = \lambda \bar{u} + k^{-1}(\lambda p - \gamma \bar{u}) = \lambda_k \bar{u} + k^{-1}\lambda_k p_k = \lambda_k(\bar{u} + t_k p_k),$$

which implies by using (5.8) the relationships

$$\begin{aligned} \langle \bar{u} + t_k p_k, \bar{x} + t_k h_k \rangle &= \bar{b} + k^{-1}\langle \bar{x}, p_k \rangle + k^{-1}\langle \bar{u}, h_k \rangle + k^{-2}\langle h_k, p_k \rangle \\ &= \bar{b} + k^{-1}(\lambda/\lambda_k)(\langle \bar{x}, p \rangle + \langle \bar{u}, h \rangle) + k^{-2}\langle h_k, p_k \rangle \\ &= \bar{b} + k^{-1}(\lambda/\lambda_k)l + k^{-2}\langle h_k, p_k \rangle \\ &= \bar{b} + t_k l_k \;\text{ for all }\; k \in I\!N. \end{aligned}$$

This shows that $(h, p, l, m) \in T_{\Lambda_2}(\bar{x}, \bar{u}, \bar{b}, \bar{z})$ via (5.7) and thus justifies the claimed reverse inclusion "$\supset$" in (5.6) in the case of $\bar{z} \neq 0$.

In the remaining case of $\bar{z} = 0$, take $h, p, l, m$ such that $m \in \mathbb{R}_{-}\{\bar{u}\}$ and that (5.8) holds. Then $m = \gamma \bar{u}$ for some $\gamma \leq 0$. Putting further

$$\begin{aligned} t_k &:= k^{-1}, \\ \lambda_k &:= k^{-1}\gamma, \\ h_k &:= h, \\ p_k &:= p, \\ l_k &:= l + k^{-1}\langle p_k, h_k \rangle, \\ m_k &:= m + \lambda_k p \;\text{ for all }\; k \in I\!N, \end{aligned}$$

observe that $t_k \downarrow 0$, $\lambda_k \to 0$, $\lambda_k \le 0$, and $(h_k,p_k,l_k,m_k) \to (h,p,l,m)$ as $k \to \infty$. Moreover

$$\bar z + t_k m_k = k^{-1}(m+\lambda_k p) = k^{-1}(\gamma\bar u + \lambda_k p) = \lambda_k(\bar u + t_k p_k)$$

and, exploiting again the additional relation (5.8), we arrive at

$$\begin{aligned}\langle \bar u + t_k p_k, \bar x + t_k h_k\rangle &= \bar b + k^{-1}\langle \bar x, p_k\rangle + k^{-1}\langle \bar u, h_k\rangle + k^{-2}\langle p_k, h_k\rangle\\ &= \bar b + k^{-1} l + k^{-2}\langle p_k, h_k\rangle\\ &= \bar b + t_k l_k \quad \text{for all} \quad k \in I\!N.\end{aligned}$$

It shows that $(h,p,l,m) \in T_{\Lambda_2}(\bar x,\bar u,\bar b,\bar z)$ via (5.7), and thus the inclusion "$\subset$" in (5.6) holds also in the case of $\bar z = 0$. This completes the proof of the proposition. △

Now we are ready to compute the contingent cone to the graph of the original set-valued mapping (5.1) arising in the sweeping process.

**Proposition 5.3 (calculating the contingent cone to the graph of $F$).** *Take any $(\bar x,\bar u,\bar b,\bar z) \in \operatorname{gph} F$ from the graph of $F$ in* (5.1). *The following assertions hold:*

**(i)** *If $\langle \bar u, \bar x\rangle < \bar b$, then*

$$T_{\operatorname{gph} F}(\bar x,\bar u,\bar b,\bar z) = \big\{(h,p,l,m)\big|\ m = 0\big\}.$$

**(ii)** *If $\langle \bar u, \bar x\rangle = \bar b$ and $\bar z \ne 0$, then*

$$T_{\operatorname{gph} F}(\bar x,\bar u,\bar b,\bar z) = \big\{(h,p,l,m)\big|\ \langle \bar u, h\rangle + \langle \bar x, p\rangle = l,\ \lambda p - m \in \mathbb{R}\{\bar u\}\big\},$$

*where $\lambda < 0$ is uniquely defined by $\bar z = \lambda \bar u$.*

**(iii)** *If $\langle \bar u, \bar x\rangle = \bar b$ and $\bar z = 0$, then*

$$\begin{aligned}T_{\operatorname{gph} F}(\bar x,\bar u,\bar b,\bar z) = \big\{(h,p,l,m\}\big|\ &\big[m = 0,\ \langle \bar u, h\rangle + \langle \bar x, p\rangle \le l\big]\\ &or\ \big[\,\langle \bar u, h\rangle + \langle \bar x, p\rangle = l,\ m \in \mathbb{R}_-\{\bar u\}\big]\big\}.\end{aligned}$$

**Proof.** Recall that due to representation (5.1) of the mapping $F$ we have $\operatorname{gph} F = \Lambda_1 \cup \Lambda_2$, and this set is closed around the reference point as follows from the arguments above. In case (i) we easily get $T_{\Lambda_1\cup\Lambda_2}(\bar x,\bar u,\bar b,\bar z) = T_{\Lambda_1}(\bar x,\bar u,\bar b,\bar z)$, and the result follows from Proposition 5.1. In case (ii) we similarly have $T_{\Lambda_1\cup\Lambda_2}(\bar x,\bar u,\bar b,\bar z) = T_{\Lambda_2}(\bar x,\bar u,\bar b,\bar z)$, where $\lambda < 0$ due to $\bar u \ne 0$ and $\bar z \ne 0$ in this case. Then the result follows from Proposition 5.2. Finally, in case (iii) we have

$$T_{\Lambda_1\cup\Lambda_2}(\bar x,\bar u,\bar b,\bar z) = T_{\Lambda_1}(\bar x,\bar u,\bar b,\bar z) \cup T_{\Lambda_2}(\bar x,\bar u,\bar b,\bar z),$$

and thus it is a combination of Proposition 5.1 and Proposition 5.2. △

Next we compute the *regular normal cone* (2.3) to the graph of $F$ by using the polarity/duality correspondence with the contingent cone computed in Proposition 5.3. Taking into account this duality we use the $*$-notation for normal vectors.

**Proposition 5.4 (calculating the regular normal cone to the graph of $F$).** *Given any $(\bar{x},\bar{u},\bar{b},\bar{z})\in \operatorname{gph} F$ for the normal cone mapping* (5.1)*, the following assertions hold:*

**(i)** *If $\langle\bar{u},\bar{x}\rangle<\bar{b}$, then*

$$\widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{u},\bar{b},\bar{z})=\big\{(h^*,p^*,l^*,m^*)\big|\ (h^*,p^*,l^*)=(0,0,0)\big\}.$$

**(ii)** *If $\langle\bar{u},\bar{x}\rangle=\bar{b}$ and $\bar{z}\neq 0$, then*

$$\widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{u},\bar{b},\bar{z})=$$

$$\big\{(h^*,p^*,l^*,m^*)\big|\ (h^*,p^*+\lambda m^*,l^*)\in\mathbb{R}\{(\bar{u},\bar{x},-1)\},\langle m^*,\bar{u}\rangle=0\big\},$$

*where the multiplier $\lambda<0$ is uniquely defined by $\bar{z}=\lambda\bar{u}$.*

**(iii)** *If $\langle\bar{u},\bar{x}\rangle=\bar{b}$ and $\bar{z}=0$, then*

$$\widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{u},\bar{b},\bar{z})=$$

$$\big\{(h^*,p^*,l^*,m^*)\big|\ (h^*,p^*,l^*)\in\mathbb{R}_+\{(\bar{u},\bar{x},-1)\},\langle m^*,\bar{u}\rangle\geq 0\big\}.$$

**Proof.** The first assertion is obvious. To justify (ii), we use the polar representation of $\widehat{N}$ via the contingent cone $T$. Taking into account that $\widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{u},\bar{b},\bar{z})=\widehat{N}_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z})$ in this case gives the expression

$$\begin{aligned}\widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{u},\bar{b},\bar{z})=&\big\{(h^*,p^*,l^*,m^*)\big|\ \langle h^*,h\rangle+\langle p^*,p\rangle+\langle l^*,l\rangle+\langle m^*,m\rangle\leq 0\\ &\forall\, h,p,l,m\ \text{ s.t. }\ \langle\bar{u},h\rangle+\langle\bar{x},p\rangle=l,\ \lambda p-m\in\mathbb{R}\{\bar{u}\}\big\}.\end{aligned}$$

Let $(h^*,p^*,l^*,m^*)\in\widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{u},\bar{b},\bar{z})=\widehat{N}_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z})$. Then

$$\langle h^*,h\rangle+\langle p^*,p\rangle+\langle l^*,l\rangle+\langle m^*,\lambda p-\gamma\bar{u}\rangle\leq 0$$

for all $h,p,l$ such that $\langle\bar{u},h\rangle+\langle\bar{x},p\rangle=l\ \forall\gamma\in\mathbb{R}$.

Putting $h=p=l=0$ here gives us $\langle m^*,-\gamma\bar{u}\rangle\leq 0$ for all $\gamma\in\mathbb{R}$, which implies that $\langle m^*,\bar{u}\rangle=0$. Then the above relationship reads as

$$\langle h^*,h\rangle+\langle p^*+\lambda m^*,p\rangle+\langle l^*,l\rangle\leq 0\ \text{ for all }\ h,p,l\ \text{ with }\ \langle\bar{u},h\rangle+\langle\bar{x},p\rangle=l.$$

Since $\bar{u} \neq 0$, the set $\{(h,p,l) | \langle \bar{u}, h\rangle + \langle \bar{x}, p\rangle = l\}$ describes a closed hyperplane whose negative polar is $\mathbb{R}\{(\bar{u},\bar{x},-1)\}$. Hence we get

$$(h^*, p^* + \lambda m^*, l^*) \in \mathbb{R}\big\{(\bar{u},\bar{x},-1)\big\} \ \text{ for all } \ (h^*,p^*,l^*,m^*) \in \widehat{N}_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z}).$$

Conversely, choose any $(h^*,p^*,l^*,m^*)$ satisfying

$$(h^*, p^* + \lambda m^*, l^*) \in \mathbb{R}\{(\bar{u},\bar{x},-1)\}$$

and such that $\langle m^*, \bar{u}\rangle = 0$ holds. Picking an arbitrary element $(h,p,l,m) \in T_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z})$, we have $\langle \bar{u}, h\rangle + \langle \bar{x}, p\rangle = l$ with $\lambda p - m \in \mathbb{R}\{\bar{u}\}$. Hence $\lambda p - m = \gamma \bar{u}$ for some $\gamma \in \mathbb{R}$ and $(h^*, p^* + \lambda m^*, l^*) = \mu(\bar{u},\bar{x},-1)$ for some $\mu \in \mathbb{R}$; therefore

$$\begin{aligned}
&\langle h^*, h\rangle + \langle p^*, p\rangle + \langle l^*, l\rangle + \langle m^*, m\rangle \\
&= \langle h^*, h\rangle + \langle p^* + \lambda m^*, p\rangle + \langle l^*, l\rangle - \gamma\langle m^*, \bar{u}\rangle \\
&= \mu(\langle \bar{u}, h\rangle + \langle \bar{x}, p\rangle - l) = 0.
\end{aligned}$$

The latter implies that $(h^*,p^*,l^*,m^*) \in \widehat{N}_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z})$, which justifies assertion (ii).

