Structure and stability of programs with probabilistic storage level constraints

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Abstract. We consider the structure of probabilistic storage level constraints as well as the stability of solutions to optimization problems involving such constraints. Apart from the general case, special structures with and without discretization are analyzed. Structures of the feasible set are characterized in terms of convexity and/or connectedness. Stability of solution sets with respect to perturbations of the probability measure is formulated by means of upper and lower semicontinuity of the associated multivalued mapping.

1 Introduction

Optimization problems involving the control of probabilistic storage levels play an important role in the application of stochastic optimization. As examples we refer to the problem of water level regulation of a lake [6] and to the optimal control of a continuous distillation process with random rate of outer inflows [4]. In this paper, we want to study issues of stability and structure for probabilistic storage level constraints. Here 'stability' refers to the behaviour of optimal solutions and optimal values of the induced programs when the true - but generally unknown - probability distribution is replaced by approximations. Results are given in terms of the Kolmogorov distance between probability measures. On the other hand, 'structure' relates to specific properties of the feasible set like convexity or connectedness or to an explicit description of this set.

To make the considered model more precise, we assume that for some future time horizon $[t_0, t_1]$ differing amounts of substance shall be extracted from a reservoir at a time-dependent rate x. At the same time, the reservoir is filled with the same substance at an inflow rate ξ, where $\xi : [t_0, t_1] \times \Omega \to \mathbb{R}$ is a stochastic process on some probability space $(\Omega, \mathcal{A}, P)$. The filling level in the reservoir is supposed to be constrained by (time-dependent) lower and upper limits m_* and m^*, respectively. It is assumed that $x, m_*, m^* \in L^\infty([t_0, t_1])$ and $\xi(\cdot, \omega) \in L^\infty([t_0, t_1])$ for P-almost all $\omega \in \Omega$. The choice of the space L^∞ is motivated by the fact that rates will be bounded in practice. With the decision variable x, we associate an objective function $g : L^\infty([t_0, t_1]) \to \mathbb{R}$. Now, the filling level of the reservoir at time t may be written as

$$m(t, \xi, x) = m_0 + \int_{t_0}^{t} \xi(\tau, \omega) d\tau - \int_{t_0}^{t} x(\tau) d\tau,$$

where m_0 denotes the initial filling level at $t = t_0$ (not depending on ξ and x). Then, the optimization problem involving stochastic storage level constraints along with some deterministic constraints defined by an abstract subset $X \subseteq L^\infty(\,[t_0, t_1])$ becomes

$$\min\{g(x) | x \in X \text{ and } m_*(t) \leq m(t, \xi, x) \leq m^*(t) \quad \forall\, t \in [t_0, t_1]\}, \qquad (1)$$

where it is assumed that $m_*(t) \leq m^*(t) \quad \forall\, t \in [t_0, t_1]$. For the decision structure of this problem it is supposed that x has to be chosen under complete ignorance of the realizations of ξ. In order to transform (1) into a tractable optimization problem, a formulation via probabilistic constraints is preferred here. This means, that an extraction profile x is feasible in case that the stochastic constraints in (1) are satisfied at least with probability $p \in (0, 1)$ according to the distribution of ξ. There are obviously many different possibilities of formulating probabilistic constraints out of those given in (1): one may consider completely joint or completely individual or partially joint and individual (w.r.t. time and upper and lower level) probabilistic constraints. Each of these possibilities will lead, in general, to different structures of the resulting optimization problem. Other modelling issues are induced by passing to the discretized version of (1), by omitting deterministic constraints or by restricting to one-sided level constraints (i.e. either w.r.t. m_* or to m^*). It is not possible to discuss all combinations of these situations. Rather, some results shall be given here for a few particular, yet relevant for applications, cases.

2 The general case

In the following, the general problem (1) will be considered under different formulations of probabilistic constraints. We start with the case of completely joint probabilistic constraints:

$$\min\{g(x) | x \in X,\, P(m_*(t) \leq m(t, \xi, x) \leq m^*(t) \quad \forall t \in [t_0, t_1]) \geq p\}. \qquad (2)$$

Here, $p \in (0, 1)$ is a given probability level. Next, the well-known concept of quasiconcave probability measures (cf. [7]) is extended in a way appropriate for our purposes:

Definition 1. A stochastic process $\eta : [t_0, t_1] \times \Omega \to \mathbb{R}$ is called *quasiconcave on intervals*, if

$$\begin{aligned} &P(\eta(t, \omega) \in [\lambda x_1(t) + (1 - \lambda) y_1(t), \lambda x_2(t) + (1 - \lambda) y_2(t)] \quad \forall t \in [t_0, t_1]) \geq \\ &\quad \min\{P(\eta(t, \omega) \in [x_1(t), x_2(t)] \quad \forall t \in [t_0, t_1]), P(\eta(t, \omega) \in [y_1(t), y_2(t)] \\ &\qquad\qquad \forall t \in [t_0, t_1])\}. \end{aligned}$$

holds true for all $\lambda \in [0, 1]$ and for all $x_1, x_2, y_1, y_2 \in \mathcal{C}^0([t_0, t_1])$ such that $x_2(t) \geq x_1(t), y_2(t) \geq y_1(t)$ for all $t \in [t_0, t_1]$.

The following results concerning the properties of the feasible set in problem (2) will be formulated in terms of the cumulative inflow process

$$\eta(t,\omega) := \int_{t_0}^{t} \xi(\tau,\omega)d\tau \quad \forall t \in [t_0,t_1] \quad \forall \omega \in \Omega. \tag{3}$$

Proposition 1. *Let η in (3) be quasiconcave on intervals and moreover let $X \subseteq L^\infty(\,[t_0,t_1])$ be convex. Then, the feasible set of problem (2) is convex.*

Proof. Obviously, all one has to show is convexity of the set

$$M := \{x \in L^\infty([t_0,t_1]) | P(m_*(t) \leq m(t,\xi,x) \leq m^*(t) \quad \forall\ t \in [t_0,t_1]) \geq p.$$

