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Metric regularity and quantitative stability in stochastic programs with probabilistic constraints

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Abstract. Introducing probabilistic constraints leads in general to nonconvex, nonsmooth or even discontinuous optimization models. In this paper, necessary and sufficient conditions for metric regularity of (several joint) probabilistic constraints are derived using recent results from nonsmooth analysis. The conditions apply to fairly general constraints and extend earlier work in this direction. Further, a verifiable sufficient condition for quadratic growth of the objective function in a more specific convex stochastic program is indicated and applied in order to obtain a new result on quantitative stability of solution sets when the underlying probability distribution is subjected to perturbations. This is used to derive bounds for the deviation of solution sets when the probability measure is replaced by empirical estimates.

Key words. stochastic programming – probabilistic constraints – metric regularity – nonsmooth analysis – quadratic growth – quantitative stability – empirical approximation

1. Introduction

When building stochastic models in decision making under (stochastic) uncertainty, the two main approaches consist in introducing future costs (e.g. for the compensation of constraint violations) and in fixing certain reliability levels for constraints. The latter approach is motivated by many problems in engineering sciences, where system reliability is an important feature (e.g. inventory control, power generation, structural design etc. [26], [31], [48]). It leads to stochastic programming problems with (so-called) probabilistic or chance constraints. To give a mathematical formulation of the model we study in this paper, let ξ be an s-dimensional random vector on some probability space $(\Omega, \mathcal{A}, P)$ and let $\xi \in H_j(x)$, $j = 1, \dots, d$, describe d constraints depending on ξ and on the decision vector $x \in \mathbb{R}^m$. Denoting by g the objective function and by C the closed subset of $\mathbb{R}^m$ expressing all deterministic constraints, we arrive at the following model:

$$\min\{g(x) \mid x \in C, P(\xi \in H_j(x)) \geq p_j,\ j = 1, \dots, d\}.$$

Here $p_j \in (0,1)$ denotes the probability (or reliability) level subject to which the constraint '$\xi \in H_j(x)$' has to be satisfied. Since different reliability requirements might be fixed for different constraints, the levels $p_j \in (0,1)$, $j = 1, \dots, d$ are allowed to be different. Later we shall prefer the following formulation of the model

$$\mathbf{P}(\mu) \qquad \min\{g(x) \mid x \in C, \mu(H_j(x)) \geq p_j,\ j = 1, \dots, d\}, \tag{1}$$

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where μ denotes the probability distribution of ξ, i.e., $\mu = P \circ \xi^{-1}$. In Section 2 the assumptions on the data $g, C, H_j\ (j = 1, \dots, d)$ are specified, so that the model is well-defined and enjoys suitable properties.

To illustrate the mathematical challenges of model (1), we look at a special instance of (1) where the only stochastic constraint is linear and takes the form $Ax \geq \xi$, i.e. $\xi \in Ax + \mathbb{R}^s_-$, with a (s, m)- matrix A, the deterministic constraint set C is a (convex) polyhedron and the objective function g is linear or (convex) quadratic. Since we have $\mu(\{\xi \mid Ax \geq \xi\}) = F_\mu(Ax)$, where F_μ denotes the probability distribution function of μ (or ξ), the specific model has the form

$$\min\{g(x) \mid x \in C, F_\mu(Ax) \geq p\} \tag{2}$$

Chance constrained models of type (2) are met in a number of applied optimization problems under uncertainty (the reader may consult [12], [36] and above all [31], and the references therein). Nevertheless, already model (2) exhibits possible nonconvexity, nondifferentiability and discontinuity properties that are induced by corresponding peculiarities of the distribution function F_μ. Conditions that imply convexity of model (2) are well understood (cf. [31] and Section 2). But, the situation is different as for differentiability properties of (2). Many multivariate distribution functions having densities are known to be nondifferentiable, e.g., classical ones like Dirichlet, Gamma (for certain parameter choices) and uniform distribution. Examples 7 and 8 in the Appendix show that the uniform distribution function of measures μ over convex and nonconvex polyhedral supports may fail to be differentiable at solutions to (2). Hence, classical tools from differentiable or convex analysis and optimization may not apply. Example 9 shows that even the existence of a continuous and bounded density does not imply the distribution function to be locally Lipschitzian (much less to be smooth). This illuminates that a smooth approach to our analysis of model (1) would significantly narrow the class of probability distributions. For that reason we will focus our analysis to nonsmooth probabilistic constraints in order to enlarge the range of applications.

In most practical applications of the stochastic programming methodology only incomplete information on the probability distribution μ (of ξ) is available. This fact and the possible need of approximations for μ in solution methods (cf. [31]) motivate a *stability analysis* of $\mathbf{P}(\mu)$ with respect to perturbations of μ in the space $\mathcal{P}(\mathbb{R}^s)$ of all Borel probability measures on $\mathbb{R}^s$ endowed with a suitable convergence (or metric). In the context of stochastic programs with probabilistic constraints, this problem was addressed in several papers, e.g. [1], [11], [12], [19], [20], [35], [36], [37], [38], [39], [40], [45], [46], [47]. In [11] a nonlinear parametric framework is adapted to study stability with respect to changes of finite dimensional parameters of the distribution μ. The convergence theory for measurable multifunctions is utilized in [39] to develop general approximation results for probabilistically constrained models. This approach is also used in [45], leading to general, satisfactory results on convergence rates of estimates for such models. Further results in this direction are given in [20]. Asymptotic properties of the optimal value based on an extended delta method are studied in [40]. Recently, a new class of nonparametric estimators that preserve convexity properties has been adapted to chance constrained models in [12]. The asymptotic behaviour of these estimates and of solution sets to stochastic programs is analysed, too. In the remaining papers quoted above, stochastic programs are viewed as parametric programs with respect to the probability measure μ. [19], [46] and [47] give qualitative stability results for constraint sets, marginal values and solutions when the measure μ is perturbed in $\mathcal{P}(\mathbb{R}^s)$ equipped with the (metrizable) topology of weak convergence ([5]). In [1], [35], [36], [37], [38] quantitative stability results for marginal values are obtained with respect to certain metric distances on $\mathcal{P}(\mathbb{R}^s)$ (the Prokhorov metric in [1] and so-called discrepancies in the other papers). The papers [35], [36], [37] also contain results on upper semicontinuity of local solution sets. For the case of $d = 1$, $C = \mathbb{R}^m$ and $H(x) = \{z \in \mathbb{R}^s \mid h(x) \geq z\}$ in problem (1) with continuously differentiable h and a probability measure μ having a locally Lipschitzian distribution function F_μ, a particular metric regularity result is given in [35] (Corollary 5.6) using the Clarke generalized gradient. This has been partially extended by allowing for a general closed subset C of $\mathbb{R}^m$ (but assuming h to be linear) in [36] (Proposition 2.1) by making use of Clarke's nonsmooth calculus. Another type of result for a nonconvex situation (with $d = 1$, C convex, h linear, but without assuming that μ has concavity properties) is developed in [38] (Theorem 4.6) and [36] (Corollary 2.2) by imposing a local growth condition on the composite function $F_\mu(h(\cdot))$ near binding feasible points.