To prove (iii), recall that $\widehat{N}_{\mathrm{gph}\,F}(\bar{x},\bar{u},\bar{b},\bar{z}) = \widehat{N}_{\Lambda_1 \cup \Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z})$ in this case and that the dual of a set union equals to the intersection of the duals. Thus

$$\begin{aligned}
\widehat{N}_{\mathrm{gph}\,F}(\bar{x},\bar{u},\bar{b},\bar{z}) =& \widehat{N}_{\Lambda_1}(\bar{x},\bar{u},\bar{b},\bar{z}) \cap \widehat{N}_{\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z}) \\
=& \big\{(h^*,p^*,l^*,m^*)\big|\ \langle h^*, h\rangle + \langle p^*, p\rangle + \langle l^*, l\rangle + \langle m^*,\ m\rangle \le 0 \\
& \ \forall h,p,l,m \ \text{ s.t. } \ m = 0 \ \text{ and } \ \langle \bar{u}, h\rangle + \langle \bar{x}, p\rangle \le l\big\} \bigcap \\
& \big\{(h^*,p^*,l^*,m^*)\big|\ \langle h^*, h\rangle + \langle p^*, p\rangle + \langle l^*, l\rangle + \langle m^*, m\rangle \le 0 \\
& \ \forall h,p,l,m \ \text{ with } \ \langle \bar{u}, h\rangle + \langle \bar{x}, p\rangle = l,\ m \in \mathbb{R}_-\{\bar{u}\}\big\}.
\end{aligned}$$

To calculate the first of the duals above, take $(h^*,p^*,l^*,m^*)$ satisfying

$$\langle h^*, h\rangle + \langle p^*, p\rangle + \langle l^*, l\rangle + \langle m^*, m\rangle \le 0$$

for all $h,p,l,m$ with $m = 0,\ \langle \bar{u}, h\rangle + \langle \bar{x}, p\rangle \le l$ and thus arrive at the relationship

$$\langle h^*, h\rangle + \langle p^*, p\rangle + \langle l^*, l\rangle \le 0 \ \text{ for all } \ h,p,l \ \text{ with } \ \langle \bar{u}, h\rangle + \langle \bar{x}, p\rangle \le l.$$

The set $\{(h,p,l) | \langle \bar{u}, h\rangle + \langle \bar{x}, p\rangle \le l\}$ describes a closed halfspace whose dual is $\mathbb{R}_+\{(\bar{u},\bar{x},-1)\}$. Hence the first dual is contained in the set $\mathbb{R}_+\{(\bar{u},\bar{x},-1)\} \times \mathbb{R}^n$.

Conversely, pick $(h^*,p^*,l^*,m^*) \in \mathbb{R}_+\{(\bar{u},\bar{x},-1)\} \times \mathbb{R}^n$ and $(h,p,l,m)$ such that $m=0$ and $\langle \bar{u},h\rangle + \langle \bar{x},p\rangle \le l$. Then $(h^*,p^*,l^*) = \mu(\bar{u},\bar{x},-1)$ for some $\mu \ge 0$, and so

$$\begin{aligned}\langle h^*,h\rangle + \langle p^*,p\rangle + \langle l^*,l\rangle + \langle m^*,m\rangle &= \langle h^*,h\rangle + \langle p^*,p\rangle + \langle l^*,l\rangle \\ &= \mu(\langle \bar{u},h\rangle + \langle \bar{x},p\rangle - l) \le 0.\end{aligned}$$

This shows that the first dual above coincides with the set $\mathbb{R}_+\{(\bar{u},\bar{x},-1)\} \times \mathbb{R}^n$.

It remains to calculate the second dual above. To proceed, take $h^*,p^*,l^*$ and $m^*$ such that

$$\langle h^*,h\rangle + \langle p^*,p\rangle + \langle l^*,l\rangle + \langle m^*,m\rangle \le 0$$

for all $h,p,l,m$ such that $\langle \bar{u},h\rangle + \langle \bar{x},p\rangle = l$ and $m \in \mathbb{R}_-\{\bar{u}\}$, which is equivalent to the relationship

$$\langle h^*,h\rangle + \langle p^*,p\rangle + \langle l^*,l\rangle + \gamma\langle m^*,\bar{u}\rangle \le 0$$

for all $\gamma \in \mathbb{R}_-$ and for all $h,p,l$ s.t. $\langle \bar{u},h\rangle + \langle \bar{x},p\rangle = l$.

Putting $h=p=l=0$, we derive from here that $\langle m^*,\bar{u}\rangle \ge 0$. Putting further $\gamma = 0$, the relationship above implies also that

$$\langle h^*,h\rangle + \langle p^*,p\rangle + \langle l^*,l\rangle \le 0 \ \ \forall h,p,l \ \text{ s.t. } \ \langle \bar{u},h\rangle + \langle \bar{x},p\rangle = l.$$

Arguing similarly to the previous case gives us $(h^*,p^*,l^*) \in \mathbb{R}\{(\bar{u},\bar{x},-1)\}$, which means that the second dual under consideration is contained in the set $\mathbb{R}\{(\bar{u},\bar{x},-1)\} \times \{m^* | \langle m^*,\bar{u}\rangle \ge 0\}$.

Conversely, for $(h^*,p^*,l^*,m^*)$ belonging to the latter set and for $(h,p,l,m)$ with $\langle \bar{u},h\rangle + \langle \bar{x},p\rangle = l$ and $m \in \mathbb{R}_-\{\bar{u}\}$ we get $(h^*,p^*,l^*) = \mu(\bar{u},\bar{x},-1)$ with some $\mu \in \mathbb{R}$ and $\gamma \in \mathbb{R}_-$ such that $m = \gamma\bar{u}$. This gives therefore that

$$\begin{aligned}\langle h^*,h\rangle + \langle p^*,p\rangle + \langle l^*,l\rangle + \langle m^*,m\rangle &= \langle h^*,h\rangle + \langle p^*,p\rangle + \langle l^*,l\rangle + \gamma\langle m^*,\bar{u}\rangle \\ &\le \langle h^*,h\rangle + \langle p^*,p\rangle + \langle l^*,l\rangle \\ &= \mu(\langle \bar{u},h\rangle + \langle \bar{x},p\rangle - l) = 0,\end{aligned}$$

which shows that the second dual coincides with the set $\mathbb{R}\{(\bar{u},\bar{x},-1)\} \times \{m^* | \langle m^*,\bar{u}\rangle \ge 0\}$. Intersection of this set with the set $\mathbb{R}_+\{(\bar{u},\bar{x})\} \times \mathbb{R}^n$ yields the asserted formula in the third case and thus completes the proof of the proposition. △

Passing to the limit from regular normals, we calculate next the limiting normal cone (2.4) to the graph of the normal cone mapping (5.1).

**Proposition 5.5 (calculating the limiting normal cone to the graph of $F$).** *Let $(\bar{x},\bar{u},\bar{b},\bar{z})\in\operatorname{gph}F$ belong to the graph of $F$ in (5.1). The following assertions hold:*

**(i)** *If $\langle\bar{u},\bar{x}\rangle<\bar{b}$, then*

$$N_{\operatorname{gph}F}(\bar{x},\bar{u},\bar{b},\bar{z})=\big\{(h^*,p^*,l^*,m^*)\big|\;(h^*,p^*,l^*)=(0,0,0)\big\}.$$

**(ii)** *If $\langle\bar{u},\bar{x}\rangle=\bar{b}$ and $\bar{z}\neq 0$, then*

$$N_{\operatorname{gph}F}(\bar{x},\bar{u},\bar{b},\bar{z})=$$

$$\big\{(h^*,p^*,l^*,m^*)\big|\;(h^*,p^*+\lambda m^*,l^*)\in\mathbb{R}\{(\bar{u},\bar{x},-1)\},\;\langle m^*,\bar{u}\rangle=0\big\},$$

*where $\lambda<0$ is uniquely defined by $\bar{z}=\lambda\bar{u}$.*

**(iii)** *If $\langle\bar{u},\bar{x}\rangle=\bar{b}$ and $\bar{z}=0$, then*

$$\begin{aligned}N_{\operatorname{gph}F}(\bar{x},\bar{u},\bar{b},\bar{z}) \quad = \quad & \big\{(h^*,p^*,l^*,m^*)\big|\;\big[h^*=p^*=l^*=0\big]\quad\text{or}\\ & \big[(h^*,p^*,l^*)\in\mathbb{R}\{(\bar{u},\bar{x},-1)\}\quad\text{and}\quad\langle m^*,\bar{u}\rangle=0\big],\quad\text{or}\\ & \big[(h^*,p^*,l^*)\in\mathbb{R}_+\{(\bar{u},\bar{x},-1)\}\quad\text{and}\quad\langle m^*,\bar{u}\rangle\geq 0\big]\big\}.\end{aligned}$$

**Proof.** It follows from the explicit formula (5.2) for the mapping $F$ that in the first two cases the limiting normal cone to the graph of $F$ agrees with the regular one, and so assertions (i) and (ii) of this proposition reduce to those in Proposition 5.4. In case (iii) we represent the limiting normal cone to the graph of $F$ as

$$N_{\operatorname{gph}F}(\bar{x},\bar{u},\bar{b},\bar{z})=N_{\Lambda_1\cup\Lambda_2}(\bar{x},\bar{u},\bar{b},\bar{z}=A\cup B\cup C$$

with the sets $A$, $B$, and $C$ defined by

$$\begin{aligned}A \quad &:= \quad \mathop{\operatorname{Lim\,sup}}_{\substack{(x,u,b,z)\to(\bar{x},\bar{u},\bar{b},0)\\(x,u,b,z)\in\Lambda_1\setminus\Lambda_2}}\widehat{N}_{\Lambda_1\cup\Lambda_2}(x,u,b,z)=\mathop{\operatorname{Lim\,sup}}_{\substack{(x,u,b,z)\to(\bar{x},\bar{u},\bar{b},0)\\(x,u,b,z)\in\Lambda_1\setminus\Lambda_2}}\widehat{N}_{\Lambda_1}(x,u,b,z)\\ B \quad &:= \quad \mathop{\operatorname{Lim\,sup}}_{\substack{(x,u,b,z)\to(\bar{x},\bar{u},\bar{b},0)\\(x,u,b,z)\in\Lambda_2\setminus\Lambda_1}}\widehat{N}_{\Lambda_1\cup\Lambda_2}(x,u,b,z)=\mathop{\operatorname{Lim\,sup}}_{\substack{(x,u,b,z)\to(\bar{x},\bar{u},\bar{b},0)\\(x,u,b,z)\in\Lambda_2\setminus\Lambda_1}}\widehat{N}_{\Lambda_2}(x,u,b,z)\\ C \quad &:= \quad \mathop{\operatorname{Lim\,sup}}_{\substack{(x,u,b,z)\to(\bar{x},\bar{u},\bar{b},0)\\(x,u,b,z)\in\Lambda_1\cap\Lambda_2}}\widehat{N}_{\Lambda_1\cup\Lambda_2}(x,u,b,z)\,.\end{aligned}$$

Let us calculate subsequently all the three sets $A$, $B$, and $C$. It immediately follows from assertion (i) of Proposition 5.4 that

$$A=\widehat{N}_{\Lambda_1}(x,u,b,z)=\{(0,0,0)\}\times\mathbb{R}^n. \tag{5.9}$$

Next we justify the formula

$$B=\big\{(h^*,p^*,l^*,m^*)\big|\;(h^*,p^*,l^*)\in\mathbb{R}\{(\bar{u},\bar{x},-1)\},\,\langle m^*,\bar{u}\rangle=0\big\}. \tag{5.10}$$

To proceed, pick any $(h^*, p^*, l^*, m^*) \in B$ and find by definition sequences $(x_k, u_k, b_k, z_k) \to (\bar{x}, \bar{u}, \bar{b}, 0)$ and $(h^*_k, p^*_k, l^*_k, m^*_k) \to (h^*, p^*, l^*, m^*)$ as $k \to \infty$ satisfying $(x_k, u_k, b_k, z_k) \in \Lambda_2 \backslash \Lambda_1$ and $(h^*_k, p^*_k, l^*_k, m^*_k) \in \widehat{N}_{\Lambda_2}(x_k, u_k, b_k, z_k)$ for all $k \in I\!N$. By Proposition 5.4(ii) we have the relationships $\langle m^*_k, u_k \rangle = 0$ and

$$(h^*_k, p^*_k + \lambda_k m^*_k, l^*_k) = \mu_k(u_k, x_k, -1), \quad k \in I\!N,$$

with $\lambda_k \leq 0$ uniquely defined by $z_k = \lambda_k u_k$ and with some $\mu_k \in \mathbb{R}$. As in the proof of the closedness of $\Lambda_2$ above, we get that $\lambda_k \to \lambda$ for some $\lambda \leq 0$ uniquely defined by $\bar{z} = \lambda \bar{u}$. Since $z_k \to 0$, it follows that $\lambda = 0$ and moreover $\langle m^*, \bar{u} \rangle = 0$. Next we see that $\mu_k(u_k, x_k, -1) \to (h^*, p^*, l^*)$. Hence $\mu_k \to -l^*$ and so

$$\mu_k u_k \to -l^* \bar{u} = h^* \ \text{ and } \ \mu_k x_k \to -l^* \bar{x} = p^* \ \text{ as } \ k \to \infty.$$

Consequently we arrive at

$$(h^*, p^*, l^*) = -l^*(\bar{u}, \bar{x}, -1) \in \mathbb{R}\{(\bar{u}, \bar{x}, -1)\},$$

which thus justifies the inclusion "$\subset$" of (5.10).