Now, let $x, y \in M$ and $\lambda \in [0,1]$ be arbitrarily given. Introducing the continuous functions

$$\alpha(t) := m_*(t) - m_0 + \int_{t_0}^{t} x(\tau)d\tau \quad \text{and} \quad \beta(t) := m_*(t) - m_0 + \int_{t_0}^{t} y(\tau)d\tau,$$

the relations $m_*(t) \leq m(t,\xi,x)$, $m(t,\xi,y) \leq m^*(t)$ may be equivalently written as

$$\begin{aligned} \eta(t,\omega) &\in [\alpha(t), \alpha(t) + m^*(t) - m_*(t)] \\ \eta(t,\omega) &\in [\beta(t), \beta(t) + m^*(t) - m_*(t)], \end{aligned}$$

respectively. Hence, the assumption $x, y \in M$ implies that

$$\begin{aligned} P(\eta(t,\omega) \in [\alpha(t), \alpha(t) + m^*(t) - m_*(t)] \quad \forall t \in [t_0,t_1]) \geq p \\ P(\eta(t,\omega) \in [\beta(t), \beta(t) + m^*(t) - m_*(t)] \quad \forall t \in [t_0,t_1]) \geq p. \end{aligned}$$

Then, the quasiconcavity property of η yields

$$\begin{aligned} & P(m_*(t) \leq m(t,\xi,\lambda x + (1-\lambda)y) \leq m^*(t) \quad \forall t \in [t_0,t_1]) = \\ & P(\eta(t,\omega) \in [\lambda\alpha(t) + (1-\lambda)\beta(t), \lambda(\alpha(t) + m^*(t) - m_*(t)) + \\ & \qquad (1-\lambda)(\beta(t) + m^*(t) - m_*(t)) \quad \forall\ t \in [t_0,t_1]) \geq \\ & \min\{P(\eta(t,\omega) \in [\alpha(t), \alpha(t) + m^*(t) - m_*(t)] \quad \forall t \in [t_0,t_1]), \\ & \qquad P(\eta(t,\omega) \in [\beta(t), \beta(t) + m^*(t) - m_*(t)] \quad \forall t \in [t_0,t_1])\} \geq p. \end{aligned}$$

Hence, $\lambda x + (1-\lambda)y \in M$.

■

The following lemma indicates a situation where quasiconcavity of η on intervals is satisfied:

Lemma 1. *Let the inflow rate be given as the simple process $\xi(t,\omega) = c(t) + \tilde{\xi}(\omega)$, where $\tilde{\xi}$ is a 1-dimensional random variable with quasiconcave distribution (e.g. normal distribution or many others). Then, the cumulative inflow process η from (3) is quasiconcave on intervals.*

Proof. Clearly, $\eta(t,\omega) = C(t) + \tilde{\xi}(\omega)(t-t_0)$. Since $C(t_0) = \eta(t_0,\omega)$ for all $\omega \in \Omega$, it follows in case

$$C(t_0) \notin [x_1(t_0), x_2(t_0)] \cap [y_1(t_0), y_2(t_0)]$$

that

$$\begin{aligned}\min\{&P(\eta(t,\omega) \in [x_1(t), x_2(t)]\ \forall t \in [t_0,t_1]),\\ &P(\eta(t,\omega) \in [y_1(t), y_2(t)]\ \forall t \in [t_0,t_1])\} = 0,\end{aligned}$$

hence the inequality of Definition 1 is trivially fulfilled. Therefore, assume now that

$$C(t_0) \in [x_1(t_0), x_2(t_0)] \cap [y_1(t_0), y_2(t_0)].$$

Then,

$$\begin{aligned}\eta(t_0,\omega) = C(t_0) \in [\lambda x_1(t_0) + (1-\lambda)y_1(t_0), \lambda x_2(t_0) + (1-\lambda)y_2(t_0)]&\\ \forall \omega \in \Omega\, \forall \lambda \in [0,1].&\quad (4)\end{aligned}$$

For the following derivation, we use the abbreviations with "/" relating to alternatives:

$$\begin{aligned}s_{x/y} &= \sup_{t\in(t_0,t_1]} (t-t_0)^{-1}(x_1/y_1(t) - C(t)) \text{ and}\\ i_{x/y} &= \inf_{t\in(t_0,t_1]} (t-t_0)^{-1}(x_2/y_2(t) - C(t)).\end{aligned}$$

Now,

$$\begin{aligned}
&P(\eta(t,\omega) \in [\lambda x_1(t) + (1-\lambda)y_1(t), \lambda x_2(t) + (1-\lambda)y_2(t)]\ \forall t \in [t_0,t_1]) = \\
&\hfill \text{(by (4))}\\
&P(\eta(t,\omega) \in [\lambda x_1(t) + (1-\lambda)y_1(t), \lambda x_2(t) + (1-\lambda)y_2(t)]\ \forall t \in (t_0,t_1]) = \\
&\quad P(\tilde{\xi}(\omega) \in [(t-t_0)^{-1}(\lambda(x_1(t) - C(t)) + (1-\lambda)(y_1(t) - C(t))),\\
&\quad (t-t_0)^{-1}(\lambda(x_2(t) - C(t)) + (1-\lambda)(y_2(t) - C(t)))]\ \forall t \in (t_0,t_1]) = \\
&\hfill P(\tilde{\xi}(\omega) \in\\
&[\sup_{t\in(t_0,t_1]} \{\lambda(t-t_0)^{-1}(x_1(t) - C(t)) + (1-\lambda)(t-t_0)^{-1}(y_1(t) - C(t))\},\\
&\inf_{t\in(t_0,t_1]} \{\lambda(t-t_0)^{-1}(x_2(t) - C(t)) + (1-\lambda)(t-t_0)^{-1}(y_2(t) - C(t))\}]) \geq \\
&\hfill P(\tilde{\xi}(\omega) \in [\lambda s_x + (1-\lambda)s_y, \lambda i_x + (1-\lambda)i_y]) = \\
&\hfill P(\tilde{\xi}(\omega) \in \lambda[s_x, i_x] + (1-\lambda)[s_y, i_y]) \geq \\
&\hfill \text{(by quasiconcavity of } \tilde{\xi})\\
&\hfill \min\{P(\tilde{\xi}(\omega) \in [s_x, i_x]), P(\tilde{\xi}(\omega) \in [s_y, i_y])\} = \\
&\hfill \min\{P(\eta(t,\omega) \in [x_1(t), x_2(t)]\ \forall t \in (t_0,t_1]),\\
&\hfill P(\eta(t,\omega) \in [y_1(t), y_2(t)]\ \forall t \in (t_0,t_1])\} =
\end{aligned}$$

$$\min\{P(\eta(t,\omega) \in [x_1(t), x_2(t)]\ \forall t \in [t_0,t_1]),$$
$$P(\eta(t,\omega) \in [y_1(t), y_2(t)]\ \forall t \in [t_0,t_1])\}$$
$$\text{(by (4))}$$