The *aim* of the present paper is to extend the results in [35], [36], [37] in *two* directions: earlier conditions on the stability of probabilistic constraint sets are considerably generalized and a novel result on the Hausdorff Hölder stability of solution sets is established. We start our analysis by stating a general quantitative stability result for $\mathbf{P}(\mu)$ (Theorem 1), which relies on the recent work by Klatte [22] and on techniques developed in [37], [38]. The crucial conditions in this result are the *metric regularity* of the probabilistic constraints and a *quadratic growth* condition for the objective function near nonisolated minima. The growth condition appears in a more general context also in [2], [6], [41] for instance, and in a slightly different framework in [24]. The aim of our analysis is to derive verifiable conditions (on the original problem $\mathbf{P}(\mu)$) for metric regularity and quadratic growth. In particular, we focus on conditions that apply to nonsmooth probabilistic constraints.

In Section 3 we shall study the case of $C \subseteq \mathbb{R}^m$ being closed and $H_j(x) = \{z \in \mathbb{R}^s \mid h_j(x) \geq z_j\}$ with $h_j : \mathbb{R}^m \to \mathbb{R}^{s_j}$, $j = 1, \ldots, d$ and $\sum_{j=1}^{d} s_j = s$ in (1). Characterizations of metric regularity will be obtained by exploiting the nonconvex subdifferential calculus by Mordukhovich ([27], [28]). Two types of sufficient conditions for metric regularity are developed. The first one represents an explicit growth condition for the composite function $\theta_\mu(x) = (\mu(H_1(x)), \ldots, \mu(H_d(x)))$ at a feasible point (Theorem 4). The second type consists of separate constraint qualifications for the function $h = (h_1, \ldots, h_d)$ relative to C and for a function Φ_μ whose components are certain marginal distribution functions of μ (Theorem 5). In case μ has a density, a more transparent and verifiable condition, which implies the constraint qualification for Φ_μ, is established (Theorem 6). This can be achieved even globally if the strict positivity region of the density contains a so-called infinity path (Theorem 7). The principal statements are illustrated by examples showing their validity and limitations. Earlier results are essentially extended.

In Section 4 we consider a convex stochastic program of the form (2) and give a criterion implying quadratic growth of the objective near the solution set. In this respect a local strong concavity property of the measure μ is essential. The methodology for proving this result (Theorem 8) is shown to extend to establishing the Hausdorff Hölder continuity for solution sets (Theorem 9). Finally, we outline in Section 5 that our quantitative stability results have immediate applications for empirical approximations of $\mathbf{P}(\mu)$. Making use of recent results in empirical process theory we derive (exponential) bounds for the distance of original and approximate solution sets (Proposition 5 and 6).

2. Quantitative stability results

In this section, we develop a framework for stability analysis of probabilistic constrained models and present a general result on the quantitative stability of marginal values and (local) solution sets. We consider the stochastic programming model $\mathbf{P}(\mu)$ formulated in the introduction

$$\mathbf{P}(\mu) \qquad \min\{g(x) \mid x \in C, \mu(H_j(x)) \geq p_j, \;\; j = 1, \ldots, d\},$$

which involves several (joint) probabilistic constraints. For the data we assume that g is a continuous mapping from $\mathbb{R}^m$ into $\mathbb{R}$, C is a nonempty, closed subset of $\mathbb{R}^m$, H_j is a set-valued mapping from R^m into $\mathbb{R}^s$ having a closed graph (for each $j = 1, \ldots, d$), $p_j \in (0, 1)$ ($j = 1, \ldots, d$) and $\mu \in \mathcal{P}(\mathbb{R}^s)$. Making use of the notations $p = (p_1, \ldots, p_d)$ and $M_p(\nu) = \{x \in C \mid \nu(H_j(x)) \geq p_j, \;\; j = 1, \ldots, d)\}$ for each $\nu \in \mathcal{P}(\mathbb{R}^s)$, the model $\mathbf{P}(\mu)$ takes the form

$$\min\{g(x) \mid x \in M_p(\mu)\}. \tag{3}$$

We note that, for $\nu \in \mathcal{P}(\mathbb{R}^s)$, the function $\nu(H_j(\cdot))$ is upper semicontinuous (cf. Proposition 3.1 in [37]).

The first step to analyse stability of (3) with respect to perturbations of μ in $\mathcal{P}(\mathbb{R}^s)$ is to identify a (suitable) metric distance on $\mathcal{P}(\mathbb{R}^s)$. Consistently with [38], [37] we consider the following distance, which is sometimes called $\mathcal{B}$-discrepancy:

$$\alpha_{\mathcal{B}}(\mu, \nu) = \sup\{|\mu(B) - \nu(B)| \mid B \in \mathcal{B}\} \tag{4}$$

Here $\mathcal{B}$ is a class of closed subsets of $\mathbb{R}^s$ such that all sets of the form $H_j(x)$ ($x \in C$; $j = 1, \ldots, d$) belong to $\mathcal{B}$ and that $\mathcal{B}$ is a determining class (i.e., it has the property that if any two measures agree on $\mathcal{B}$, then they coincide). Convergence of a sequence of probability measures with respect to the metric $\alpha_{\mathcal{B}}$ means its uniform convergence on $\mathcal{B}$. Necessary and sufficient conditions on $\mathcal{B}$ such that weak convergence of probability measures implies uniform convergence on $\mathcal{B}$ usually refer to certain uniformity properties of the class $\mathcal{B}$ with respect to the limit measure ([4]) or to the sequential compactness of $\mathcal{B}$, viewed as a subset of the hyperspace of closed subsets of $\mathbb{R}^s$ equipped with a suitable topology ([25]). In particular, if $\mathcal{B}$ is a subclass of all convex Borel sets, then the uniform convergence on $\mathcal{B}$ to the limit measure μ is implied by its weak convergence and the condition $\mu(\partial B) = 0$ for all $B \in \mathcal{B}$ (∂B denoting the topological boundary of B). A special class $\mathcal{B}$ to be considered in the following is

$$\mathcal{B}_K = \{\times_{j=1}^d B_j \mid B_j \in \{\mathbb{R}^{s_j}, z_j + \mathbb{R}_-^{s_j}\}, z_j \in \mathbb{R}^{s_j}\}, \quad \text{where } \sum_{j=1}^d s_j = s.$$

The induced $\mathcal{B}$- discrepancy is denoted as α_K. For $d = 1$ it reduces to the Kolmogorov distance $\alpha_K(\mu, \nu) = \|F_\mu - F_\nu\|_\infty = \sup_{z\in\mathbb{R}^s} |F_\mu(z) - F_\nu(z)|$, where $F_\nu(z) = \nu(z + \mathbb{R}^s_-)$ is the distribution function of ν.