To prove the converse inclusion "$\supset$" in (5.10), let $(h^*, p^*, l^*, m^*)$ be such that $(h^*, p^*, l^*) \in \mathbb{R}\{(\bar{u}, \bar{x}, -1)\}$ and $\langle m^*, \bar{u} \rangle = 0$. Combining this with

$$(\bar{x}, \bar{u}, \bar{b}, -k^{-1}\bar{u}) \in \Lambda_2 \backslash \Lambda_1, \quad k \in I\!N,$$

gives us $(\bar{x}, \bar{u}, \bar{b}, -k^{-1}\bar{u}) \to (\bar{x}, \bar{u}, \bar{b}, 0)$ as $k \to \infty$. It follows from Proposition 5.4(ii) that

$$(h^*, p^* + k^{-1}m^*, l^*, m^*) \in \widehat{N}_{\Lambda_2}(\bar{x}, \bar{u}, \bar{b}, -k^{-1}\bar{u}), \quad k \in I\!N.$$

Indeed, taking $-k^{-1}$ for $\lambda$ in Proposition 5.4(ii), we derive the claimed relationship from

$$(h^*, p^* + k^{-1}m^* + (-k^{-1}m^*), l^*) = (h^*, p^*, l^*) \in \mathbb{R}\{(\bar{u}, \bar{x}, -1)\}.$$

By the obvious convergence

$$(h^*, p^* + k^{-1}m^*, l^*, m^*) \to (h^*, p^*, l^*, m^*) \ \text{ as } \ k \to \infty$$

we arrive at $(h^*, p^*, l^*, m^*) \in B$ and thus get representation (5.10).

To justify assertion (iii), it remains to show that

$$C = \big\{(h^*, p^*, l^*, m^*) \big|\, (h^*, p^*, l^*) \in \mathbb{R}_+\{(\bar{u}, \bar{x}, -1)\}, \ \langle m^*, \bar{u} \rangle \geq 0\big\}. \quad (5.11)$$

The proof of the inclusion "$\subset$" in (5.11) follows exactly the same lines as the proof of the corresponding inclusion in (5.10). The converse inclusion in

(5.11) is evident since the right-hand side of (5.11) equals $\widehat{N}_{\Lambda_1\cup\Lambda_2}(\bar{x},\bar{u},\bar{b},0)$ according to Proposition 5.4(iii). On the other hand, we have

$$\widehat{N}_{\Lambda_1\cup\Lambda_2}(\bar{x},\bar{u},\bar{b},0)\subset C$$

by definition of the set $C$ and due to $(\bar{x},\bar{u},\bar{b},0)\in\Lambda_1\cap\Lambda_2$. Unifying the above representations in $A\cup B\cup C$ yields the claimed formula in case (iii) and completes the proof of the proposition. △

Finally, we arrive at the main result of this section, which gives us exact formulas for calculating the coderivative (2.7) of the normal cone mapping (5.1) and plays a significant role in the implementation of the method of discrete approximations to derive necessary optimality conditions for the controlled sweeping process (1.4) in the subsequent sections.

**Theorem 5.6 (calculating the coderivative of the normal cone mapping).** *Let $(\bar{x},\bar{u},\bar{b},\bar{z})\in\operatorname{gph}F$ for the normal cone mapping* (5.1). *The following assertions hold:*

**(i)** *If $\langle\bar{u},\bar{x}\rangle<\bar{b}$, then*

$$D^*F(\bar{x},\bar{u},\bar{b},\bar{z})(m^*)=\{(0,0,0)\}.$$

**(ii)** *If $\langle\bar{u},\bar{x}\rangle=\bar{b}$ and $\bar{z}\neq 0$, then*

$$D^*F(\bar{x},\bar{u},\bar{b},\bar{z})(m^*)=\begin{cases}\mathbb{R}\{(\bar{u},\bar{x},-1)\}-\{(0,\lambda m^*,0)\} & \text{for}\quad \langle m^*,\bar{u}\rangle=0,\\ \emptyset \quad \text{for}\quad \langle m^*,\bar{u}\rangle\neq 0,\end{cases}$$

*where $\lambda<0$ is uniquely defined by $\bar{z}=\lambda\bar{u}$.*

**(iii)** *If $\langle\bar{u},\bar{x}\rangle=\bar{b}$ and $\bar{z}=0$, then*

$$D^*F(\bar{x},\bar{u},\bar{b},\bar{z})(m^*)=\begin{cases}\mathbb{R}\{(\bar{u},\bar{x},-1)\} \quad \text{for}\quad \langle m^*,\bar{u}\rangle=0,\\ \mathbb{R}_+\{(\bar{u},\bar{x},-1)\} \quad \text{for}\quad \langle m^*,\bar{u}\rangle<0,\\ \{(0,0,0)\} \quad \text{for}\quad \langle m^*,\bar{u}\rangle>0.\end{cases}$$

**Proof.** It follows directly from definition (2.7) of the coderivative and the calculation of the limiting normal cone (2.4) in Proposition 5.5. △

Now we consider the *truncation* of the normal cone mapping (5.1) defined by

$$F_1(x,u,b):=F(x,u,b)\cap I\!B=-N\big(x;C(u,b)\big)\cap I\!B \tag{5.12}$$

with the controlled moving set $C(u,b)=\{y\in\mathbb{R}^n|\ \langle u,y\rangle\leq b\}$. The next proposition calculates the coderivative of $F_1$ by using Theorem 5.6 and the calculus result of Proposition 4.2.

**Proposition 5.7 (coderivative of the truncated mapping).** *Let*

$$(\bar{x},\bar{u},\bar{b},\bar{z}) \in \operatorname{gph} F_1$$

*for the truncated mapping* (5.12). *Then the coderivative of* $F_1$ *is expressed via the coderivative of* $F$ *in Theorem* 5.6 *as*

$$D^*F_1\left(\bar{x},\bar{u},\bar{b},\bar{z}\right)(m^*) = \begin{cases} D^*F\left(\bar{x},\bar{u},\bar{b},\bar{z}\right)(m^*) & \text{if } \|\bar{z}\| < 1, \\ \bigcup_{t\geq 0} D^*F\left(\bar{x},\bar{u},\bar{b},\bar{z}\right)(m^*+t\bar{z}) & \text{if } \|\bar{z}\| = 1. \end{cases} \tag{5.13}$$

**Proof.** The formula in the case of $\|\bar{z}\| < 1$ in (5.13) is obvious, since in this case the mapping $F_1$ in (5.12) is not different locally from $F$. In the case of $\|z\| = 1$ we apply Proposition 4.2 to our mapping $F_1$, which has the structure of (4.4). To proceed, let us check that the qualification condition

$$0 \notin D^*F(\bar{x},\bar{u},\bar{b},\bar{z})(\bar{z}) \ \text{ if } \ \|\bar{z}\| = 1 \tag{5.14}$$

holds for the mapping $F$ from (5.1). Indeed, we employ Theorem 5.6 in case (ii), since the requirement $\|\bar{z}\| = 1$ automatically yields that $\langle \bar{u},\bar{x}\rangle = \bar{b}$. Thus the negation of the qualification condition (5.14) reads in this case as

$$(\alpha\bar{u},\alpha\bar{x}-\lambda\bar{z},-\alpha) = 0 \ \text{ if } \ \langle \bar{z},\bar{u}\rangle = 0 \ \text{ for some } \ \alpha \in \mathbb{R} \tag{5.15}$$

with $\lambda < 0$ satisfying $\bar{z} = \lambda\bar{u}$. It follows from (5.15) that $\lambda\bar{z} = 0$, which contradicts the conditions $\lambda < 0$ and $\bar{z} \neq 0$. Finally, we employ formula (4.5) from Proposition 4.2 that gives us the second expression in (5.13) and completes the proof of the proposition. △

Thus the coderivative of $F_1$ is calculated by the explicit formulas of Theorem 5.6 if $\|\bar{z}\| < 1$. In the case of $\|\bar{z}\| = 1$ in Proposition 5.7 we arrive at the following formulas.

**Corollary 5.8 (calculating the coderivative of the truncated mapping).** *Let* $(\bar{x},\bar{u},\bar{b},\bar{z}) \in \operatorname{gph} F_1$ *with* $\|\bar{z}\| = 1$ *in* (5.12). *Then we have*

$$D^*F_1(\bar{x},\bar{u},\bar{b},\bar{z})(m^*) =$$

$$\begin{cases} \mathbb{R}\big\{(\bar{u},\bar{x},-1)\big\} - \big\{\big(0,\lambda(m^*-\lambda^2\langle m^*,\bar{u}\rangle\bar{u})\big)\big\} & \text{if } \ \langle m^*,\bar{u}\rangle \geq 0, \\ \emptyset & \text{if } \ \langle m^*,\bar{u}\rangle < 0, \end{cases}$$

*where the number* $\lambda < 0$ *is uniquely defined by* $\bar{z} = \lambda\bar{u}$.

**Proof.** As mentioned above, in the setting of $\|\bar{z}\| = 1$ we automatically have $\langle \bar{u},\bar{x}\rangle = \bar{b}$ and thus case (ii) of Theorem 5.6 is applied. Observe first that

$$\langle m^*+t\bar{z},\bar{u}\rangle = \langle m^*,\bar{u}\rangle + t/\lambda \ \text{ for all } \ t \geq 0. \tag{5.16}$$

Since $\lambda < 0$, the assumption $\langle m^*, \bar{u}\rangle < 0$ yields that $\langle m^* + t\bar{z}, \bar{u}\rangle < 0$ for all $t \geq 0$, and thus the combination of (5.13) with Theorem 5.6(ii) implies that $D^*F_1\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)(m^*) = \emptyset$. Under the other assumption of $\langle m^*, \bar{u}\rangle \geq 0$ we get from (5.16) that

$$\langle m^* + t\bar{z}, \bar{u}\rangle = 0 \Longleftrightarrow t = -\lambda \langle m^*, \bar{u}\rangle \geq 0.$$

This implies in turn by (5.13) and Theorem 5.6(ii) that

$$\begin{aligned} D^*F_1\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)(m^*) &= D^*F\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)\left(m^* - \lambda \langle m^*, \bar{u}\rangle \bar{z}\right) \\ &= D^*F\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)\left(m^* - \lambda^2 \langle m^*, \bar{u}\rangle \bar{u}\right) \\ &= \mathbb{R}\big\{(\bar{u}, \bar{x}, -1)\big\} - \big\{\big(0, \lambda(m^* - \lambda^2\langle m^*, \bar{u}\rangle\bar{u}), 0\big)\big\}, \end{aligned}$$

which completes the proof of the corollary. △

Another useful consequence of Proposition 5.7 via Corollary 5.8 important in the sequel is a full characterization of the *kernel* for $D^*F_1(\bar{x}, \bar{u}, \bar{b}, \bar{z})$ we get next.