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Next, let us consider problem (2) with a time discretization $\{t^1,\dots,t^N\} \subseteq [t_0,t_1]$:

$$\min\{g(x) | x \in X,\ P(m_*(t^i) \le m(t^i,\xi,x) \le m^*(t^i) \quad i = 1,\dots,N) \ge p\}. \quad (5)$$

A quasiconcavity property adapted to this structure would result when the inequality in Definition 1 is required to hold for all finite selections $\{t^1,\dots,t^N\} \subseteq [t_0,t_1]$ rather than for all $t \in [t_0,t_1]$. In that case, we want to call the process η in Def. 1 *quasiconcave on finite selections of intervals*. Repeating the same lines of argumentation as in Prop. 1 and Lemma 1, we get

Proposition 2. *Let η in (3) be quasiconcave on finite selections of intervals (as it is true again for the simple process considered in Lemma 1) and $X \subseteq L^\infty([t_0,t_1])$ be convex. Then, the feasible set of problem (5) is convex.*

If the structure of problem (5) is slightly simplified, a characterization of the feasible set in terms of the weaker connectedness property can be given without any assumptions on the stochastic process ξ or its cumulant η, respectively:

Proposition 3. *Assume that $X = L^\infty([t_0,t_1])$ (i.e., no deterministic constraints) and only one-sided limitations (either $m_*(t)$ or $m^*(t)$) occur. Then, the feasible set of (5) is (path-) connected.*

The proof of this proposition relies on the following more general result from [5]:

Theorem 1. *Let Y be a Banach space, $(\Omega, \mathcal{A}, P)$ a probability space, $\tilde{\xi} : \Omega \to \mathbb{R}^s$ a random vector, $p \in (0,1)$ a probability level and $h : Y \to \mathbb{R}^m$, $g : \mathbb{R}^s \to \mathbb{R}^m$ mappings defining a constraint set*

$$M = \{x \in Y \mid P(h(x) \ge g(\tilde{\xi}(\omega))) \ge p\}.$$

Let g be Borel measurable and h be continuous and have strictly quasiconcave components h_i. If the constraint qualification

$$\operatorname{Im} h \cap (t\mathbf{1} + \mathbb{R}^m_+) \neq \emptyset \quad \forall t \in \mathbb{R} \quad (6)$$

is satisfied, where 'Im' denotes the image and $\mathbf{1} = (1,\dots,1)$, then M is path-connected.

Proof. (Prop. 3)
Without loss of generality, we assume that only the upper limit m^* occurs (the proof running similar for the alternative case), hence the constraint set in (5) becomes in this special case

$$M = \{x \in L^\infty([t_0, t_1]) \mid$$
$$P(\int_{t_0}^{t^i} x(\tau)d\tau + m^*(t^i) - m_0 \geq \int_{t_0}^{t^i} \xi(\tau,\omega)d\tau \quad i = 1,\ldots,N) \geq p\}.$$

This fits the structure of Theorem 1 with

$$Y := L^\infty([t_0, t_1]),$$
$$h_i(x) = \int_{t_0}^{t^i} x(\tau)d\tau + m^*(t^i) - m_0,\ \tilde{\xi}_i(\omega) = \int_{t_0}^{t^i} \xi(\tau,\omega)d\tau \ (i = 1,\ldots,N),$$

where $N = m$ and $g(z) = z$. Now, h is a continuous, affine linear mapping, hence its components are concave and, consequently, strictly quasiconcave. In order to check the constraint qualification (6), let $t \in \mathbb{R}$ be arbitrary. Putting

$$x(\tau) := \begin{cases} t/(t^1 - t_0) & \tau \in [t_0, t^1] \\ 0 & \tau \in [t^1, t_1] \end{cases},$$

one has $x \in L^\infty([t_0, t_1])$ and $h_i(x) = t$ for $i = 1,\ldots,N = m$, hence $x \in \operatorname{Im} h \cap (t\mathbf{1} + \mathbb{R}^m_+)$.

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Note that the property of path-connectedness in Prop. 3 is slightly stronger than topological connectedness. Also note that the result of this proposition was derived under no assumptions on the stochastic parameter. On the other hand, it is easy to find counter-examples for this result in case that, for instance, deterministic constraints are included that strictly shrink the whole space (Ex. 4.2 in [5]). The reason for this is that, unlike convexity, connectedness is not stable under intersections.

Finally, let us consider problem (2) but with individual probabilistic constraints with respect to time:

$$\min\{g(x) | x \in X,\ P(m_*(t) \leq m(t,\xi,x) \leq m^*(t)) \geq p \quad \forall t \in [t_0, t_1]\}. \tag{7}$$

Proposition 4. *The following holds true:*

1. *If the cumulative inflow process η in (3) is quasiconcave on finite selections of intervals and if X is convex, then the feasible set of (7) is convex.*
2. *If only one-sided limitations occur in (7), then the feasible set of (7) is convex without any assumptions on ξ or η.*

Proof. Concerning a), the assumption on η implies the set

$$\{x \mid P(m_*(\bar{t}) \leq m(\bar{t}, \xi, x) \leq m^*(\bar{t})) \geq p\}$$

to be convex for each fixed $\bar{t} \in [t_0, t_1]$ (analogous to the proof of of Prop. 1) but with $t = \bar{t}$ fixed. Now, the result follows from the stabilty of convexity under intersections (with respect to all $t \in [t_0, t_1]$ and X). For b), we assume without loss of generality, that only upper limitations occur. Then, we may write for each fixed $\bar{t} \in [t_0, t_1]$

$$P(m(\bar{t}, \xi, x) \leq m^*(\bar{t})) = P(\int_{t_0}^{\bar{t}} \xi(\tau, \omega) d\tau \leq m^*(\bar{t}) - m_0 + \int_{t_0}^{\bar{t}} x(\tau) d\tau)$$
$$= P(\tilde{\xi}(\omega) \leq l(x) + \alpha) = F_{\tilde{\xi}}(l(x) + \alpha),$$

where

$$l(x) = \int_{t_0}^{\bar{t}} x(\tau) d\tau, \ \alpha = m^*(\bar{t}) - m_0, \ \tilde{\xi}(\omega) = \int_{t_0}^{\bar{t}} \xi(\tau, \omega) d\tau$$

and $F_{\tilde{\xi}}$ is the distribution function of the 1-dimensional random variable $\tilde{\xi}$. Hence,

$$\{x \mid P(m(\bar{t}, \xi, x) \leq m^*(\bar{t})) \geq p\} = \{x \mid l(x) + \alpha) = (F_{\tilde{\xi}})^{-1}(p)\},$$

where $(F_{\tilde{\xi}})^{-1}(p)$ denotes the p-quantile of $\tilde{\xi}$. As l is a linear function of x, this last set is convex. Now, the result follows again from stability of convexity under intersections.