A special feature of model (3) is that we have to take into account its possible nonconvexity. Even when the original model is convex (cf. e.g. Corollary 1), perturbations of μ (e.g. by discrete measures) lead to nonconvex perturbed programs. Hence, an appropriate concept for the stability analysis of (3) has to take into account the perturbation of sets of local minimizers. Here we make use of the concepts developed in [21], [32] and, in particular, of so-called complete minimizing sets (CLM sets). Given $V \subseteq \mathbb{R}^m$, we put for each $\nu \in \mathcal{P}(\mathbb{R}^s)$

$$\varphi_V(\nu) = \inf\{g(x) \mid x \in M_p(\nu) \cap cl\,V\}$$
$$\Psi_V(\nu) = \operatorname{argmin}\{g(x) \mid x \in M_p(\nu) \cap cl\,V\} = \{x \in M_p(\nu) \cap cl\,V \mid g(x) = \varphi_V(\nu)\},$$

where $cl\,V$ denotes the closure of V. For $V = \mathbb{R}^m$ we shall briefly write $\varphi(\nu)$ and $\Psi(\nu)$ for the resulting global optimal value function and the set of global minimizers. Given $\mu \in \mathcal{P}(\mathbb{R}^s)$, we call a nonempty subset X of $\mathbb{R}^m$ a CLM set for (3) with respect to V, if V is an open subset of $\mathbb{R}^m$ containing X and $X = \Psi_V(\mu)$. For a discussion of CLM sets we refer to [32], but mention that nonempty sets of global minimizers, isolated local minimizers and sets of non-isolated local minimizers around which g satisfies a quadratic growth condition (cf. e.g. [6], [41], [22]) are examples of CLM sets.

To state our quantitative stability result, we still need a stability property for the probabilistic constraint in (3). We put $\theta_\mu : \mathbb{R}^m \to \mathbb{R}^d$, $\theta^j_\mu(x) = \mu(H_j(x))$ for each $x \in \mathbb{R}^m$, $j = 1, \ldots, d$, and $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$. Consistently with the general definition given in Section 3 we say here that the probabilistic constraint function $\theta_\mu(\cdot) - p$ is metrically regular with respect to C at some $x^0 \in M_p(\mu)$ if there are constants $a > 0$ and $\varepsilon > 0$ such that

$$\operatorname{dist}(x, M_{p-y}(\mu)) \leq a \cdot \operatorname{dist}(\theta_\mu(x) - p, \mathbb{R}^d_+ - y) = a\|\max\{0, p - y - \theta_\mu(x)\}\|$$

for all $(x, y) \in (C \cap B_\varepsilon(x^0)) \times B_\varepsilon(0)$. Here (and in all what follows) $B_\varepsilon(x)$ denotes the closed ball with radius ε around x. The following general stability result will serve as an orientation for the further development of our analysis.

Theorem 1. *In addition to the general conditions, assume that*

(i) X *is a CLM set for* $\mathbf{P}(\mu)$ *with respect to a bounded set* V *(i.e.,* $X = \Psi_V(\mu)$ *and* X *is compact),*
(ii) g *is locally Lipschitz continuous,*
(iii) the probabilistic constraint function $\theta_\mu(\cdot) - p$ *is metrically regular with respect to* C *at each* $x^0 \in X$.

Then there are constants $L > 0$ *and* $\delta > 0$ *such that the set-valued mapping* Ψ_V *from* $(\mathcal{P}(\mathbb{R}^s), \alpha_{\mathcal{B}})$ *to* $\mathbb{R}^m$ *is upper semicontinuous at* μ, $\Psi_V(\nu)$ *is a CLM set for* $\mathbf{P}(\nu)$ *with respect to* V *and* $|\varphi_V(\mu) - \varphi_V(\nu)| \leq L \cdot \alpha_{\mathcal{B}}(\mu, \nu)$ *holds whenever* $\nu \in \mathcal{P}(\mathbb{R}^s)$, $\alpha_{\mathcal{B}}(\mu, \nu) < \delta$.

If, moreover, the following quadratic growth condition is satisfied *(iv) there exists a constant* $c > 0$ *such that we have*

$$g(x) \geq \varphi_V(\mu) + c \cdot \text{dist}\,(x, \Psi_V(\mu))^2 \;\; \forall x \in M_p(\mu) \cap V,$$

then Ψ_V *is upper Hölder continuous at* μ *with rate* $1/2$*, i.e.,*

$$\sup_{x \in \Psi_V(\nu)} \text{dist}\,(x, \Psi_V(\mu)) \leq L \cdot \alpha_{\mathcal{B}}(\mu, \nu)^{1/2} \text{ whenever } \nu \in \mathcal{P}(\mathbb{R}^s),\; \alpha_{\mathcal{B}}(\mu, \nu) < \delta.$$

Proof. The first part of the assertion is proved in Theorem 3.2 of [37]. It remains to note that condition *(iii)* is equivalent to the fact that the set-valued mapping $q \mapsto M_q(\mu)$ from R^d to $\mathbb{R}^m$ is pseudo-Lipschitzian at each pair $(x^0, p) \in X \times \{p\}$ (cf. [29], Theorem 1.5). On the other hand, the latter property is equivalent to the local Lipschitz continuity of the function $(x, q) \mapsto \text{dist}\,(x, M_q(\mu))$ from $\mathbb{R}^m \times \mathbb{R}^d$ to $\mathbb{R}$ at each $(x^0, p) \in X \times \{p\}$ (see Theorem 2.3 in [34]), which is assumed in [37]. The second part of the result follows from Theorem 2.2 in [22] by using the same arguments as by deriving Theorem 3.2 in [37] from Proposition 1 and Theorem 1 in [21] (see also Theorem 2.5 in [35]).

□

All assumptions *(i)*-*(iv)* in the theorem concern the original (or unperturbed) problem $\mathbf{P}(\mu)$. While *(i)* and *(ii)* do not require further discussion, the conditions *(iii)* and *(iv)* are decisive and deserve verification.