**Corollary 5.9 (characterization of the coderivative kernel for the truncated mapping).** *Let* $(\bar{x}, \bar{u}, \bar{b}, \bar{z}) \in \operatorname{gph} F_1$ *for the truncated mapping* (5.12)*. The following hold:*

**(i)** *If* $\bar{z} = 0$*, then*

$$(0,0,0) \in D^*F_1\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)(m^*) \quad \textit{for all} \quad m^* \in \mathbb{R}^n.$$

**(ii)** *If* $0 < \|\bar{z}\| < 1$*, then*

$$(0,0,0) \in D^*F_1\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)(m^*) \Longleftrightarrow m^* = 0.$$

**(iii)** *If* $\|\bar{z}\| = 1$*, then*

$$(0,0,0) \in D^*F_1\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)(m^*) \Longleftrightarrow m^* \in \mathbb{R}_+\{\bar{u}\}.$$

**Proof.** The result for $\bar{z} = 0$ in (i) follows immediately from Proposition 5.7 and cases (i) and (iii) in Theorem 5.6. In case (ii) of this corollary we get from Proposition 5.7 and Theorem 5.6(ii) that

$$(0,0,0) \in D^*F_1\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)(m^*) \Longrightarrow (0,0,0) = \mu\left(\bar{u}, \bar{x}, -1\right) - (0, \lambda m^*, 0)$$

for some $\mu \in \mathbb{R}$ and $\lambda < 0$ uniquely defined by $\bar{z} = \lambda\bar{u}$. This yields that $\mu = 0$ and hence $m^* = 0$. Conversely, it follows from $m^* = 0$ by Proposition 5.7 and Theorem 5.6(ii) that

$$D^*F_1\left(\bar{x}, \bar{u}, \bar{b}, \bar{z}\right)(0) = \mathbb{R}\big\{\,(\bar{u}, \bar{x}, -1)\,\big\} \ni (0,0,0).$$

Finally, if $\|\bar{z}\| = 1$ in case (iii), then Corollary 5.8 ensures that

$$(0,0,0) \in D^*F_1\left(\bar{x},\bar{u},\bar{b},\bar{z}\right)(m^*) \Longrightarrow \langle m^*, \bar{u}\rangle \geq 0$$

and

$$(0,0,0) = \mu\left(\bar{u},\bar{x},-1\right) - \left(0, \lambda\left(m^* - \lambda^2 \langle m^*, \bar{u}\rangle \bar{u}\right), 0\right)$$

for some $\mu \in \mathbb{R}$ and $\lambda < 0$ uniquely defined by $\bar{z} = \lambda \bar{u}$. As before, we get from the above that $\mu = 0$ and hence

$$m^* = \lambda^2 \langle m^*, \bar{u}\rangle \bar{u} \in \mathbb{R}_+ \{\bar{u}\}.$$

Assuming conversely that $m^* \in \mathbb{R}_+\{\bar{u}\}$ implies that $\langle m^*, \bar{u}\rangle \geq 0$ and $m^* = \lambda^2 \langle m^*, \bar{u}\rangle \bar{u}$. Thus applying Corollary 5.8 yields in this case that

$$D^*F_1\left(\bar{x},\bar{u},\bar{b},\bar{z}\right)(m^*) = \mathbb{R}\big\{ (\bar{u},\bar{x},-1) \big\} \ni (0,0,0),$$

which completes the proof of this corollary. △

# 6 Well-Posed Discrete Approximations

In this section we develop the method of discrete approximations to study the control sweeping process (1.4) and the dynamic optimization problem for it formulated in Section 1. Our *standing assumptions* are those in Theorem 4.1, and we always suppose with no loss of generality that $M = 1$ therein. Along with the original normal cone mapping $F(x,u,b)$ from (5.1), equivalently represented in (5.2), we consider its truncation $F_1$ defined by (5.12).

Our first result is about a certain *strong approximation* of an arbitrary feasible solution to the differential inclusion in (1.4), or equivalently in (1.6), by feasible solutions to the corresponding *finite-difference* inclusions piecewise linearly extended to the continuous-time interval. This result is of its own interest while playing a crucial role in the justification *well-posedness* (stability, convergence) of discrete approximations and the implementation of this method to deriving necessary optimality conditions for the controlled sweeping process.

To the best of our knowledge, the strong approximation type obtained for general differential inclusions (see [5, 11, 13] and the references therein) cannot be applied to the sweeping system under consideration since the Lipschitz continuity assumption of [11, 13] and the "modified one-sided Lipschitz condition" (MOSL) of [5] are not satisfied in this framework. The proof of the following theorem is based on the extension of the approach in [11] to *non-Lipschitzian* and *discontinuous* differential inclusions with taking into account a specific structure of the sweeping process (1.4).

**Theorem 6.1 (discrete approximations of feasible solutions).** *Let $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$ be an arbitrary feasible solution to the controlled sweeping system* (1.4)*, and let*

$$\Delta_k := \{0 = t_0^k < t_1^k < \ldots < t_k^k = T\} \ \text{with} \ h_k := \max_{0 \le j \le k-1} \{t_{j+1}^k - t_j^k\} \downarrow 0 \tag{6.1}$$

*as $k \to \infty$ be an arbitrary partition of $[0,T]$ for each $k \in I\!N$. Then there exists a sequence of piecewise linear function $z^k(\cdot) = (x^k(t), u^k(t), b^k(t))$ on $[0,T]$ with $\|u^k(t_j^k)\| = 1$ for $j = 0, \ldots, k-1$, satisfying the discretized inclusions*

$$x^k(t) = x^k(t_j) + (t - t_j)v_j^k, \ \ x(0) = x_0, \ \ t_j^k \le t \le t_{j+1}^k, \quad j = 0, \ldots, k-1, \tag{6.2}$$

*with $v_j^k \in F_1(z^k(t_j^k))$ on $\Delta_k$ for all $k \in I\!N$ and such that*

$$z^k(t) \to \bar{z}(t) \ \ uniformly \ \ on \ \ [0,T] \ \ and \ \ \int_0^T \|\dot{z}^k(t) - \dot{\bar{z}}(t)\|\, dt \to 0 \ \ as \ \ k \to \infty. \tag{6.3}$$

*The latter implies the convergence $\dot{z}^k(t) \to \dot{\bar{z}}(t)$ of some subsequence for a.e. $t \in [0,T]$.*

**Proof.** By Theorem 4.1 with $M = 1$ we have the estimates

$$\|\dot{\bar{z}}(t)\| \le 1 \ \text{ a.e. } \ t \in [0,T] \ \text{ and } \ \|\bar{z}(t)\| \le \|x_0\| + T \ \text{ on } \ [0,T]. \tag{6.4}$$

Thus $\bar{z}(\cdot)$ is in fact a feasible solution to the truncated differential inclusion with the replacement of $F(\cdot) = -N(x(\cdot); C(\cdot))$ in (1.4) (and equivalently in (1.6)) by $F_1(\cdot)$ from (5.12). Due to the density of step functions in $L^1([0,T]; \mathbb{R}^{2n})$ we approximate $(\dot{\bar{x}}(t), (\dot{\bar{u}}(t))$ strongly in $L^1([0,T]; \mathbb{R}^{2n})$ by a sequence of step functions $(w_1^k(t), w_2^k(t))$, which are bounded in $L^1([0,T]; \mathbb{R}^{2n})$ and constant on the intervals $[t_j^k, t_{j+1}^k)$, $j = 0, \ldots, k-1$, from the sequence of partitions (6.1). Then define the sequence of pairs

$$\big(y_1^k(t), y_2^k(t)\big) := \bar{z}(0) + \int_0^t \big(w_1^k(s), w_2^k(s)\big)\, ds, \quad t \in [0,T],$$

and easily observe by construction and the Lebesgue dominated convergence theorem that

$$\big(y_1^k(t), y_2^k(t)\big) \to \big(\bar{x}(t), \bar{u}(t)\big) \ \text{ uniformly } \ \text{in } t \in [0,T] \ \text{ as } \ k \to \infty.$$

Define next $y_3^k(\cdot)$ to be piecewise linear on $[0,T]$ and satisfying the following conditions on $\Delta_k$: $\big[\langle \bar{x}(t_j^k), \bar{u}(t_j^k)\rangle = \bar{b}(t_j^k)\big] \Longrightarrow \big[\langle y_1^k(t_j^k), y_2^k(t_j^k)\rangle = y_3^k(t_j^k)\big]$ and

$\big[\langle\bar{x}(t_j^k),\bar{u}(t_j^k)\rangle < \bar{b}(t_j^k)\big] \Longrightarrow \big[\langle y_1^k(t_j^k),y_2^k(t_j^k)\rangle < y_3^k(t_j^k)\big]$. Then we have the convergence $y^k(t) := (y_1^k(t),y_2^k(t),y_3^k(t)) \to \bar{z}(t) := (\bar{x}(t),\bar{u}(t),\bar{b}(t))$ uniformly on $[0,T]$ and $w^k(t) := (w_1^k(t),w_2^k(t),w_3^k(t)) \to (\dot{\bar{x}}(t),\dot{\bar{u}}(t),\dot{\bar{b}}(t))$ strongly in $L^1([0,T];\mathbb{R}^{2n+1})$ as $k\to\infty$.

Fix any $\varepsilon > 0$. Since $w^k(t) \to \dot{\bar{z}}(t)$ a.e. on $[0,T]$ along a subsequence of $k\to\infty$ and since $w^k(\cdot)$ are piecewise constant functions while $\bar{z}(\cdot)$ is a feasible solution to (1.4), for any $k \in I\!N$ and $j \in \{0,\ldots,k-1\}$ we can select a numerical sequence $\{s_m\}_{m\in I\!N} \subset [t_j^k,t_{j+1}^k)$ converging to $t_j^k$ and such that

$$\Big\|w_1^k(s_m)-\dot{\bar{x}}(s_m)\Big\| \le \varepsilon/2 \ \text{ whenever } \ j=0,\ldots,k-1 \ \text{ and } \ k\in I\!N$$

and that the differential inclusion in (1.4) holds at $t = s_m$ for every $m \in I\!N$. Recall that $w_1^k(\cdot)$ is a constant function on $[t_j^k,t_{j+1}^k)$, i.e., $w_1^k \equiv a_j^k$ on $[t_j^k,t_{j+1}^k)$. Then we have $\mathrm{dist}(a_j^k;F(\bar{z}(s_m)) \le \varepsilon/2$ for $j = 0,\ldots,k-1$. Letting now $m\to\infty$ gives us $\mathrm{dist}(a_j^k;F_1(\bar{z}(t_j^k)) \le \varepsilon/2$. Observe that the above constructions of $y_3^k(\cdot)$ and $y^k(\cdot)$ imply that $d_{\mathbb{R}^n}(F_1(y^k(t_j^k);F_1(\bar{z}(t_j^k)) \le \varepsilon/2$ for all $k \in I\!N$ sufficiently large, where $d_{R^n}(\cdot;\cdot)$ stands for the Hausdorff distance between compact subsets in $\mathbb{R}^n$. This ensures the estimate

$$\mathrm{dist}\big(a_j^k;F(y^k(t_j^k))\big) \le \varepsilon \ \text{ for all } \ j=0,\ldots,k-1 \ \text{ and } \ k\in I\!N. \tag{6.5}$$