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3 The discretized problem

In this section, we consider the fully discretized version of the original problem (1)

$$\min\{g(x) \mid x \in X, \ m_* \leq \tilde{m}_0 + L\xi - Lx \leq m^*\}.$$

Here, we assume that the interval $[t_0, t_1]$ is decomposed in m subintervals defined by t^i $(i = 1, \ldots, m-1)$ such that $t_0 =: t^0 < t^1 \cdots < t^{m-1} < t^m := t_1$. By $\xi : \Omega \to \mathbb{R}^m$ we denote the m-dimensional random vector the components ξ_i of which indicate the amount of inflow to the reservoir during the interval $[t^{i-1}, t^i]$. Similarly, $x \in \mathbb{R}^m$ refers to the discretized extraction of subsatnce from the reservoir. The vectors $m_*, m^* \in \mathbb{R}^m$ represent the lower and upper bounds of the reservoir, whereas $\tilde{m}_0 \in \mathbb{R}^m$ contains the initial level m_0 in all its components. Furthermore, we put

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

such that the components of $L\xi$ and Lx yield the cumulative amounts of inflow and extracted substance. Finally, $X \subseteq \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}$ refer to abstract deterministic constraint and an objective, respectively.

In the following, the above system is subjected to joint probabilistic constraints:

$$\min\{g(x) \mid x \in X \quad \text{and} \quad P(m_* \le \tilde{m}_0 + L\xi - Lx \le m^*) \ge p\}, \tag{8}$$

where $p \in (0,1)$. Concerning the structure of the feasible set of (8), we have the following result along the lines of the previous section:

Lemma 2. *It holds that*

1. *If in (8) X is convex and ξ has a quasiconcave distribution (e.g., multivarite normal), then the feasible set of (8) is convex.*
2. *If in (8) $X = \mathbb{R}^m$ and only one-sided limits occur (either m_* or m^*), then the feasible set of (8) is path-connected, no matter what the distribution of ξ is.*

Proof. For 1., put $\eta := L\xi$ and note that quasiconcavity of the distribution of ξ implies η to have quasiconcave distribution too. This implies convexity of the set

$$\{x \mid P(\eta \in Lx - \tilde{m}_0 + [m_*, m^*]) \ge p\},$$

where brackets refer to a rectangle in $\mathbb{R}^m$. From here, the result follows immediately. Concerning 2., it is assumed without loss of generality that only upper limits m^* occur in (8). We refer to Theorem 1 by setting $Y := \mathbb{R}^m$, $h(x) = Lx$ and $g(z) = Lz + \tilde{m}_0 - m^*$. Now, the result follows upon noting that the constraint qualification (6) reduces in our special case to the positive linear independence of the rows of L (cf. [5], Lemma 4.1), which certainly holds true since L is regular.

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In the rest of this section, the focus will be on stability aspects of problem (8). We are interested in how sensitive solutions or optimal values of (8) behave with respect to small changes of the underlying random vector ξ. This issue is of importance as one usually has only partial or no information about ξ and thus is forced to solve (8) on the basis of appropriate approximations (e.g. empirical ones). Then, it is natural to ask for conditions to be imposed on the data of (8) such that approximations will converge. From the conceptual point of view it will be convenient equivalently to rewrite (8) in the form

$$\min\{g(x) | x \in X, \, \mu(H(x)) \ge p\}, \tag{9}$$

where $\mu := P \circ \xi^{-1}$ is the distribution of ξ and $H(z) := \{z \in \mathbb{R}^m \mid m_* \le \tilde{m}_0 + Lz - Lx \le m^*\}$. In [1] and [2], we studied qualitative and quantitative solution stability of problem (9) in a general (nonsmooth) as well in a convex setting, where μ was supposed to be a Borel probability measure on $\mathbb{R}^s$ and $H : \mathbb{R}^m \rightrightarrows \mathbb{R}^s$ a multifunction with closed graph. Stability of solutions was analyzed with respect to perturbations of the probability measure μ. To this aim, the mappings

$$\begin{aligned} \Psi(\nu) &:= \arg\min\{g(x)|x \in X, \nu\left(H(x)\right) \geq p\} \quad \text{and} \\ \varphi(\nu) &:= \inf\{g(x)|x \in X, \nu\left(H(x)\right) \geq p\} \end{aligned} \tag{10}$$

were introduced. They assign to each (arbitrary) probability measure ν on $\mathbb{R}^s$ the resulting solution set or optimal value of a perturbed problem resulting from (9) after replacing μ by ν (i.e., $\Psi(\mu)$ and $\varphi(\mu)$ denote the solution set and optimal value, respectively, of the original problem). Closeness of probability measures was measured by the so-called discrepancy distance

$$\alpha_{\mathcal{B}}(\nu_1, \nu_2) = \sup_{B \in \mathcal{B}} |\nu_1(B) - \nu_2(B)|, \tag{11}$$

where $\mathcal{B} = \{H(x)|x \in X\} \cup \{z + \mathbb{R}^s_- | z \in \mathbb{R}^s\}$. In the special case where $H(x) = h(x) + \mathbb{R}^s_-$ for some single-valued mapping h, the collection $\mathcal{B}$ reduces just to its second part (cells) and the discrepancy distance becomes the Kolmogorov distance

$$d_K(\nu_1, \nu_2) = \sup_{z \in \mathbb{R}^s} |F_{\nu_1}(z) - F_{\nu_2}(z)|, \tag{12}$$

where F_{ν_1} and F_{ν_2} refer to the distribution functions of ν_1 and ν_2.