The following corollaries illustrate the potentials of the approach considered here. To simplify the presentation, it is assumed in all corollaries that the objective function g and the set C of deterministic constraints in (1) are convex and that (1) contains one probabilistic constraint only (i.e. $d = 1$) and has the form

$$\min\{g(x) \mid x \in C,\; \mu(H(x)) \geq p\},$$

where H is a set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^s$ having closed graph, $p \in (0, 1)$ and $\mu \in \mathcal{P}(\mathbb{R}^s)$. The first two corollaries are concerned with the convex case that H has convex graph and μ carries a certain concavity property ('r-concavity') whereas the last corollary deals with a nonconvex situation where H and μ do not enjoy convexity assumptions. To introduce the notion of an r-concave probability measure ($r \in [-\infty, \infty]$) we define first the generalized mean function m_r on $R_+ \times \mathbb{R}_+ \times [0, 1]$ as follows:

$$m_r(a, b; \lambda) = \begin{cases} (\lambda a^r + (1-\lambda) b^r)^{1/r} & \text{if } r \in (0, \infty) \text{ or } r \in (-\infty, 0), ab > 0 \\ 0 & \text{if } ab = 0, r \in (-\infty, 0) \\ a^\lambda b^{1-\lambda} & \text{if } r = 0 \\ \max\{a, b\} & \text{if } r = \infty \\ \min\{a, b\} & \text{if } r = -\infty \end{cases} \tag{5}$$

The measure $\mu \in \mathcal{P}(\mathbb{R}^s)$ is called *r-concave*, $r \in [-\infty, \infty]$ ([7], [31]), if the inequality $\mu(\lambda B_1 + (1-\lambda) B_2) \geq m_r(\mu(B_1), \mu(B_2); \lambda)$ holds for all $\lambda \in [0, 1]$ and all convex Borel subsets B_1, B_2 of $\mathbb{R}^s$ such that $\lambda B_1 + (1-\lambda) B_2$ is Borel. For $r = 0$ and $r = -\infty$, μ is also called logarithmic concave and quasi-concave, respectively ([30]). Since $m_r(a, b; \lambda)$ is increasing in r if all the other variables are fixed, the sets of all r-concave probability measures are increasing if r is decreasing. It is known (cf. [7], [9], [30], [31]) that $\mu \in \mathcal{P}(\mathbb{R}^s)$ is r-concave for some $r \in [-\infty, 1/s]$ if μ has a density f_μ such that

$$f_\mu(\lambda z + (1-\lambda)\tilde{z}) \geq m_{r(s)}(f_\mu(z), f_\mu(\tilde{z}); \lambda), \tag{6}$$

where $r(s) = r(1-rs)^{-1}$ holds for all $\lambda \in [0,1]$ and $z, \tilde{z} \in \mathbb{R}^s$. We mention that e.g. the uniform distribution (on some bounded convex set), the (nondegenerate) multivariate normal distribution, the Dirichlet distribution, the multivariate Student and Pareto distributions are r-concave for some $r \in (-\infty, \infty]$ (see [7], Chapter 4 in [31]).

Corollary 1. *Assume that H has convex graph, that μ is r-concave for some $r \in (-\infty, \infty]$ and that there exists an element $\bar{x} \in C$ such that the strict inequality $\mu(H(\bar{x})) > p$ holds. Let $\Psi(\mu)$ be nonempty and bounded, V be an open bounded neighbourhood of $\Psi(\mu)$ and $\mathcal{B} = \{H(x), z + \mathbb{R}^s_- \mid x \in C, z \in \mathbb{R}^s\}$.*

Then there are $L > 0$, $\delta > 0$ such that the set-valued mapping Ψ_V from $(\mathcal{P}(\mathbb{R}^s), \alpha_{\mathcal{B}})$ to $\mathbb{R}^m$ is upper semicontinuous at μ with $\Psi_V(\mu) = \Psi(\mu)$ and $\Psi_V(\nu)$ being a CLM set for **P**(ν) *with respect to V, and $|\varphi(\mu) - \varphi_V(\nu)| \leq L\alpha_{\mathcal{B}}(\mu, \nu)$ holds whenever $\nu \in \mathcal{P}(\mathbb{R}^s)$, $\alpha_{\mathcal{B}}(\mu, \nu) \leq \delta$.*

The result is an immediate consequence of Theorem 1, since the assumptions on H and μ imply the metric regularity condition *(iii)* (Corollary 3.7. in [37]). In order to avoid handling of sets of local solutions and to make the presentation more transparent, we assume for the remainder of this section that C is (convex) compact and the open bounded neighbourhood V is chosen such that $C \subseteq V$, hence $\Psi_V = \Psi$ and $\varphi_V = \varphi$. The next result states Hölder stability of solution sets for a model with quadratic objective and linear probabilistic constraints and is proved in Section 4.

Corollary 2. *Let g be (linear or) quadratic, C be polyhedral, H have the form $H(x) = Ax + \mathbb{R}^s_-$, $x \in \mathbb{R}^m$ with some (s,m)- matrix A and μ be r-concave for some $r \in (-\infty, \infty]$. Assume that $\Psi(\mu) \cap \operatorname{argmin}\{g(x) \mid x \in C\} = \emptyset$ and that there exists an $\bar{x} \in C$ with $F_\mu(A\bar{x}) > p$. Moreover, assume that F^r_μ is strongly convex on some convex neighbourhood of $A\Psi(\mu)$. Then there are constants $L > 0$, $\delta > 0$ such that*

$$d_H(\Psi(\mu), \Psi(\nu)) \leq L\|F_\mu - F_\nu\|^{1/2}_\infty \text{ whenever } \nu \in \mathcal{P}(\mathbb{R}^s),\ \|F_\mu - F_\nu\|_\infty < \delta$$

(here d_H denotes the Hausdorff distance on subsets of $\mathbb{R}^m$).

The emphasis in the following Corollary is on the nonsmoothness of the measure.

Corollary 3. *Let $H(x) = h(x) + \mathbb{R}^s_-$, where $h \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^s)$ and assume that for all global minimizers $x^0 \in \Psi(\mu)$ the following conditions are satisfied:*

(i) $\forall \lambda \in \mathbb{R}^s_+ \setminus \{0\}\ \exists c \in C: \quad \sum_{j=1}^{s} \lambda_j \langle \nabla h_j(x^0), c - x^0 \rangle > 0$

(ii) μ has a density f_μ and, if $F_\mu(h(x^0)) = p$, then there exists some $z \in \mathbb{R}^s$ such that $z - h(x^0)$ belongs to the boundary of $\mathbb{R}^s_-$ and f_μ is bounded below by some positive number for almost all z' in some neighborhood of z.

Then there are constants $L > 0$, $\delta > 0$ *such that the set-valued mapping* Ψ *from* $(\mathcal{P}(\mathbb{R}^s), \alpha_K)$ *to* $\mathbb{R}^m$ *is upper semicontinuous at* μ *and* $|\varphi(\nu) - \varphi(\mu)| \le L\alpha_K(\mu, \nu)$ *for* $\nu \in \mathcal{P}(\mathbb{R}^s)$ *with* $\alpha_K(\mu, \nu) < \delta$*.*

The proof follows from Theorem 5, Theorem 6 and Proposition 3. Another result with emphasis on a possible nonsmoothness of the function h will be given later (cf. Corollary 5).