Fix $k \in I\!N$ and define $u^k(t) := y_2^k(t)$ for $t \in [0,T]$. By normalization we can always achieve the constraints $\|u^k(t)\| = 1$ on $\Delta_k$. Construct now required trajectories $x^k(t)$ for inclusions (6.2) and the control functions $b^k(t)$ on $\Delta_k$ denoting for simplicity $t_j := t_j^k$, $j = 0,\ldots,k-1$. We proceed by induction assuming that $x^k(t_j)$ is known and that $\|x^k(t_j)-y_1^k(t_j)\| \le h_k\varepsilon$; it is always the case for $j = 0$. Then choose $b^k(t_j) := \langle x^k(t_j),u^k(t_j)\rangle$ if $y_3^k(t_j) = \langle y_1^k(t_j),y_2^k(t_j)\rangle$ and $b^k(t_j) < \langle x^k(t_j),u^k(t_j)\rangle$ if $y_3^k(t_j) < \langle y_1^k(t_j),y_2^k(t_j)\rangle$. By

$$\begin{aligned}|\langle x^k(t_j),u^k(t_j)\rangle-\langle y_1^k(t_j),y_2^k(t_j)\rangle| &= |\langle x^k(t_j),y_2^k(t_j)\rangle-\langle y_1^k(t_j),y_2^k(t_j)\rangle|\\ &= |\langle x^k(t_j)-y_1^k(t_j),y_2^k(t_j)\rangle|\\ &\le \|x^k(t_j)-y_1^k(t_j)\|\cdot\|y_2(t_j)\| \le h_k\varepsilon\end{aligned}$$

we have $|b^k(t_j)-y_3^k(t_j)| < 2h_k\varepsilon$. Extend now the discrete trajectory $x^k(\cdot)$ to the whole interval $(t_j,t_{j+1}]$ as follows. Select a Euclidean projection

$$v_j^k \in \Pi\big(a_j^k;F_1\big(y^k(t_j)\big)\big)$$

and by $v_j^k \in F_1(y^k(t_j))$ get from (6.5) that $\|v_j^k-a_j^k\| \le \varepsilon$. It follows from the above choice of $b^k(t_j)$, from $y_2^k(t_j) = u^k(t_j)$, and from the normal cone structure of the mapping $F_1$ (exploited also in the proof of Proposition 6.2) that

$$v_j^k \in F_1\big(y^k(t_j)\big) = F_1\big(x^k(t_j),u^k(t_j),b^k(t_j)\big),$$

and thus inclusion (6.7) is satisfied at $t_j$. Extending next $x^k(t) := x^k(t_j) + (t - t_j)v_j^k$ to the interval $t \in [t_j, t_{j+1}]$, observe that

$$\|\dot{y}_1^k(t) - \dot{x}^k(t)\| = \|a_j^k - v_j^k\| \le \varepsilon \ \text{ for all } \ t \in (t_j, t_{j+1}),$$

which implies in turn that

$$\|y_1^k(t) - x^k(t\| \le |t - t_j| \cdot \|a_j^k - v_j^k\| \le h_k\varepsilon \ \text{ for all } \ t \in [t_j, t_{j+1}].$$

This allows us to define $b^k(t_{j+1})$ in the same way as $b^k(t_j)$ and get the induction estimate $|b^k(t_{j+1}) - y_3^k(t_{j+1})| < 2h_k\varepsilon$. Extending further $b^k(t)$ linearly to $[t_j, t_{j+1}]$ yields that

$$\begin{aligned}
|\dot{b}^k(t) - \dot{y}_3^k(t)| &= \Big|\frac{b^k(t_{j+1}) - b^k(t_j)}{t_{j+1} - t_j} - \frac{y_3^k(t_{j+1}) - y_3^k(t_j)}{t_{j+1} - t_j}\Big| \\
&= \Big|\frac{b^k(t_{j+1}) - y_3^k(t_{j+1})}{t_{j+1} - t_j} - \frac{b^k(t_j) - y_3^k(t_j)}{t_{j+1} - t_j}\Big| \\
&\le \frac{|b^k(t_{j+1}) - y_3^k(t_{j+1})|}{t_{j+1} - t_j} + \frac{|b^k(t_j) - y_3^k(t_j)|}{t_{j+1} - t_j} \\
&\le \frac{2h_k\varepsilon}{h_k} + \frac{2h_k\varepsilon}{h_k} = 4\varepsilon, \quad t \in (t_j, t_{j+1}).
\end{aligned}$$

Finally, putting $z^k(t) := (x^k(t), u^k(t), b^k(t)),\ t \in [0, T]$, we conclude from the constructions and arguments above that $z^k(\cdot) \to \bar{x}(\cdot)$ in the norm of $L^1([0, T]; \mathbb{R}^{2n+1})$ as $k \to \infty$, which justifies (6.3) and completes the proof of the theorem. △

Note that Theorem 6.1 concerns just the controlled sweeping process (1.4) while not its optimization. It establishes the approximation of *any* feasible solution to (1.4). Having this result in hand, we are able to construct a sequence of discrete-time optimization problems whose optimal solutions exist and strongly approximate a given optimal solution to the original dynamic optimization problem (1.3)–(1.4) labeled as $(P)$ from now on.

Let $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$ be an optimal solution to problem $(P)$, and let $F_1$ be defined in (5.12). We construct the following sequence of discrete-time optimization problem $(P_k)$, $k \in I\!N$, with $h_k \downarrow 0$ as $k \to \infty$:

$$\begin{aligned}
&\text{minimize} \quad J_k[z^k] := \varphi(x_k^k) + h_k \sum_{j=0}^{k-1} \ell\Big(x_j^k, \frac{x_{j+1}^k - x_j^k}{h_k}\Big) \\
&+ \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Big\| \frac{(x_{j+1}^k, u_{j+1}^k, b_{j+1}^k) - (x_j^k, u_j^k, b_j^k)}{h_k} - \Big(\dot{\bar{x}}(t), \dot{\bar{u}}(t), \dot{\bar{b}}(t)\Big)\Big\|^2 dt
\end{aligned} \tag{6.6}$$

over elements $z^k := (x_0^k, x_1^k, \ldots, x_k^k, u_0^k, u_1^k, \ldots, u_{k-1}^k, b_0^k, b_1^k, \ldots, b_{k-1}^k)$ satisfying the constraints

$$x_{j+1}^k \in x_j^k + h_k F_1(x_j^k, u_j^k, b_j^k), \quad j = 0, \ldots, k-1, \quad \text{with} \quad x_0^k = x_0, \tag{6.7}$$

$$\|u_j^k\| = 1 \quad \text{for} \quad j = 0, \ldots, k-1. \tag{6.8}$$

Note that the index $j = 0, \ldots, k$ plays a role of the discrete time in $(P_k)$ and that inclusions (6.7) correspond to those in (6.2) at the partition points.

First of all we need to make sure that that problems $(P_k)$ admit optimal solutions. It is not difficult to check under general assumptions imposed on the cost functions.

**Proposition 6.2 (existence of discrete optimal solutions).** *Suppose that the cost functions $\varphi$ and $\ell$ in* (6.6) *are lower semicontinuous. Then for each $k \in I\!N$ problem $(P_k)$ admits an optimal solution.*

**Proof.** Fix $k \in I\!N$ and observe that the set of feasible solutions to $(P_k)$ is obviously nonempty and bounded by (5.12), (6.7), and (6.8). Thus it remains to show that it is closed and then to apply the classical Weierstrass existence theorem. To proceed, take a sequence of

$$z_m = (x_0, x_{1m}, \ldots, x_{mk}, u_{0m}, \ldots, u_{(k-1)m}, b_{0m}, \ldots, b_{(k-1)m})$$

convergent to $z = (x_0, x_1, \ldots, x_k, u_0, \ldots, u_{k-1}, b_0, \ldots, b_{k-1})$ as $m \to \infty$ omitting the upper index "$k$" for simplicity. We need to check that $z$ is feasible to $(P_k)$ provided that all $z_m$ have this property. This only requires checking that the components of the limiting vector $z$ satisfy inclusions (6.7) for all $j$; it is obvious for constraints (6.8). Consider the two possible cases for every $j = 0, \ldots, k-1$:

**(a)** If $\langle x_j, u_j \rangle < b_j$, then for $m \in I\!N$ sufficiently large we also have $\langle x_{jm}, u_{jm} \rangle < b_{jm}$. It immediately follows from the normal cone definition that $x_{(j+1)m} = x_{jm}$ and thus $x_{j+1} = x_j$ for large $m \in I\!N$, i.e., inclusion (6.7) is satisfied for the limiting discrete trajectory.

**(b)** Let $\langle x_j, u_j \rangle = b_j$. Taking into account that

$$F_1(x, u, b) = \big\{ -\alpha u \big|\ 0 \le \alpha \le 1 \big\} \quad \text{whenever} \quad \langle x, u \rangle = b \tag{6.9}$$

by the construction of (5.12) and that the triple $(x_{jm}, u_{jm}, b_{jm})$ satisfies (6.7), we get that

$$x_{(j+1)m} = x_{jm} + h_k(-\alpha_m)u_{jm} \quad \text{for all} \quad m \in I\!N$$

along a sequence of $0 \le \alpha_m \le 1$, which converges without loss of generality to some number $\alpha \in [0, 1]$. This implies that $x_{j+1} = x_j + h_k(-\alpha)u_j$ by passing to the limit at $m \to \infty$. Employing (6.9) again, we justify that $z$ is a feasible solution to problem $(P_k)$ for each $k \in I\!N$, which thus completes the proof of the proposition. $\triangle$

Our next goal is to establish an appropriate strong convergence of optimal solutions of discrete approximations $(P_k)$ to the given optimal solution $\bar{z}(\cdot)$ of the original sweeping control problem $(P)$. To proceed, we need some amount of *relaxation stability* of the original problem. Along with $(P)$, consider the *relaxed sweeping control problem* $(R)$ given by

$$\text{minimize } \widehat{J}[z] := \varphi\big(x(T)\big) + \int_0^T \widehat{\ell}_F\big(x(t), \dot{x}(t)\big)\,dt$$

subject to all the constraints in the controlled sweeping process (1.4), where $\widehat{\ell}_F(z,v)$ is the *convexification* of $\ell_F$ in the $v$ variable, i.e., the largest convex and lower semicontinuous function majorized by $\ell_F(z,\cdot)$ for each $x$, and where

$$\ell_F(z,v) := \ell(x,v) + \delta\big(v; F_1(z)\big)$$

is defined via the set indicator function $\delta(v;F)$ equal to 0 if $v \in F$ and to $\infty$ otherwise. Denoting by $J_P$ and $\widehat{J}_R$ the optimal value (infimum) of the cost functionals in $(P)$ and $(P_R)$, respectively, we always have that $\widehat{J}_R \le J_P$. Furthermore, it follows from Theorem 3.3 and its proof that the minimum is achieved in $(R)$ under our standing assumptions of Theorem 4.1 provided in addition that the terminal cost function $\varphi$ is lower semicontinuous.

We say that the original problem $(P)$ is *stable with respect to relaxation* if the equality $J_P = \widehat{J}_R$ holds. Note that it is always the case when the running cost $\ell(x,\cdot)$ is lower semicontinuous and convex in the velocity variable for each $x$. Also, this property is known to be automatically satisfied (as yet another manifestation of "hidden convexity" of continuous-time control systems) for nonconvex differential inclusions with no endpoint constraints under Lipschitzian or MOSL assumptions; see [5, 11, 13] for precise results, discussions, and references. As mentioned above, the latter Lipschitz-type assumptions are not fulfilled for the sweeping process under consideration. However, we *conjecture* that the relaxation stability automatically holds for $(P)$ without any convexity of $\ell(x,\cdot)$ due to specific features of the controlled sweeping process exploited partly in the proof of Theorem 6.1; so far we keep relaxation stability as an assumption in the next theorem.