Theorem 2. *Consider the optimization problem (9) under the following assumptions*

1. *$X \subseteq \mathbb{R}^m$ is closed and convex, $H : \mathbb{R}^m \rightrightarrows \mathbb{R}^s$ has closed and convex graph, $g : \mathbb{R}^m \to \mathbb{R}$ is convex and μ is an r-concave probability measure on $\mathbb{R}^s$ for some $r < 0$.*
2. *The solution set $\Psi(\mu)$ of problem (9) is nonempty and bounded.*
3. *There exists some $x^* \in X$ such that $\mu(H(x^*)) > p$ (Slater's condition).*

Then, for the mappings defined in (11), it holds that Ψ is upper semicontinuous at μ and that φ is locally upper Lipschitzian at μ.

The first property means that for each open set $W \supseteq \Psi(\mu)$ there exists some $\delta > 0$ such that $\Psi(\nu) \subseteq W$ whenever $\alpha_{\mathcal{B}}(\mu, \nu) < \delta$. The second property ensures the existence of constants $L, \delta > 0$ such that $|\varphi(\nu) - \varphi(\mu)| \leq K\alpha_{\mathcal{B}}(\mu, \nu)$, whenever $\alpha_{\mathcal{B}}(\mu, \nu) < \delta$. The upper semicontinuity statement for the solution set mapping in Theorem 2 implies that any sequence of solutions to problem (9) with μ replaced by ν_n and $\alpha_{\mathcal{B}}(\nu_n, \mu) \to 0$ will have a cluster point belonging to the original solution set $\Psi(\mu)$. Note that, although the original measure μ was supposed to be r-concave for some $r < 0$, no such restriction is imposed on the perturbed measures, hence empirical approximations (not being r-concave) are included in this stability result. Applying it to problem (8), we have

Corollary 1. *In problem (8) assume that*

1. *X is closed and convex, g is convex and ξ has an r-concave distribution for some $r < 0$ (e.g., multivarite normal).*
2. *The solution set of (8) is nonempty and bounded.*
3. *There exists some $x^* \in X$ such that $P(m_* \leq \tilde{m}_0 + L\xi - Lx^* \leq m^*) > p$.*

Then, for the mappings Ψ and φ in (11), it holds that Ψ is upper semicontinuous at μ and that φ is locally upper Lipschitzian at μ with respect to the discrepancy distance (11).

The proof of the corollary is just a reformulation of Theorem 2 in terms of the data of problem (8) upon noting that the graph of H is evidently convex and closed there.

Of course assumption 2. of Corollary 1 is automatically fulfilled if X is a compact set. We note that quantitative stability results such as the upper Lipschitz property of the optimal value function obtained in Corollary 1 can be used to obtain large deviation results with respect to empirical approximations of the original probability measure (see, e.g., [8]).

The upper semicontinuity result obtained so far for the solution set mapping guarantees solutions of approximating problems to converge towards *some* solution of the original problem. However, as the solutions of problem (8) may be not unique, the question arises if *every* solution of problem (8) can be obtained as a limit of solutions to approximating problems. In other words, this issue relates to the lower semicontituity of Ψ at μ. Simple examples show that the assumptions of Theorem 2 are not sufficient to entail lower semicontituity (see, e.g. the preprint version of [2]. In order to indicate the necessary extensions yielding both, upper and lower semicontinuity of Ψ at once, we introduce the following concept:

Definition 2. (9) is called a *special convex problem* if

1. g is either affine linear or strictly convex.
2. X is either a polyhedral or strictly convex and closed set.
3. $H(x) = h(x) + \mathbb{R}^s_-$, where the components h_i are either affine linear or strictly concave.
4. μ is an r-concave probability measure on $\mathbb{R}^s$ for some $r < 0$ such that the rth power $(F_\mu)^r$ of the associated distribution function is strictly convex.

The following statement is a consequence of a result in [2] (Theorem 2):

Theorem 3. *Let (9) be a special convex problem and assume further that*

1. *The solution set $\Psi(\mu)$ of problem (9) is nonempty and bounded.*
2. *There exists some $x^* \in X$ such that $F_\mu(h(x^*)) > p$.*

Then it holds that Ψ is Hausdorff continuous at μ with respect to the Kolmogorov distance (12), i.e., for all $\varepsilon > 0$ there exists some $\delta > 0$ such that $d_H(\Psi(\mu), \Psi(\nu)) < \varepsilon$ holds true for all probability measures on $\mathbb{R}^s$ such that $d_K(\mu, \nu) < \delta$. Here, d_H refers to the Hausdorff distance between closed sets.

Before applying the last theorem to our specific problem (8), we need the following result:

Lemma 3. *Let μ be an r-concave probability measure on $\mathbb{R}^s$ such that $r < 0$, $\operatorname{supp} \mu = \mathbb{R}^s$ and the associated distribution function F_μ is analytic. Then $(F_\mu)^{r'}$ is strictly convex for all $r' < r$.*

Proof. Note first that F_μ is strictly positive due to $\operatorname{supp} \mu = \mathbb{R}^s$. Let $r' < r$ be arbitrary. Then, μ is r'-concave since μ is r-concave. Consequently, both $(F_\mu)^r$ and $(F_\mu)^{r'}$ are convex functions. Assume that $(F_\mu)^{r'}$ fails to be strictly convex. Then, there exist $x, y \in \mathbb{R}^s$ such that $x \neq y$ and

$$(F_\mu)^{r'}(\lambda x + (1-\lambda)y) = \lambda (F_\mu)^{r'}(x) + (1-\lambda)(F_\mu)^{r'}(y) \quad \forall \lambda \in [0,1]. \quad (13)$$

If $F_\mu(x) \neq F_\mu(y)$, then we may raise (13) with $\lambda = 1/2$ to the power $\alpha := r/r'$. Upon noting that $0 < \alpha < 1$, we derive that

$$(F_\mu)^r \left(\frac{x+y}{2}\right) = \left(\frac{(F_\mu)^{r'}(x) + (F_\mu)^{r'}(y)}{2}\right)^\alpha > \frac{(F_\mu)^r(x) + (F_\mu)^r(y)}{2}$$

from the strict concavity of the function t^α and from $(F_\mu)^{r'}(x) \neq (F_\mu)^{r'}(y)$. This, however, contradicts the convexity of $(F_\mu)^r$. Hence, $F_\mu(x) = F_\mu(y)$ and it follows from (13) that $(F_\mu)^{r'}$ is constant on the line segment $[x, y]$. The same is true, of course, for the distribuition function F_μ itself. With F_μ being an analytic function by assumption, it follows that F_μ is constant on a whole line in $\mathbb{R}^s$. Since F_μ is strictly positive everywhere, its constant value on the line is different from zero. Taking an appropriate sense of direction, at least one coordinate tends towards $-\infty$ when moving along the line. This implies any distribution function to decrease towards zero in contradiction to F taking a constant, strictly positive value along the entire line.