3. Metric regularity of probabilistic constraints

The importance of metric regularity as a stability concept in stochastic programming has been outlined in Section 2 (Theorem 1). In this section we study a specific class of probabilistic constraints by putting

$$H_j(x) = \{z \in \mathbb{R}^s \mid h_j(x) \ge z_j\} \quad x \in \mathbb{R}^m;\ j = 1, \dots, d$$

in the general model $\mathbf{P}(\mu)$ formulated in Section 1. Here we assume that $z_j \in \mathbb{R}^{s_j}, h_j : \mathbb{R}^m \to \mathbb{R}^{s_j}, z = (z_1, \dots, z_d) \in \mathbb{R}^s = \mathbb{R}^{s_1} \times \cdots \times \mathbb{R}^{s_d}$. Then the probabilistic constraint becomes

$$M = \{x \in C \mid \mu(\{z \in \mathbb{R}^s \mid h_j(x) \ge z_j\}) \ge p_j\} \quad (j = 1, \dots, d), \tag{7}$$

where $C \subseteq \mathbb{R}^m$ is closed, $\mu \in \mathcal{P}(\mathbb{R}^s)$ is a probability measure on R^s and $p_j \in (0, 1)$ are prescribed probability levels. For the following it will be more convenient to transform (7) into the equivalent description

$$M = \{x \in C \mid \Phi_\mu(h(x)) \ge p\}, \tag{8}$$

where $h = (h_1, \dots, h_d) : \mathbb{R}^m \to \mathbb{R}^s$ and $p = (p_1, \dots, p_d)$ refer to the entities introduced above. The mapping $\Phi_\mu = (\Phi_\mu^1, \dots, \Phi_\mu^d) : \mathbb{R}^s \to \mathbb{R}^d$ comprises the marginal distribution functions of μ as its components:

$$\Phi_\mu^j(y) = F_\mu(\infty, \dots, \infty, \overset{\overset{j}{\downarrow}}{y_j}, \infty, \dots, \infty) \quad (j = 1, \dots, d),$$

where $y = (y_1, \dots, y_d) \in \mathbb{R}^s, y_j \in \mathbb{R}^{s_j} (j = 1, \dots, d)$. Note that Φ_μ is a nondecreasing mapping which, in case of $d = 1$, reduces to the usual distribution function F_μ. Since the multifunctions H_j have closed graph, the components Φ_μ^j are upper semicontinuous (cf. Section 2).

The aim of this section is to formulate sufficient characterizations of metric regularity in a general nonsmooth framework. As the main tool the subdifferential calculus by Mordukhovich [28] shall be applied. This offers certain advantages over using the corresponding (larger in general) concepts by Clarke [10]. In particular, the Mordukhovich coderivative yields an equivalent criterion for metric regularity [27]. It turns out that, for instance in the case of a single locally Lipschitzian inequality $f(x) \le 0$, which is binding at some feasible point $\bar{x}$, an equivalent characterization of metric regularity by a relation like $0 \notin \partial f(\bar{x})$ requires the departure of ∂ from the framework of convexity. In fact, it is shown in [13] that Mordukhovich's subdifferential of Lipschitzian functions may be homeomorphic to any compact subset of $\mathbb{R}^n$.

3.1. Basics from nonsmooth analysis

In this section, some basic concepts for characterizing metric regularity in a nonsmooth setting shall be recalled. Let X, Y, Z be arbitrary sets. For multifunctions $\Phi : X \rightrightarrows Y$, $\Theta : Y \rightrightarrows Z$ put

$$\begin{aligned}
\operatorname{Ker}\Phi &= \{x \in X \mid 0 \in \Phi(x)\}\\
\operatorname{Im}\Phi &= \{y \in Y \mid y \in \Phi(x), x \in X\}\\
\operatorname{Gph}\Phi &= \{(x, y) \in X \times Y \mid y \in \Phi(x)\}\\
\Phi^{-1}(y) &= \{x \in X \mid y \in \Phi(x)\}\\
\Theta \circ \Phi(x) &= \bigcup_{y \in \Phi(x)} \Theta(y) \quad (x \in X), \text{ and if } X = \mathbb{R}^{n_1}, Y = \mathbb{R}^{n_2} :\\
\limsup_{x \to x^0} \Phi(x) &= \{y \in Y \mid \exists x_n \to x^0 \, \exists y_n \to y : y_n \in \Phi(x_n)\}.
\end{aligned}$$

Now let X, Y be two normed spaces. A multifunction $\Phi : X \rightrightarrows Y$ is called *metrically regular* at some point $(x^0, y^0) \in \operatorname{Gph}\Phi$ if there are constants $a > 0$ and $\varepsilon > 0$ such that

$$\operatorname{dist}(x, \Phi^{-1}(y)) \le a \cdot \operatorname{dist}(y, \Phi(x)) \quad \forall (x, y) \in B_\varepsilon(x^0) \times B_\varepsilon(y^0).$$

The abstract form of constraint sets writes as $C \cap F^{-1}(K)$, where $C \subseteq X$ and $K \subseteq Y$ are closed subsets of the respective spaces (K usually being a closed convex cone) and $F : X \to Y$ is the constraint function. Then, F is said to be metrically regular with respect to C at some feasible point $x^0 \in C \cap F^{-1}(K)$ if the associated multifunction

$$\Phi(x) = \begin{cases} -F(x) + K & \text{for } x \in C \\ \emptyset & \text{else} \end{cases}$$

is metrically regular at $(x^0, 0)$. It is easily seen that this is equivalent to the conventional definition of metric regularity for constrained systems:

$$\begin{aligned}
&\exists \varepsilon > 0 \, \exists a > 0 \, \forall (x, y) \in (C \cap B_\varepsilon(x^0)) \times B_\varepsilon(0) :\\
&\qquad \operatorname{dist}(x, C \cap F^{-1}(K - y)) \le a \cdot \operatorname{dist}(F(x), K - y)
\end{aligned}$$

Note that in this relation only the constraints given by F are subject to perturbations y whereas C is considered to be a fixed set of unperturbed constraints.