**Theorem 6.3 (strong convergence of discrete solutions).** *Let* $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$ *be an optimal solution to problem* $(P)$*, which is stable with respect to relaxation. Assume in addition that both terminal and running costs in* (1.3) *are continuous at* $\bar{x}(T)$ *and at* $(\bar{x}(t), \dot{\bar{x}}(t))$ *a.e.* $t \in [0,T]$*, respectively. Then any sequence of optimal solutions* $\bar{z}^k(t) = (\bar{x}^k(t), \bar{u}^k(t), \bar{b}^k(t))$ *to* *discrete problems* $(P_k)$ *piecewise linearly extended to* $[0,T]$ *converges to* $\bar z(t)$ *strongly in the space* $W^{1,p}([0,T];\mathbb{R}^{2n+1})$ *as* $k\to\infty$ *whenever* $p\in[1,\infty)$.

**Proof.** Due to the uniform boundedness results of Theorem 4.1 the strong convergence $\bar z^k(\cdot)\to\bar z(\cdot)$ in $W^{1,p}([0,T];\mathbb{R}^{2n+1})$ as $k\to\infty$ is invariant with respect to all $p\in[1,\infty)$; so it is in fact sufficient to show that

$$\lim_{k\to\infty}\int_0^T \big\|\dot{\bar z}^k(t)-\dot{\bar z}(t)\big\|\,dt=0. \tag{6.10}$$

Arguing by contradiction, suppose that (6.10) does not hold and then, using the classical Dunford theorem on the weak compactness in $L^1([0,T];\mathbb{R}^{2n+1})$, find a number $\gamma>0$ and a function $v(\cdot)\in L^1([0,T];\mathbb{R}^{2n+1})$ such that

$$\lim_{k\to\infty}\int_0^T \big\|\dot{\bar z}^k(t)-\dot{\bar z}(t)\big\|\,dt\to\gamma \ \text{ and } \ \dot{\bar z}^k(\cdot)\to v(\cdot) \tag{6.11}$$

weakly in $L^1([0,T];\mathbb{R}^{2n+1})$ along a subsequence of $k\in I\!N$, which we identify as usual with the whole natural series. Defining an absolutely continuous function $\widetilde z\colon[0,T]\to\mathbb{R}^{2n+1}$ by

$$\widetilde z(t):=\big(x_0,\bar u(0),\bar b(0)\big)+\int_0^T v(s)\,ds \ \text{ for all } \ t\in[0,T],$$

we easily get that $\dot{\widetilde z}(t)=v(t)$ a.e. $t\in[0,T]$ and that $\dot{\bar z}^k(\cdot)\to\dot{\widetilde z}(\cdot)$ weakly in $L^1([0,T];\mathbb{R}^{2n+1})$ as $k\to\infty$. Furthermore, the convexity of the values of $F_1(\cdot)$ in (5.12) and the classical Mazur theorem on weak closure imply that $\widetilde z(\cdot)$ is a feasible trajectory to problem $(P)$ and hence to its relaxation $(R)$.

Let $\{z^k(\cdot)\}_{k\in I\!N}$ be a sequence of feasible solutions to $(P_k)$ strongly approximating $\bar z(\cdot)$ by Theorem 6.1. Since $\bar z_k(\cdot)$ is an optimal solution to $(P_k)$ for each $k\in I\!N$, we have

$$J_k[\bar z^k]\le J_k[z^k],\quad k\in I\!N. \tag{6.12}$$

It follows from the strong convergence in Theorem 6.1, the continuity assumptions of this theorem, and the Lebesgue dominated convergence theorem that

$$J_k[z_k]\to J[\bar z] \ \text{ as } \ k\to\infty.$$

On the other hand, the arguments above and the construction of $(R)$ ensure that

$$\widehat J_R[\widetilde z]=\varphi\big(\widetilde x(T)\big)+\int_0^T \widehat\ell\big(\widetilde x(t),\dot{\widetilde x}(t)\big)\,dt\le\liminf_{k\to\infty}J[\bar z^k]$$

for the first component of $\widetilde{z}(\cdot)$. Passing finally to the limit in (6.12) as $k \to \infty$ with taking into account (6.11) and the relaxation stability of $(P)$ , we arrive at

$$\widehat{J}[\widetilde{z}] + \gamma \le J[\bar{z}] = \widehat{J}[\bar{z}], \quad \text{i.e.} \quad \widehat{J}[\widetilde{z}] < \widehat{J}[\bar{z}],$$

which contradicts the optimality of $\bar{z}(\cdot)$ in the relaxed control problem and thus justifies (6.10). This completes the proof of the theorem. △

# 7 Necessary Optimality Conditions

The concluding section of the paper is devoted to deriving necessary optimality conditions for the controlled sweeping problem $(P)$ by using the method of discrete approximations [11, 13]. Employing the well-posedness of discrete approximation problems $(P_k)$ and strong convergence of their optimal solutions established in the previous section, our further procedure is first to obtain necessary conditions for optimal solutions to $(P_k)$ and then to derive optimality conditions for $(P)$ by passing to the limit from those for $(P_k)$. The implementation of the second step in [11, 13] is strongly based on Lipschitzian properties of differential inclusions, which is not the case for the sweeping process. Here we develop another approach that utilizes the constructive coderivative calculations given in Section 5.

Let us begin with deriving necessary conditions for optimal solutions to problems $(P_k)$ defined in (6.6)–(6.8), where $F_1 \colon \mathbb{R}^{2n+1} \rightrightarrows \mathbb{R}^n$ is an arbitrary set-valued mapping of closed graph while its special structure in (5.12) is not exploited so far. For simplicity we assume in what follows that the cost functions $\varphi$ and $\ell$ are locally Lipschitzian around the points in question, although these assumptions can be subsequently relaxed to lower semicontinuity.

**Theorem 7.1 (necessary optimality conditions for discrete approximations).** *Fix $k \in I\!N$ and let*

$$\bar{z}^k = (\bar{x}_0, \bar{x}_1^k, \ldots, \bar{x}_k^k, \bar{u}_0^k, \bar{u}_1^k, \ldots, \bar{u}_{k-1}^k, \bar{b}_0^k, \bar{b}_1^k, \ldots, \bar{b}_{k-1}^k)$$

*be an optimal solution to problem $(P_k)$, where $F_1 \colon \mathbb{R}^{2n+1} \rightrightarrows \mathbb{R}^n$ is an arbitrary closed-graph mapping. Then there exist $\lambda^k \ge 0$, $\xi_i^k \in \mathbb{R}$ $(i = 0, \ldots, k-1)$, and $p^k = (p_0^k, \ldots, p_k^k) \in \mathbb{R}^{(k+1)n}$, not equal to zero simultaneously, such that*

$$-p_k^k \in \lambda^k \partial\varphi(\bar{x}_k^k) \quad \text{and}$$

$$\Big(\frac{p_{j+1}^k - p_j^k}{h_k}, p_{j+1}^k - \frac{1}{h_k}\lambda^k\theta_j^k, -\frac{2}{h_k}\xi_j^k\bar{u}_j^k, 0\Big) \in \lambda^k(w_j^k, v_j^k, 0, 0) +$$

$$N\Big(\Big(\bar{x}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}, \bar{u}_j^k, \bar{b}_j^k\Big); \operatorname{gph} F_1\Big)$$

*with some* $(w_j^k, v_j^k) \in \partial\ell\big(\bar{x}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\big)$ *for* $j = 0, \ldots, k-1$*, where*

$$\theta_j^k := 2 \int_{t_j}^{t_{j+1}} \Big(\frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k} - \dot{\bar{x}}(t)\Big)\, dt. \tag{7.1}$$

**Proof.** Consider an extended new variable

$$z := (x_0, x_1, \ldots, x_k, y_0, \ldots, y_{k-1}, u_0, \ldots, u_{k-1}, b_0, \ldots, b_{k-1}) \in \mathbb{R}^{(3k+1)n+k}$$

with the fixed initial vector $x_0$ and define for it the following problem of mathematical programming $(MP)$ with many equality and geometric constraints:

$$\text{minimize} \quad \varphi_0[z] := \varphi(x_k) + h_k \sum_{j=0}^{k-1} \ell(x_j, y_j)$$

$$+\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left\| \frac{(x_{j+1}, u_{j+1}, b_{j+1}) - (x_j, u_j, b_j)}{h_k} - \Big(\dot{\bar{x}}(t), \dot{\bar{u}}(t), \dot{\bar{b}}(t)\Big) \right\|^2 dt \ \text{ subject to}$$

$$h_j(z) := \|u_j\|^2 - 1 = 0 \ \text{ for } \ j = 0, \ldots, k-1,$$

$$g_j(z) := x_{j+1} - x_j - h_k y_j = 0 \ \text{ for } \ j = 0, \ldots, k-1,$$

$$z \in \Delta_j := \big\{z \big|\ y_j \in F_1(x_j, u_j, b_j)\big\} \ \text{ for } \ j = 0, \ldots, k-1, \ \text{ and} \tag{7.2}$$

$$z \in \Delta_k := \big\{z \big|\ x_0 \ \text{ is fixed}\big\}.$$

It is clear that problem $(MP)$ is equivalent to $(P_k)$ for each fixed number $k \in I\!N$. Necessary optimality conditions for problem $(MP)$ in terms of the limiting normal cone (2.4) are given, e.g., in [11, Proposition 5.1]; see also [13, 16] for more discussions and references. Applying this result to the given optimal solution $\bar{z} = \bar{z}^k$ of $P_k$ in the form $(MP)$ and omitting the upper index "$k$" for simplicity, we find numbers $\mu_0 \geq 0$ and $\xi_j \in \mathbb{R}$ $(j = 0, \ldots, k-1)$ as well as vectors $\psi_j \in \mathbb{R}^n$ $(j = 0, \ldots, k-1)$ and $z_j^* \in \mathbb{R}^{(3k+1)n+k}$ $(j = 0, \ldots, k)$, not equal to zero simultaneously, such that

$$z_j^* \in N(\bar{z}; \Delta_j) \ \text{ for } \ j = 0, \ldots, k \ \text{ and} \tag{7.3}$$

$$-z_0^* - \ldots - z_k^* \in \partial\big(\mu_0\varphi_0\big)(\bar{z}) + \sum_{j=0}^{k-1} \big(\nabla g_j(\bar{z})\big)^* \psi_j + \sum_{j=0}^{k-1} \big(\nabla h_j(\bar{z})\big)^* \xi_j. \tag{7.4}$$

Letting $\lambda^k := \mu_0 \geq 0$ and denoting

$$z_j^* = \big(x_{0j}^*, \ldots, x_{kj}^*, y_{0j}^*, \ldots, y_{(k-1)j}^*, u_{0j}^*, \ldots, u_{(k-1)j}^*, b_{0j}^*, \ldots, b_{(k-1)j}^*\big)$$

in $\mathbb{R}^{(3k+1)n+k}$ for $j=0,\ldots,k$, we have from the structures of the sets $\Delta_j$ above that the first component of $z_k^*$ is arbitrary while the others are zero and that the inclusions in (7.3) are equivalent to

$$(x_{jj}^*,y_{jj}^*,u_{jj}^*,b_{jj}^*)\in N\Big(\Big(\bar{x}_j^k,\frac{\bar{x}_{j+1}^k-\bar{x}_j^k}{h_k},\bar{u}_j^k,\bar{b}_j^k\Big);\operatorname{gph}F_1\Big),$$

$$x_{ij}^*=y_{ij}^*=u_{ij}^*=b_{ij}^*=0,\ \text{ and}$$

$$x_{kj}^*=u_{kj}^*=u_{kj}^*=0\ \text{ for every } i,j=1,\ldots,k-1\ \text{ with }\ i\neq j.$$

Taking this into account, we get from (7.4) the following relationships:

$$-x_{jj}^*=\lambda^k h_k w_j^k+\psi_{j-1}-\psi_j\ \text{ for }\ j=1,\ldots,k-1,$$

$$-y_{jj}^*=\lambda^k h_k v_j^k+\lambda^k\theta_j^k-h_k\psi_j\ \text{ for }\ j=1,\ldots,k-1,$$

$$-\psi_{k-1}\in\lambda^k\partial\varphi(\bar{x}_k^k),\ \text{ and}$$

$$-u_{jj}^*=2\xi_j\bar{u}_j^k\ \text{ for }\ j=1,\ldots,k-1,$$

where $(w_j^k,v_j^k)\in\partial\ell\big(\bar{x}_j^k,\frac{\bar{x}_{j+1}^k-\bar{x}_j^k}{h_k}\big)$ for every $j=0,\ldots,k-1$, and where the numbers $\theta_j^k$ are defined in (7.1). Denoting finally

$$p_0^k:=0,\quad p_k^k:=\psi_{k-1},\ \text{ and }\ p_j^k:=\psi_{j-1}\ \text{ for }\ j=1,\ldots,k-1,$$

we arrive at all the necessary optimality conditions claimed in the theorem. $\triangle$

The next result is a consequence of Theorem 7.1 and the precise coderivative calculations in Section 5 for set-valued mappings arising in the controlled sweeping process.