∎

Recalling that the nondegenerate multivariate normal distribution is r-concave for all $r < 0$, we have the following corollary to Lemma 3:

Corollary 2. *Any negative power of the distribution function of some non-degenrate multivariate normal distribution is strictly convex.*

Now we are in a position to formulate the desired continuity result. It will relate to problem (8) without abstract deterministic constraints ($X = \mathbb{R}^m$) and with one-sided limits only (upper limits without loss of generality). Finally, we pass in (8) from the inflow ξ to the cumulative inflow $\eta = L\xi$. This means, we consider the problem

$$\min\{g(x) \mid P(\tilde{m}_0 + \eta - Lx \leq m^*) \geq p\} \tag{14}$$

with $p \in (0,1)$ along with its equivalent reformulation (9), where μ is the distribution of η and $H(x) = \{z \in \mathbb{R}^m \mid \tilde{m}_0 + z - Lx \leq m^*\}$.

Theorem 4. *In (14) assume that g is affine linear or strictly convex, that η has a nondegenerate m-dimensional normal distribution and that the solution set of (14) is nonempty and bounded. Then, the solution set mapping Ψ defined in (11) is Hausdorff continuous at μ with respect to the Kolmogorov distance.*

Proof. Obviously, it holds that $H(x) = h(x) + \mathbb{R}^m_-$ with $h(x) := Lx + m^* - \tilde{m}_0$. Furthermore, since the distribution η has a nondegenerate m-dimensional normal distribution μ, we have that μ is (-1)-concave, for instance, and by Corollary 2, that $1/F_\mu$ is strictly convex. Consequently, (9) is a special convex problem in the sense of Definition 2. Finally, since $p < 1$, there exists some $t \in \mathbb{R}$ such that $F_\mu(t, \ldots, t) > p$. Now, setting

$$x_i^* := \max\{t + \tilde{m}_0 - \min_{j=1,\ldots,m} m_j^*, 0\} \quad (i = 1, \ldots, m),$$

one obtains

$$x_1^* + \cdots + x_i^* + m_i^* - \tilde{m}_0 \geq x_i^* + m_i^* - \tilde{m}_0 \geq t \quad (i = 1, \ldots, m).$$

Hence, $h(x^*) \geq (t, \ldots, t)$ and $F_\mu(h(x^*)) \geq F_\mu(t, \ldots, t) > p$ due to distribution functions being nondecreasing with respect to the partial order of $\mathbb{R}^m$. Now, Theorem 3 applies.

■

We note that the result of Theorem 4 can also be formulated in the presence of abstract deterministic constraints ($X \neq \mathbb{R}^m$) or for a probability level $p = 1$ but then, assumption 2. of Theorem 3 is not automatically satisfied as in Theorem 4 but has to be verified. An example given in [2] shows that no quantitative stability result for the solution set mapping can be obtained (in the sense of Hölder-Hausdorff continuity at any rate) under the assumptions of Theorem 3. To get, for instance, Hölder-Hausdorff continuity of Ψ at rate $1/2$, one would have further to restrict the convexity requirements of the data in Problem (9) (g convex-quadratic, X polyhedral, μ r-concave with some $r < 0$ such that F_μ^r is *strongly* convex). This issue was considered in detail in [1] and [3].

4 Fixed inflow profiles with stochastic initial time

In this section, we analyze the structure of the continuous problem (1) for a specific inflow process. We shall assume that this process has a fixed deterministic profile which is realized at a random starting time. The profile is given by a nonnegative function $v \in L_+^\infty([0,d])$ (determined, e.g., by a fixed operating pattern of some machine), where d denotes the duration of inflow. In the simplest case, the profile is constant over $[0,d]$. Defining ξ' as the random initial time, the inflow process becomes

$$\xi(t,\omega) := \begin{cases} v(t-\xi'(\omega)) & \text{if } t \in [\xi'(\omega), \xi'(\omega)+d] \\ 0 & \text{else} \end{cases} .$$

Accordingly, the cumulative inflow is

$$\eta(t,\omega) := \begin{cases} 0 & \text{if } t \leq \xi'(\omega) \\ V(t-\xi'(\omega)) & \text{if } t \in [\xi'(\omega), \xi'(\omega)+d] \\ V(d) & \text{if } t \geq \xi'(\omega)+d \end{cases} ,$$

where $V(t) = \int_0^t v(\tau)d\tau$. Since, by assumption on v, V is a nonnegative, (Lipschitz-) continuous and nondecreasing function, η maybe represented as

$$\eta(t,\omega) = \min\{\max\{0, V(t-\xi'(\omega))\}, V(d)\}. \tag{15}$$

Finally, we put $z(t) = \int_{t_0}^t x(\tau)d\tau$ for the cumulative extraction from the reservoir. For simplicity, we disregard additional abstract constraints in (1) and take the level constraints as constants m_*, m^*. Then, passing from (1) to probabilistic constraints, one has the following four options of jointly or separately treating constraints with respect to time and to the levels:

$$\begin{array}{ll} 1) & P(m_* \leq m(t,\xi,x) \leq m^* \quad \forall t \in [t_0,t_1]) \geq p \\ 2) & P(m_* \leq m(t,\xi,x) \quad \forall t \in [t_0,t_1]) \geq p \quad \text{and} \\ & P(m(t,\xi,x) \leq m^* \quad \forall t \in [t_0,t_1]) \geq p \\ 3) & P(m_* \leq m(t,\xi,x) \leq m^*) \geq p \quad \forall t \in [t_0,t_1] \\ 4) & P(m_* \leq m(t,\xi,x)) \geq p \quad \forall t \in [t_0,t_1] \quad \text{and} \\ & P(m(t,\xi,x) \leq m^*) \geq p \quad \forall t \in [t_0,t_1] \end{array} \tag{16}$$

In general, the choice of some alternative as in (16) has to be justified at the model building stage. In this concrete case, however, several of the constraint sets turn out to be equivalent. From the very definition of the probabilistic constraints in (16) one immediately derives the following implications:

$$\begin{array}{ccc} 1) & \Longrightarrow & 2) \\ \Downarrow & & \Downarrow \\ 3) & \Longrightarrow & 4) \end{array} \tag{17}$$