For some closed subset $S \subseteq \mathbb{R}^n$ and $x^0 \in S$ the following concepts are defined:

$$\begin{aligned}
T(S; x^0) &= \limsup_{t \downarrow 0} t^{-1}(S - \{x^0\}) \quad \text{(contingent cone)}\\
T_c(S; x^0) &= \{h \in \mathbb{R}^n \mid \forall x_n \to x^0 \ (\{x_n\} \subseteq S) \ \forall t_n \downarrow 0 \ \exists h_n \to h : \ x_n + t_n h_n \in S\}\\
&\quad \text{(Clarke's tangent cone)}\\
T^0(S; x^0) &= \{x^* \in \mathbb{R}^n \mid \langle x^*, h\rangle \le 0 \ \forall h \in T(S; x^0)\} \quad \text{(Fréchet normal cone)}\\
N_a(S; x^0) &= \limsup_{\substack{x \to x^0 \\ x \in S}} T^0(S; x) \quad \text{(approximate normal cone)}\\
N_c(S; x^0) &= \{x^* \in \mathbb{R}^n \mid \langle x^*, h\rangle \le 0 \ \forall h \in T_c(S; x^0)\} \quad \text{(Clarke's normal cone)}
\end{aligned}$$

The normal cone N_a induces the approximate subdifferential for lower semicontinuous functions $f : \mathbb{R}^n \to \mathbb{R}$:

$$\partial_a f(x^0) = \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N_a(\text{Epi } f; (x^0, f(x^0)))\},$$

where Epi refers to the epigraph. For locally Lipschitzian functions Clarke's subdifferential ∂_c relates to ∂_a as

$$\partial_c f(x^0) = \overline{conv}\, \partial_a f(x^0). \tag{9}$$

A closed subset $S \subseteq \mathbb{R}^n$ is called regular at $x^0 \in S$ in the sense of Clarke, if $T(S; x^0) = T_c(S; x^0)$. Similarly, a locally Lipschitzian function f is called regular at $x^0 \in \mathbb{R}^n$ in the sense of Clarke, if $T(\text{Epi } f; (x^0, f(x^0))) = T_c(\text{Epi } f; (x^0, f(x^0)))$. In case of the mentioned kinds of regularity it holds that $N_c(S; x^0) = N_a(S; x^0)$ and $\partial_c f(x^0) = \partial_a f(x^0)$.

A multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with closed graph and some point $(x^0, y^0) \in \text{Gph}\,\Phi$ induces a multifunction $D_a^*\Phi(x^0, y^0) : \mathbb{R}^m \rightrightarrows R^n$ defined via

$$D_a^*\Phi(x^0, y^0)(y^*) = \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N_a(\text{Gph}\,\Phi; (x^0, y^0))\},$$

which is called the approximate coderivative of Φ at (x^0, y^0). For single valued, locally Lipschitzian functions $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ one has (see [16], Proposition 8):

$$D_a^*\Phi(x, \Phi(x))(y^*) = \partial_a \langle y^*, \Phi\rangle(x) \quad \forall x \in \mathbb{R}^n\ \forall y^* \in \mathbb{R}^m \tag{10}$$

The following results are due to Mordukhovich (compare [27], [28]) and will be substantially exploited in this section:

Theorem 2. *A multifunction* $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *with closed graph is metrically regular at some point* $(x^0, y^0) \in \text{Gph}\,\Phi$ *if and only if Ker* $D_a^*\Phi(x^0, y^0) = \{0\}$.

Theorem 3. *Let the multifunctions* $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *and* $\Theta : \mathbb{R}^m \rightrightarrows \mathbb{R}^k$ *have closed graph and* $(\bar{x}, \bar{z}) \in \text{Gph}\,(\Theta \circ \Phi)$. *Suppose that the multifunction* $M : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ *defined by*

$$M(x, z) = \Phi(x) \cap \Theta^{-1}(z)$$

is locally bounded around $(\bar{x}, \bar{z})$ *and that the condition*

$$D_a^*\Theta(y, \bar{z})(0) \cap \text{Ker}\, D_a^*\Phi(\bar{x}, y) = \{0\} \quad \forall y \in M(\bar{x}, \bar{z})$$

holds. Then one has

$$D_a^*(\Theta \circ \Phi)(\bar{x}, \bar{z})(z^*) \subseteq \bigcup_{y \in \Phi(\bar{x}) \cap \Theta^{-1}(\bar{z})} D_a^*\Phi(\bar{x}, y) \circ D_a^*\Theta(y, \bar{z})(z^*) \quad \forall z^* \in \mathbb{R}^k$$

Lemma 1. *Let* $S_1, S_2 \subseteq \mathbb{R}^n$ *be closed sets with* $\bar{x} \in S_1 \cap S_2$ *and* $N_a(S_1; \bar{x}) \cap -N_a(S_2; \bar{x}) = \{0\}$. *Then*

$$N_a(S_1 \cap S_2; \bar{x}) \subseteq N_a(S_1; \bar{x}) + N_a(S_2; \bar{x}),$$

where equality holds if S_1, S_2 *are regular in the sense of Clarke.*

3.2. An explicit growth condition

Before dealing with the chance constraint (8) we start our considerations with general constraint sets described by finitely many inequalities:

$$P = \{x \in C \mid F(x) \geq 0\} \quad F : \mathbb{R}^n \to \mathbb{R}^k,\ C \subseteq \mathbb{R}^n\ (C \text{ closed}). \tag{11}$$

Obviously, (8) fits into this type of constraints. For a feasible point $x^0 \in P$ denote by

$$I = \{i \in \{1, \ldots, k\} \mid F_i(x^0) = 0\}$$
$$J = \{i \in \{1, \ldots, k\} \mid F_i \text{ is not continuous at } x^0\}$$

the sets of active and noncontinuity indices, respectively, at x^0, where the F_i refer to the components of F. The following definition provides an explicit growth condition on the components of F which will imply metric regularity.

Definition 1. *We say that the constraint mapping $F : \mathbb{R}^n \to \mathbb{R}^k$ in (11) is growing at some feasible point $x^0 \in P$ with respect to C if*

(i) F_i is upper semicontinuous in a neighbourhood of x^0 for $i \in \{1, \ldots, k\}$
(ii) there exists an $\rho > 0$ such that the following local growth condition is fulfilled:

$$\exists \eta > 0\ \forall x \in B_\eta(x^0) \cap C\ \forall \varepsilon > 0\ \exists y \in B_\varepsilon(x) \cap C :$$
$$F_i(y) > F_i(x) + \rho\|y - x\| \quad \forall i \in I \cup J.$$

Note that, for continuous F, this is merely a growth condition imposed on the active components at x^0.

Lemma 2. *Let $x^0 \in P$ be a feasible point of (11). If F is growing at x^0 with respect to C, then F is metrically regular at x^0 with respect to C.*

Proof. According to Section 3.1 one has to verify metric regularity of the multifunction

$$\Phi(x) = \begin{cases} -F(x) + \mathbb{R}^k_+ & \text{if } x \in C \\ \emptyset & \text{else} \end{cases}$$

at the point $(x^0, 0) \in \operatorname{Gph} \Phi$. Choose a number γ with $0 < \gamma < \eta$ (where η refers to Definition 1) which, according to the definition of the index sets I and J, satisfies

$$F_i(z) > \gamma \quad \forall i \notin I \cup J\ \forall z \in \operatorname{int} B_\gamma(x^0) \tag{12}$$

For computing Fréchet normal cones T^0 in a neighbourhood of $(x^0, 0)$, fix an arbitrary $(x, b) \in (\operatorname{int} B_\gamma(x^0) \times \operatorname{int} B_\gamma(0)) \cap \operatorname{Gph} \Phi$. Then $x \in C$ and $b \geq -F(x)$ by definition of Φ.