**Corollary 7.2 (necessary optimality conditions for the discretized sweeping process).** *Let the mapping $F_1$ in the framework of Theorem* 7.1 *be defined by* (5.12)*. Then for each $k\in I\!N$ we have the relationships*

$$-p_k^k\in\partial\varphi(\bar{x}_k^k), \tag{7.5}$$

$$\frac{p_{j+1}^k-p_j^k}{h_k}=w_j^k,\quad \xi_j^k=0\ \text{ for }\ j=0,\ldots,k-1,\quad \text{and} \tag{7.6}$$

$$\big(w_j^k,v_j^k\big)\in\partial\ell\Big(\bar{x}_j^k,\frac{\bar{x}_{j+1}^k-\bar{x}_j^k}{h_k}\Big),\quad j=0,\ldots,k-1, \tag{7.7}$$

*in the necessary optimality conditions of Theorem* 7.1.

**Proof.** It follows from the necessary conditions via the normal cone in Theorem 7.1 and the coderivative definition (2.7) that

$$\Big(\frac{p^k_{j+1}-p^k_j}{h_k}-\lambda^k w^k_j,-\frac{2}{h_k}\xi^k_j\bar u^k_j,0\Big)\in$$

$$D^*F_1\Big(\bar x^k_j,\bar u^k_j,\bar b^k_j,\frac{\bar x^k_{j+1}-\bar x^k_j}{h_k}\Big)\Big(-p^k_{j+1}+\frac{1}{h_k}\lambda^k\theta^k_j+\lambda^k v^k_j\Big)$$

for all $j=0,\ldots,k-1$. Fix $j$ and apply to the latter inclusion Proposition 5.7 and the explicit coderivative calculations of Theorem 5.6. Then we have the following three cases:

**(i)** If $\langle\bar x^k_j,\bar u^k_j\rangle<\bar b^k_j$, then $\frac{p^k_{j+1}-p^k_j}{h_k}=\lambda^k w^k_j$ and $\xi^k_j=0$.

**(ii)** If $\langle\bar x^k_j,\bar u^k_j\rangle=\bar b^k_j$ and $\frac{\bar x^k_{j+1}-\bar x^k_j}{h_k}\neq 0$, then there are numbers $\nu\in\mathbb{R}$ and $t\ge 0$ such that

$$\Big\langle p^k_{j+1}-\frac{1}{h_k}\lambda^k\theta^k_j-\lambda^k v^k_j-t\frac{\bar x^k_{j+1}-\bar x^k_j}{h_k},\bar u^k_j\Big\rangle=0 \quad\text{and}$$

$$\Big(\frac{p^k_{j+1}-p^k_j}{h_k}-\lambda^k w^k_j,-\frac{2}{h_k}\xi^k_j\bar u^k_j,0\Big)=$$

$$\nu\big(\bar u^k_j,\bar x^k_j,-1\big)-\Big(0,\lambda\Big(-p^k_{j+1}+\frac{1}{h_k}\lambda^k\theta^k_j+\lambda^k v^k_j+t\frac{\bar x^k_{j+1}-\bar x^k_j}{h_k}\Big),0\Big),$$

where the number $\lambda<0$ is uniquely defined by $\frac{\bar x^k_{j+1}-\bar x^k_j}{h_k}=\lambda\bar u^k_j$. The latter inclusion implies that $\nu=0$, and hence we get $\frac{p^k_{j+1}-p^k_j}{h_k}=\lambda^k w^k_j$ and the equalities

$$-\frac{2}{h_k}\xi^k_j\bar u^k_j=\lambda\Big(-p^k_{j+1}+\frac{1}{h_k}\lambda^k\theta^k_j+\lambda^k v^k_j+t\frac{\bar x^k_{j+1}-\bar x^k_j}{h_k}\Big),$$

$$\Big\langle\bar u^k_j,-p^k_{j+1}+\frac{1}{h_k}\lambda^k\theta^k_j+\lambda^k v^k_j+t\frac{\bar x^k_{j+1}-\bar x^k_j}{h_k}\Big\rangle=0,$$

which ensure in turn that $\xi^k_j=0$. Our final case is:

**(iii)** $\langle\bar x^k_j,\bar u^k_j\rangle=\bar b^k_j$ and $\frac{\bar x^k_{j+1}-\bar x^k_j}{h_k}=0$. Employing Theorem 5.6(iii) leads us to the same conclusion as in the previous case (ii).

Thus in all the three cases (i)–(iii) we get the necessary optimality conditions of Theorem 7.1 with $p^k_0=0$ and $\xi^k_j=0$ for $j=0,\ldots,k-1$. Due to the above nontriviality $(\lambda^k,p^k_1,\ldots,p^k_k)\neq 0$ these conditions ensure that $\lambda^k>0$, which allows us to set $\lambda^k=1$ by normalization. Hence we arrive at (7.5)–(7.7) and complete the proof of the theorem. △

Now we are ready to derive necessary conditions for optimal solutions of the original problem $(P)$ by passing to the limit from discrete approximations.

**Theorem 7.3 (coderivative optimality conditions for the controlled sweeping process).** *Let* $\bar z(\cdot) = (\bar x(\cdot), \bar u(\cdot), \bar b(\cdot))$ *be an optimal solution to problem* $(P)$*, which is assumed to be stable with respect to relaxation. Then there are functions* $p\colon [0,T] \to \mathbb{R}^n$ *absolutely continuous on* $[0,T]$ *and* $(w(\cdot), v(\cdot)) \in L^\infty([0,T];\mathbb{R}^{2n})$ *such that*

$$(0,0,0) \in D^*F_1\Big(\bar x(t), \bar u(t), \bar b(t), \dot{\bar x}(t)\Big)\big(v(t)-p(t)\big) \quad a.e. \quad t \in [0,T], \tag{7.8}$$

$$p(t) = p(T) + \int_T^t w(s)\,ds \quad with \quad -p(T) \in \partial\varphi\big(\bar x(T)\big), \quad and \tag{7.9}$$

$$\big(w(t), v(t)\big) \in \operatorname{co}\partial\ell\big(\bar x(t), \dot{\bar x}(t)\big) \quad a.e. \quad t \in [0,T], \tag{7.10}$$

*where the coderivative of* $F_1$ *from* (5.12) *is calculated in Proposition* 5.7 *and Theorem* 5.6.

**Proof.** Given the optimal solution $\bar z(\cdot)$ to the original problem $(P)$, we construct its discrete approximations $(P_k)$ whose optimal solutions $\bar z^k(\cdot) = (\bar x^k(\cdot), \bar u^k(\cdot), \bar b^k(\cdot))$ strongly converge to $\bar z(\cdot)$ as $k \to \infty$ by Theorem 6.3. Applying necessary optimality conditions for $z^k(\cdot)$ from Theorem 7.1 and Corollary 7.2 allows us to find dual elements $p^k = (p_0^k, \ldots, p_k^k)$, $v^k = (v_0^k, \ldots, v_{k-1}^k)$, and $w^k = (w_0^k, \ldots, w_{k-1}^k)$ satisfying the relationships (7.5), (7.6), and

$$(0,0,0) \in D^*F_1\Big(\bar x_j^k, \bar u_j^k, \bar b_j^k, \frac{\bar x_{j+1}^k - \bar x_j^k}{h_k}\Big)\Big(-p_{j+1}^k - \frac{1}{h_k}\theta_j^k + v_j^k\Big),\ j = 0, \ldots, k-1, \tag{7.11}$$

where the quantities $\theta_j^k$ are defined in (7.1). For each $k \in I\!N$ we extend the discrete arcs $p^k(\cdot)$ piecewise linearly to the whole interval $[0,T]$ similarly to $z^k(\cdot)$, while for $w^k(\cdot)$ and $v^k(\cdot)$ we consider their piecewise constant extensions to $[0,T]$. It follows from (7.7), the well-known boundedness of the limiting subdifferential $\partial\ell$ by the Lipschitz constant of $\ell$, and standard functional analysis that the sequence of $(w^k(t), v^k(t))$ is weakly compact in $L^2([0,T];\mathbb{R}^{2n})$. Hence we suppose with no relabeling that

$$w^k(t) \to w(t) \ \text{ and } \ v^k(t) \to v(t) \ \text{ weakly in } \ L^2([0,T];\mathbb{R}^n) \ \text{ as } \ k \to \infty$$

for some $w(\cdot), v(\cdot) \in L^\infty([0,T];\mathbb{R}^n)$ due to the uniform boundedness of $v^k(t)$ and $w^k(t)$. The classical Mazur theorem ensures that there are convex combinations of $v^k(t)$ and $w^k(t)$, which converge to $v(t)$ and $w(t)$, respectively, strongly in $L^2([0,T];\mathbb{R}^n)$ and thus a.e. on $[0,T]$ along some subsequences. Furthermore, it follows from the first relationship in (7.6) that the corresponding convex combinations of the piecewise constantly extended "discrete derivatives" $\frac{p_{j+1}^k - p_j^k}{h_k}$ of $p^k(t)$ converge to $w(t)$ a.e. on $[0,T]$. Using the boundedness of $\{p^k(T)\}$ by (7.5) and the Lipschitz continuity of $\varphi$ and then the

Newton-Leibniz formula, we conclude that the sequence $\{p^k(t)\}$ converges uniformly on $[0,T]$ to the function

$$p(t) := p(T) + \int_T^t w(s)\,ds \quad \text{for} \quad 0 \le t \le T$$

absolutely continuous on $[0,T]$ with the transversality condition $-p(T) \in \partial\ell(\bar{x}(t),\dot{x}(t))$, which follows by passing to the limit in (7.5) due to the well-known robustness (closed graph) property of the limiting subdifferential. Employing further this robustness property of the subdifferential $\partial\ell$ and passing to the limit in (7.7) along the a.e. convergent sequences of convex combinations of $v^k(t)$ and $w^k(t)$, we arrive at the inclusion (7.10).