For the concrete problem, a much more precise characterization can be obtained:

Theorem 5. *We have the following assertions:*

1. *In (16), it always holds that* 2) $\Longleftrightarrow$ 4), *and the common constraint set (in x) is convex and has the explicit form*
$$\varphi^1(t) \leq \int_{t_0}^{t} x(\tau)d\tau \leq \varphi^2(t) \quad \forall t \in [t_0, t_1], \tag{18}$$
where $\varphi^{1(2)}(t) := m_0 - m^*(m_*) + \min\{\max\{0, V(t - q_1(q_2))\}, V(d)\}$, *and* $q_1 := \sup\{t \mid P(\xi' \leq t) \leq 1 - p\}$ *and* $q_2 := \inf\{t \mid P(\xi' \leq t) \geq p\}$.
2. *In case that* $V(d) \leq m^* - m_*$ *(moderate inflow), one even has* 2) $\Longleftrightarrow$ 3) $\Longleftrightarrow$ 4) *and, hence, the constraint set of 3) is convex too and has the explicit form given in 1.*
3. *If the inflow profile v is constant and if ξ' has a quasiconcave distribution, then the constraint set of 3) is convex.*

Proof. Ad 1.
We start by making explicit the first constraint of 2). Exploiting (15), it follows that:

$$\begin{aligned} P(m_* \leq m(t, \xi(\omega), x) \quad \forall t \in [t_0, t_1]) &= \\ P(m_* \leq m_0 + \eta(t, \omega) - z(t) \quad \forall t \in [t_0, t_1]) &= \\ P(z(t) \leq m_0 - m_* + \min\{\max\{0, V(t - \xi'(\omega))\}, V(d)\} \quad \forall t \in [t_0, t_1]) & \end{aligned} \tag{19}$$

Define $\Delta(\tau, t) := m_0 - m_* + \min\{\max\{0, V(t - \tau)\}, V(d)\}$ and

$$\alpha := \sup\{\tau \mid z(t) \leq \Delta(\tau, t) \quad \forall t \in [t_0, t_1]\}.$$

Note that, since V is nondecreasing and continuous, one has that Δ is continuous too and $\Delta(\tau_1, t) \geq \Delta(\tau_2, t)$ for all $t \in [t_0, t_1]$, whenever $\tau_1 \leq \tau_2$. Using this last relation along with (19), we end up at

$$\begin{aligned} &P(m_* \leq m(t, \xi(\omega), x) \quad \forall t \in [t_0, t_1]) \geq p \Longleftrightarrow \\ &\quad P(z(t) \leq \Delta(\xi'(\omega), t) \quad \forall t \in [t_0, t_1]) \geq p \Longleftrightarrow P(\xi'(\omega) \leq \alpha) \geq p \Longleftrightarrow \\ &\alpha \geq q_2 \Longleftrightarrow z(t) \leq \Delta(q_2, t) \quad \forall t \in [t_0, t_1] \Longleftrightarrow \\ &\qquad\qquad z(t) \leq \varphi^2(t) \quad \forall t \in [t_0, t_1], \end{aligned}$$

where q_2 and φ^2 are defined in the statement of 1. This gives the second inequality in (18). Similarly, the first inequality follows from the second constraint in 2). The convexity of the feasible set of $x \in L^\infty([t_0, t_1])$ satisfying (18) is obvious. It remains to show that 4) implies 2). Assume that 2) is violated, hence, without loss of generality, the first constraint of 2) is concerned: $P(m_* \leq m(t, \xi(\omega), x) \quad \forall t \in [t_0, t_1]) < p$. Now, by the chain of equivalences derived above, it may be deduced that $z(\bar{t}) > \varphi^2(\bar{t})$ for some $\bar{t} \in [t_0, t_1]$. As a consequence, $z(\bar{t}) > \Delta(q_2, \bar{t})$. Denoting $\alpha' := \sup\{\tau \mid z(\bar{t}) \leq \Delta(\tau, \bar{t})\}$,

one obtains that $\alpha' < q_2$, hence $P(\xi'(\omega) \le \alpha') < p$. This, in turn, implies that $P(z(\bar t) \le \Delta(\xi'(\omega), \bar t)) < p$, which amounts to the relation $P(m_* \le m(\bar t, \xi, x)) < p$. Hence, we have arrived at a violation of (4).
ad 2.
According to diagram (17) and taking into account the previous result, it suffices to show that 4)$\Longrightarrow$3). Assuming violation of 3), there exists some $\bar t \in [t_0, t_1]$ such that $P(m_* \le m(t, \xi, x) \le m^*) < p$. Consider first the case that $z(\bar t) \ge m_0 - m^* + V(d)$. Then,

$$z(\bar t) \ge m_0 - m^* + \min\{\max\{0, V(t - \xi'(\omega))\}, V(d)\} \quad \forall \omega \in \Omega,$$

hence, $m^* \ge m_0 + \eta(\bar t, \omega) - z(\bar t) = m(\bar t, \xi(\omega), x)$ for all $\omega \in \Omega$. Therefore, 4) is violated:

$$P(m_* \le m(\bar t, \xi, x)) = P(m_* \le m(\bar t, \xi, x) \le m^*) < p.$$

In the alternative case that $z(\bar t) < m_0 - m^* + V(d)$, the assumption of moderate inflow in 2. yields that

$$z(\bar t) < m_0 - m_* \le m_0 - m_* + \min\{\max\{0, V(t - \xi'(\omega))\}, V(d)\} \quad \forall \omega \in \Omega,$$

whence, $m_* \le m_0 + \eta(\bar t, \omega) - z(\bar t) = m(\bar t, \xi(\omega), x)$ for all $\omega \in \Omega$. Consequently,

$$P(m(\bar t, \xi, x) \le m^*) = P(m_* \le m(\bar t, \xi, x) \le m^*) < p.$$

Summarizing, the violation of 3) implies the violation of 4), as was to be shown.
ad 3.
Fixing an arbitrary $\bar t \in [t_0, t_1]$, it suffices to show that the set

$$\{x \mid P(m_* \le m(\bar t, \xi, x) \le m^*) \ge p\}$$

is convex. By the very definitions, we have

$$\begin{aligned} P(m_* \le m(\bar t, \xi(\omega), x) \le m^*) &= \\ P(m_0 - m^* + \eta(\bar t, \omega) \le z(\bar t) \le m_0 - m_* + \eta(\bar t, \omega)). & \end{aligned} \tag{20}$$