Let us first consider the case $I \cup J \neq \emptyset$. By Definition 1 there exists a sequence $y_l \to x$ ($y_l \in C$), such that $F_i(y_l) > F_i(x) + \rho\|y_l - x\|\ \forall i \in I \cup J$. Clearly $y_l \neq x$. We show that the vector

$$\begin{pmatrix} x \\ b \end{pmatrix} + \|y_l - x\| \begin{pmatrix} (y_l - x)/\|y_l - x\| \\ -\rho\mathbf{1} \end{pmatrix} = \begin{pmatrix} y_l \\ b - \rho\|y_l - x\|\mathbf{1} \end{pmatrix} \tag{13}$$

with $\mathbf{1} = (1, \dots, 1)^T$ belongs to $\operatorname{Gph}\Phi$ for l large enough. In fact, if $i \in I \cup J$, then

$$[b]_i - \rho\|y_l - x\| \geq -F_i(x) - \rho\|y_l - x\| > -F_i(y_l),$$

where the $[b]_i$ denote the corresponding components of b. On the other hand, taking for instance the Euclidean norm, $b \in \operatorname{int} B_\gamma(0)$ implies $[b]_i > -\gamma$, hence $[b]_i - \rho\|y_l - x\| > -\gamma$ for $i = 1, \dots, k$ and large l. In particular, relation (12) makes also the indices $i \notin I \cup J$ satisfy $[b]_i - \rho\|y_l - x\| > -F_i(y_l)$ (l large enough). Combining both cases one arrives at $b - \rho\|y_l - x\|\mathbf{1} \in -F(y_l) + \mathbb{R}^k_+$, which together with $y_l \in C$ yields $(y_l, b - \rho\|y_l - x\|\mathbf{1}) \in \operatorname{Gph}\Phi$. Without loss of generality, we assume $(y_l - x)/\|y_l - x\| \to \xi$, so (13) shows that $(\xi, -\rho\mathbf{1})$ belongs to the contingent cone $T(\operatorname{Gph}\Phi; (x, b))$. Consequently,

$$\langle(\xi, -\rho\mathbf{1}), (\xi^*, y^*)\rangle = \langle\xi, \xi^*\rangle - \rho\langle\mathbf{1}, y^*\rangle \leq 0 \quad \forall(\xi^*, y^*) \in T^0(\operatorname{Gph}\Phi; (x, b))$$

Due to $\|\xi\| = 1$ this means $\|\xi^*\| \geq \langle-\xi, \xi^*\rangle \geq -\rho\langle\mathbf{1}, y^*\rangle$.

Now turn to the case $I \cup J = \emptyset$. Here $(x, b) + \delta(0, -\rho\mathbf{1}) \in \operatorname{Gph}\Phi$ for sufficiently small $\delta > 0$ (compare (12) and recall $[b]_i > -\gamma$ for the components of b). So $(0, -\rho\mathbf{1}) \in T(\operatorname{Gph}\Phi; (x, b))$, and applying an arbitrary normal vector (ξ^*, y^*) to this provides the inequality $-\rho\langle\mathbf{1}, y^*\rangle \leq 0$. Summarizing, one has

$$-\rho\langle\mathbf{1}, y^*\rangle \leq \|\xi^*\| \tag{14}$$
$$\forall(\xi^*, y^*) \in T^0(\operatorname{Gph}\Phi; (x, b)) \ \forall(x, b) \in (\operatorname{int} B_\gamma(x^0) \times \operatorname{int} B_\gamma(0)) \cap \operatorname{Gph}\Phi$$

in any case. Consider any $z^* \in \operatorname{Ker} D_a^*\Phi(x^0, 0)$. Local upper semicontinuity of all components F_i together with the closedness of C imply the closedness (near $(x^0, 0)$) of $\operatorname{Gph}\Phi$. By virtue of Theorem 2 the lemma is proved if we can show that $z^* = 0$. By definition

$$(0, -z^*) \in N_a(\operatorname{Gph}\Phi; (x^0, 0)) = \limsup_{\substack{(x,b)\to(x^0,0)\\(x,b)\in\operatorname{Gph}\Phi}} T^0(\operatorname{Gph}\Phi; (x, b))$$

so there are sequences

$$(x_l, b_l) \to (x^0, 0),\ (x_l, b_l) \in \operatorname{Gph}\Phi,\ (\xi_l^*, y_l^*) \to (0, -z^*),\ (\xi_l^*, y_l^*) \in T^0(\operatorname{Gph}\Phi; (x_l, b_l)).$$

Along with (14) this leads to $-\rho\langle\mathbf{1}, -z^*\rangle \leq 0$, or, because ρ is positive, to $\langle\mathbf{1}, z^*\rangle \leq 0$. On the other hand, $b_l \geq -F(x_l)$ implies $(0, e_j) \in T(\operatorname{Gph}\Phi; (x_l, b_l))$ for arbitrary standard unit vectors $e_j \in \mathbb{R}^k$, $(j = 1, \dots, k)$, hence $y_l^* \leq 0$. By continuity, $z^* \geq 0$, so the desired relation $z^* = 0$ follows.

□

The reverse direction of Lemma 2 does not hold in general, as one can see from the example $C = \mathbb{R}$, $F(x) = |x|$ if $x \neq 0$ and $F(0) = 1$. While F is upper semicontinuous, it fails to be growing at 0. On the other hand one computes

$$N_a(\operatorname{Epi}(-F); (0, 0)) = \{(x, y) \in \mathbb{R}^2 \mid y \in \{0, -|x|\}\}$$

hence, $\operatorname{Ker} D_a^*\Phi(0,0) = \{0\}$ for the multifunction $\Phi = -F + I\!R_+$, so Φ is metrically regular at $(0,0)$ due to Theorem 2 and, therefore, F is metrically regular at 0. For the continuous case, however, the growth condition of Definition 1 is an equivalent characterization of metric regularity in the constraint system (11) (cf. [14]):

Lemma 3. *In (11) assume that F is continuous and that $x^0 \in P$. Then metric regularity of F at x^0 w.r.t. C implies F to be growing at x^0 w.r.t. C.*

Now we apply the above results to the characterization of metric regularity of the probabilistic constraint (8).