To complete the proof of the theorem, it remains passing to the limit in the coderivative inclusion (7.11) as $k \to \infty$. To proceed, define first

$$\theta_k(t) := \frac{\theta_j^k}{h_k} \quad \text{for} \quad t \in [t_j, t_{j+1}), \quad j = 0, \ldots, k-1,$$

and observe by (7.1) and Theorem 6.3 the $L^1$-convergence of these extensions:

$$\begin{aligned} \int_0^T |\theta^k(t)|\,dt &= \sum_{j=0}^{k-1} |\theta_j^k| \le 2\sum_{j=0}^{k-1} \Big\|\dot{\bar{x}}(t) - \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\Big\|\,dt \\ &= \int_0^T \|\dot{\bar{x}}(t) - \dot{\bar{x}}^k(t)\|\,dt \to 0 \quad \text{as} \quad k \to \infty, \end{aligned}$$

which implies the a.e. on $[0,T]$ convergence of a subsequence. Now we are able to pass to the limit in inclusion (7.11) extended to the whole interval $[0,T]$ by taking into account the established pointwise convergence of all the sequences therein, the robustness of the coderivative with respect to all of its variables, and the coderivative structure in Proposition 5.7 and Theorem 5.6 that allows us to replace the weakly convergent sequence of $v^k(t)$ by strongly convergent sequence of their convex combination. Thus we arrive at (7.7) and complete the proof of the theorem. △

Next we present several consequences of Theorem 7.3. The first one imposes the differentiability assumption on the running cost.

**Corollary 7.4 (coderivative optimality conditions for the controlled sweeping process with smooth running costs).** *Suppose that in the framework of Theorem* 7.3 *the running cost* $\ell(\cdot,\cdot)$ *is strictly differentiable at* $(\bar{x}(t),\dot{\bar{x}}(t))$ *for a.e.* $t \in [0,T]$. *Then*

$$(0,0,0) \in$$
$$D^*F_1\left(\bar{x}(t),\bar{u}(t),\bar{b}(t),\dot{\bar{x}}(t)\right)\left(\int_t^T \nabla_x\ell\big(\bar{x}(s),\dot{\bar{x}}(s)\big)\,ds + \nabla_v\ell\big(\bar{x}(t),\dot{\bar{x}}(t)\big) - p\right)$$

*for a.e.* $t \in [0,T]$ *with some constant* $p \in -\partial\varphi\big(\bar{x}(T)\big)$.

**Proof.** By the strict differentiability assumption on $\ell$ we have

$$\partial\ell\big(\bar{x}(t),\dot{\bar{x}}(t)\big) = \Big\{\nabla\ell_x\big(x(t),\dot{\bar{x}}(t)\big),\nabla\ell_v\big(x(t),\dot{\bar{x}}(t)\big)\Big\}.$$

Then the result readily follows from Theorem 7.3. △

Taking further into account a particular *coderivative kernel* form of inclusion (7.8) in Theorem 7.3 and its specification in Corollary 7.4 and then applying the coderivative kernel expressions given in Corollary 5.9, we can derive from these results explicit necessary optimality conditions for $(P)$ formulated entirely via the initial data of controlled sweeping process. Let us present some consequences of this type .

**Corollary 7.5 (explicit optimality conditions for the sweeping process with smooth running costs).** *In the setting of Corollary* 7.4 *the following hold for a.e.* $t \in [0,T]$ *with some constant* $p \in -\partial\varphi(\bar{x}(T))$*:*

**(i)** *If* $0 < \|\dot{\bar{x}}(t)\| < 1$*, then we have*

$$\nabla_v\ell\big((\bar{x}(t),\dot{\bar{x}}(t)\big) + \int_t^T \nabla_x\ell\big((\bar{x}(s),\dot{\bar{x}}(s)\big)\,ds = p.$$

**(ii)** *If* $\|\dot{\bar{x}}(t)\| = 1$*, then we have*

$$\nabla_v\ell\big((\bar{x}(t),\dot{\bar{x}}(t)\big) + \int_t^T \nabla_x\ell\big((\bar{x}(s),\dot{\bar{x}}(s)\big)\,ds \in p + \mathbb{R}_+\big\{\bar{u}(t)\big\}.$$

**Proof.** Follows from Corollary 7.4 and Corollary 5.9. △

The next corollary characterizes optimal solutions to problem $(P)$ with no running costs.

**Corollary 7.6 (characterizations of optimal solutions for problems with terminal costs).** *Let* $\ell = 0$ *in the framework of Theorem* 7.3*. Then for a.e.* $t \in [0,T]$ *exactly one of the following three cases holds:*

**(i)** $\dot{\bar{x}}(t) = 0$.

**(ii)** $0 < \|\dot{\bar{x}}(t)\| < 1$ *and* $0 \in \partial\varphi\big(\bar{x}(T))\big)$.

**(iii)** $\|\dot{\bar{x}}(t)\| = 1$ *and* $\mathbb{R}_+\{\bar{u}(t)\} \cap \partial\varphi\big(\bar{x}(T)\big) \neq \emptyset$.

**Proof.** Follows from Theorem 7.3 and Corollary 5.9. △

To conclude this paper, we present three examples showing how the optimality conditions obtained above allow us to find optimal solutions to the controlled sweeping process.

Our first example concerns the optimal control problem $(P)$ with a smooth running cost.

**Example 7.7 (calculating optimal solutions for the sweeping process with running costs).** Consider the controlled sweeping process (1.4) with the cost functional

$$\text{minimize} \quad J[x] := \varphi\big(x(T)\big) + \int_0^T \big\|\dot{x}(t)\big\|^2\, dt,$$

i.e., with $\ell(x,v) = \|v\|^2$ in $(P)$. Then the coderivative inclusion (7.8) is written as

$$(0,0,0) \in D^*F_1\left(\bar{x}(t), \bar{u}(t), \bar{b}(t), \dot{\bar{x}}(t)\right)(2\dot{\bar{x}}(t) - p) \quad a.e. \quad t \in [0,T]$$

with some $p \in -\partial\varphi(\bar{x}(T))$. Let us examine all the possibilities for optimal solutions to this problem based on the results of Corollary 7.5.

Consider first the case of $p \neq 0$. If $0 < \|\dot{x}(t)\| < 1$, then by Corollary 7.5(i) we have $2\dot{\bar{x}}(t) - p = 0$, i.e., $\dot{\bar{x}}(t) = \frac{p}{2}$. This implies that

$$\bar{u}(t) = -\frac{\dot{\bar{x}}(t)}{\|\dot{\bar{x}}(t)\|} = -\frac{p}{\|p\|},$$

and by $\dot{\bar{x}}(t) \neq 0$ it gives $\bar{b}(t) = \langle \bar{x}(t), \bar{u}(t)\rangle$.

If $\|\dot{x}(t)\| = 1$ in the case of $p \neq 0$, then by Corollary 7.5 there is $\alpha \geq 0$ such that

$$2\dot{\bar{x}}(t) - p = \alpha\bar{u}(t) \;\text{ for this }\; t \in [0,T]. \tag{7.12}$$

It follows from the structure of the controlled sweeping process in (1.4) that the vectors $\bar{u}(t)$ and $\dot{\bar{x}}(t)$ are parallel and have the opposite directions. Then we conclude from (7.12) that the vectors $\dot{\bar{x}}(t)$ and $p$ have the same direction. Since $\|\dot{x}(t)\| = 1$, it gives that

$$\dot{\bar{x}}(t) = \frac{p}{\|p\|}, \quad \bar{u}(t) = -\frac{p}{\|p\|}, \;\text{ and }\; \bar{b}(t) = \langle \bar{x}(t), \bar{u}(t)\rangle.$$

Consider finally the remaining case of $p = 0$. Then by Corollary 7.5 we have that either $\|\dot{\bar{x}}(t)\| = 1$ or $\|\dot{\bar{x}}(t)\| = 0$ for a.e. $t \in [0,T]$. If $\|\dot{\bar{x}}(t)\| = 1$ in this case, then there is $\alpha \geq 0$ such that $2\dot{\bar{x}}(t) = \alpha\bar{u}(t)$. As mentioned above, the vectors $\bar{u}(t)$ and $\dot{\bar{x}}(t)$ have the opposite directions. This leads to $\dot{\bar{x}}(t) = 0$, a contradiction. Thus $\bar{x}(t) = 0$ for all $t \in [0,T]$ and we arrive at the conclusion of $\bar{x}(t) \equiv x_0$ on $[0,T]$, which completes our consideration.

The next example concerns problem $(P)$ with a specific while rather general terminal cost that may not be smooth.

**Example 7.8 (calculating optimal solutions for the sweeping process with nonsmooth terminal costs).** Consider problem $(P)$ with

$$\ell(x,v) = \|v\|^2$$

and with terminal cost given by the distance square

$$\varphi(x) := \mathrm{dist}^2(x;K), \quad x \in \mathbb{R},$$

where $K$ is a closed set not containing the origin, and where the dynamics is given by (1.4) with $x_0 = 0$. Recall that we always assume that $M = 1$ in (4.1). It is well known (see, e.g., [16, Example 8.53]) that the limiting subdifferential of the distance function at points $x \notin K$ is given by the exact formula

$$\partial \mathrm{dist}(x;K) = \frac{x - \Pi(x;K)}{\mathrm{dist}(x;K)}$$

via the (generally multivalued) Euclidean projector, and so we get by the elementary subdifferential chain rule that

$$\partial \mathrm{dist}^2(x;K) = 2\big(x - \Pi(x;K)\big), \quad x \notin K.$$

Invoking the calculations in Example 7.7 gives us that optimal trajectories for this problem have constant velocities and follow any of the steepest descent direction of $\mathrm{dist}(x;K)$. Thus for every $w \in \Pi(0;K)$ there exists an optimal trajectory in the direction $w$. Setting $p := -2(x(1) - w)$, we see that the velocity of the corresponding optimal trajectory in the direction $z$ is given by $p/2$ if $0 < \|w\| \le 2$ and by $w/\|w\|$ if $\|w\| > 2$; the case $x(1) = w \in K$ is impossible. To conclude our consideration, observe that the terminal point $(T = 1)$ of the optimal trajectory is $x(1) = w/2$ in the first case and $w/\|w\|$ in the second one.

Note that in both Example 7.7 and Example 7.8 optimal trajectories $x(\cdot)$ of the sweeping process happen to be of constant velocity. Our final example shows that it is not always the case even for the two-dimensional sweeping process with smooth cost functions, where the running cost does not depend on the velocity variable.

**Example 7.9 (optimal sweeping trajectories with variable velocities).** Consider problem $(P)$ with the cost functions

$$\varphi(x) := \Big(x_1 + \frac{1}{\pi}\Big)^2 + x_2^2 \;\text{ and }\; \ell(x) := \Big(x_1^2 + x_2^2 - \frac{1}{\pi^2}\Big)^2, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

the initial condition $x_0 = (1/\pi, 0)$, and the terminal time $T = 1$. Since $J[\bar{x}] = 0$ for the minimizing functional at the trajectory

$$\bar{x}(t) = \frac{1}{\pi}\Big(\cos(\pi t), \sin(\pi t)\Big), \quad t \in [0,1],$$

this trajectory is optimal to $(P)$. Observe that $\bar{x}(\cdot)$ satisfies the necessary optimality condition from Corollary 7.5(ii) with $p = 0$. The corresponding optimal controls are

$$\bar{u}(t) = \frac{1}{\pi}\Big( -\sin(\pi t), \cos(\pi t)\Big) \quad \text{and} \quad \bar{b}(t) = 0 \quad \text{for all} \quad t \in [0,1].$$

Observe further that every optimal trajectory $\widetilde{x}(\cdot)$ must satisfy the condition $\|\widetilde{x}(t)\| = 1/\pi$ for a.e. $t \in [0,1]$; this follows from Corollary 7.5(i). In fact, if $\|\widetilde{x}(t)\| \neq 1/\pi$ on a set of positive measure, then $p$ cannot be constant. Hence, the necessary optimality condition given by Corollary 7.5(ii) becomes

$$(0,0) \in 2\big(\big(x_1(1) + 1/\pi\big), x_2(1)\big) + \mathbb{R}_+\{\bar{u}(t)\},$$

which yields, since $\bar{u}(t)$ cannot be constant, that $\big(x_1(1), x_2(1)\big) = (-1/\pi, 0)$. This implies that $\|\dot{\widetilde{x}}(t)\| = 1$ for a.e. $t \in [0,1]$, and so the unique optimal trajectory to $(P)$ is $\bar{x}(\cdot)$.

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