In case that $z(\bar t) \ge m_0 - m^* + V(d)$, it follows from (20) and $\eta(\bar t, \omega) \le V(d)$ (for all $\omega \in \Omega$) that

$$\begin{aligned} P(m_* \le m(\bar t, \xi(\omega), x) \le m^*) = P(z(\bar t) \le m_0 - m_* + \eta(\bar t, \omega)) &= \\ P(m_* \le m(\bar t, \xi(\omega), x)). & \end{aligned}$$

Now, repeating the chain of equivalences proved in part 1. of this Theorem but with $t = \bar t$ fixed, it can be shown that

$$P(m_* \le m(\bar t, \xi(\omega), x)) \ge p \Longleftrightarrow z(\bar t) \le \varphi^2(\bar t).$$

Then, the desired convexity property results from the definition of z. A symmetric argument applies to the case $z(\bar{t}) \leq m_0 - m_*$, hence we may assume now that (due to $m_* \leq m^*$)

$$m_0 - m^* \leq m_0 - m_* < z(\bar{t}) < m_0 - m^* + V(d) \leq m_0 - m_* + V(d). \quad (21)$$

Since the inflow profile v was supposed to be constant here, we have $\eta(\bar{t}, \omega) = \min\{\max\{0, c(\bar{t} - \xi'(\omega))\}, cd\}$ for some $c > 0$. In particular, $V(d) = cd$, and combination of (20) with (21) gives

$$\begin{aligned} & P(m_* \leq m(\bar{t}, \xi(\omega), x) \leq m^*) \\ &= P(z(\bar{t}) \in [m_0 - m^* + c(\bar{t} - \xi'(\omega)), m_0 - m_* + c(\bar{t} - \xi'(\omega))]) \\ &= P(\xi'(\omega) \in [c^{-1}(m_0 - m^* - z(\bar{t})) + \bar{t}, c^{-1}(m_0 - m_* - z(\bar{t})) + \bar{t}]). \end{aligned}$$

We arrive at $\{x \mid P(m_* \leq m(\bar{t}, \xi, x) \leq m^*) \geq p\} = \{x \mid \theta(z(\bar{t})) \geq p\}$, where

$$\theta(\alpha) := P(\xi'(\omega) \in [c^{-1}(m_0 - m^* - \alpha) + \bar{t}, c^{-1}(m_0 - m_* - \alpha) + \bar{t}]) \quad \forall \alpha \in \mathbb{R}.$$

Now the desired convexity result holds certainly true if θ is a quasiconcave function. This, however, follows immediately from the imposed quasiconcavity of the distribution of ξ'.

■

We note that the choice of equal probability levels in 2) and 4) of (16) just served the purpose of compatability with 3) in statement 2. of Theorem 5. Statement 1. of Theorem 5 remains true for unequal p levels in 2) and 4) respectively, upon modifying appropriately the definitions of q_1 and q_2. Finally, it is not difficult to construct counter-examples, illustrating that 3) can define a disconnected feasible set contradicting a full equivalence in (17).

The explicit representation in (18) allows to derive maximum probability levels p such that the resulting constraint set is nonempty. This is an important information for the appropriate choice of p in the modelling stage.

Corollary 3. *In case of moderate inflow ($V(d) \leq m^* - m_*$), the (coinciding) constraint sets defined by 2), 3) or 4) in (16) are nonempty for all probability levels p. In the opposite case ($V(d) \geq m^* - m_*$), define*

$$p^{max} = \sup\{p \in (0,1) \mid \exists x \text{ such that 2) (and 4)) in (16) are fulfilled}\}.$$

Assume that the inflow profile v is constant: $v(t) = c > 0 \, \forall t \in [0, d]$. Then, $p^{max} \geq 0.5$. If, moreover, ξ' has a distribution function F which is symmetric, continuous and strictly increasing on $F^{-1}(0,1)$ (like normal, t- or uniform distribution), then $p^{max} = F(F^{-1}(0.5) + (m^* - m_*)/(2c))$.

Proof. According to Theorem 5, the constraint sets defined by 2) and 4) and, in case of moderate inflow also by 3) in (16) have the explicit form (18).

Consequently, the constraint set is nonempty if and only if $\varphi^1(t) \leq \varphi^2(t)$ for all $t \in [t_0, t_1]$. Indeed, if this relation holds true, then

$$x(t) := \begin{cases} 0 & t \leq q_1 \\ v(t-q_1) & q_1 \leq t \leq q_1 + d \\ 0 & t > q_1 + d \end{cases} \Longrightarrow \varphi^1(t) = \int_{t_0}^{t} x(\tau) d\tau,$$

hence, x is one feasible solution. More precisely, the inequality above means that the constraint set is nonempty if and only if

$$\sup_{t \in [t_0, t_1]} \{\min\{\max\{0, V(t-q_1))\}, V(d)\} - \min\{\max\{0, V(t-q_2))\}, V(d)\}\} \leq m^* - m_*.$$

Since the supremum on the left is always less than or equal to $V(d)$, the assertion on the case of moderate inflow follows immediately. For the constant inflow profile indicated in the statement of this corollary, one has $V(t) = ct$ and the supremum above calculates as $\min\{c(q_2 - q_1), cd\}$. Hence, for non-moderate inflow, one arrives at the condition $q_2 - q_1 \leq c^{-1}(m^* - m_*)$ for the considered constraint set to be nonempty. From the definitions of q_1, q_2 (see statement of Theorem 5) it follows, that the last relation always holds true for $p = 0.5$ (then: $q_1 \geq q_2$), hence $p^{max} \geq 0.5$. If the distribution function F of ξ' is continuous and strictly increasing on $F^{-1}(0,1)$, then - by $p \in (0,1)$ - one has $q_1 = F^{-1}(1-p)$ and $q_2 = F^{-1}(p)$. Hence, the condition above gets the form $F^{-1}(p) - F^{-1}(1-p) \leq c^{-1}(m^* - m_*)$. Now, symmetry of F means that $F^{-1}(p) - F^{-1}(0.5) = F^{-1}(0.5) - F^{-1}(1-p)$. Summarizing, the constraint set defined by 3) and 4) in (16) is nonempty if and only if $F^{-1}(p) \leq (2c)^{-1}(m^* - m_*) + F^{-1}(0.5)$, whence the last assertion of the corollary.

■

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