Theorem 4. *In the probabilistic constraint (8) let h be continuous and $x^0 \in M(\mu)$ some feasible point. Suppose there exist $\rho > 0, \eta > 0$ such that for all components Φ_μ^j of Φ_μ that are not continuous at $h(x^0)$ or that are binding (i.e., $\Phi_\mu^j(h(x^0)) = p_j$) the growth condition*

$$\forall x \in B_\eta(x^0) \cap C \ \forall \varepsilon > 0 \ \exists y \in B_\varepsilon(x) \cap C : \quad \Phi_\mu^j(h(y)) > \Phi_\mu^j(h(x)) + \rho\|y - x\|$$

is fulfilled. Then the constraint function $\Phi_\mu(h(\cdot)) - p$ is metrically regular at x^0 w.r.t. C. If, moreover, Φ_μ is continuous, then the growth condition above, imposed on the binding components Φ_μ^j, is equivalent with metric regularity of $\Phi_\mu(h(\cdot)) - p$ at x^0 w.r.t. C.

Proof. Recall that the components of Φ_μ are automatically upper semicontinuous, hence the composition $\Phi_\mu(h(\cdot)) - p$ enjoys the same property. Apply Lemma 2. For the second part apply Lemma 3.

□

Two examples shall illustrate the potential and the limitations of Theorem 4.

Example 1. In the chance constraint (8) let $m = 2,\ s = d = 1,\ p = 0.5,\ h(x_1, x_2) = x_1 + x_2$. Let μ be the uniform distribution over the interval $[-0.5, 0.5]$ and take

$$C = \{(x_1, x_2) \in I\!R^2 \mid x_1 \geq 0,\ x_1^3 \leq x_2 \leq x_1^2\}$$

Obviously one has $\Phi_\mu(y) = F_\mu(y) = y + 0.5 \quad \forall y \in (-0.5, 0.5)$. The point of interest is $x^0 = (0,0) \in C$. Then, in a small neighbourhood of this point, it holds that $F_\mu(h(x_1, x_2)) = x_1 + x_2 + 0.5$. In particular, the constraint is binding at x^0. Evidently, the second statement of Theorem 4 applies, so we know that checking metric regularity is equivalent to verifying the growth condition of Theorem 4. Now, fix any $x \in C$ near x^0. One may find a point $y \in C,\ y \neq x$ arbitrarily close to x such that $y - x \in I\!R_+^2$. Then, $F_\mu(h(y)) - F_\mu(h(x)) = y_1 + y_2 - (x_1 + x_2) = \|y - x\|_1$, therefore $F_\mu(h(\cdot)) - p$ is growing with $\rho = 1/2$ at x^0 w.r.t C, hence metric regularity of $F_\mu(h(\cdot)) - p$ holds at x^0 w.r.t. C.

In [36] (Corollary 2.2) a sufficient growth condition for metric regularity of the constraint function $\Phi_\mu(h(\cdot)) - p$ was proposed for the special case $d = 1,\ \Phi_\mu = F_\mu$ continuous, h linear and C convex. Essentially, growth was required along line segments in C. Note that in Example 1 there are no (nontrivial) line segments emanating from x^0 and entirely contained in C, so the mentioned condition does not work here although, apart from nonconvexity of C, the remaining assumptions are fulfilled. Furthermore, even if C is convex and F_μ continuous, but h violates linearity (e.g. being piecewise differentiable), this condition does no longer hold true. This illustrates the extension obtained by Theorem 4.

The next example indicates a situation where metric regularity of chance constraints cannot be recovered from the growth condition of Theorem 4 (compare Remark 2.5 in [36]).

Example 2. In (8), let $d = 1$, $C = \mathbb{R}^m$, h continuous and $\mu \in \mathcal{P}(\mathbb{R}^s)$ be a discrete measure with finite support. Suppose $p \in (0,1)$ to fulfill $\inf_{z\in\mathbb{R}^s} |F_\mu(z) - p| > 0$. Then the constraint function $F_\mu(h(\cdot)) - p$ is metrically regular at all feasible x^0, whereas it is not growing at all x^0 for which F_μ is not continuous at $h(x^0)$.

3.3. Separate constraint qualifications

While metric regularity of the probabilistic constraint (8) has been characterized in terms of the composite function $\Phi_\mu \circ h$ so far, we now want to formulate separate constraint qualifications for the two single functions that are easier to verify and to interpret. First, an auxiliary result is needed:

Proposition 1. *Let $F : \mathbb{R}^n \to \mathbb{R}^k$ have upper semicontinuous components and be nondecreasing at $x^0 \in \mathbb{R}^n$. Then the associated multifunction $\phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ defined by $\phi(x) = -F(x) + \mathbb{R}^k_+$ satisfies* $\operatorname{Im} D^*_a\phi(x^0, y) \subseteq \mathbb{R}^n_- \ \forall y \in \phi(x^0)$.

Proof. First note that $\operatorname{Gph}\phi$ is closed due to the upper semicontinuity of F. Consider arbitrary $y \in \phi(x^0)$ and $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^k$ such that $x^* \in D^*_a\phi(x^0, y)(y^*)$. This means $(x^*, -y^*) \in N_a(\operatorname{Gph}\phi; (x^0, y))$ and, by definition, there are sequences $(x_l, y_l) \to (x^0, y)$, $((x_l, y_l) \in \operatorname{Gph}\phi)$ and $(x_l^*, -y_l^*) \to (x^*, -y^*)$ $((x_l^*, -y_l^*) \in T^0(\operatorname{Gph}\phi; (x_l, y_l)))$. Since F is nondecreasing at x^0, one has $(e_j, 0) \in T(\operatorname{Gph}\phi; (x_l, y_l))$ for all standard unit vectors $e_j \in \mathbb{R}^n$ and for all $l \in \mathbb{N}$. It follows that $\langle (x_l^*, -y_l^*), (e_j, 0)\rangle = (x_l^*)_j \le 0$ for $j = 1, \ldots, n$, hence $x_l^* \le 0$ and $x^* \in \mathbb{R}^n_-$, as desired.

□

It is interesting to note that, as a consequence of the last proposition, one has the equivalence (see [14])

$$0 \in \partial_a F(x) \quad \Longleftrightarrow \quad 0 \in \partial_c F(x)$$

for any nondecreasing $F : \mathbb{R}^n \to \mathbb{R}$. In particular, this holds for distribution functions.

Now, with the constraint functions Φ_μ and h from the definition of the probabilistic constraint in (8) we associate the following two multifunctions $\Gamma_1 : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ and $\Gamma_2 : \mathbb{R}^m \rightrightarrows \mathbb{R}^s$ via

$$\Gamma_1(z) = p - \Phi_\mu(z) + \mathbb{R}^d_+ \quad \text{and} \quad \Gamma_2(x) = \begin{cases} h(x) & x \in C \\ \emptyset & \textit{else} \end{cases}$$

Then, their composition is $\Gamma = \Gamma_1 \circ \Gamma_2 : I\!R^m \rightrightarrows I\!R^d$ with

$$\Gamma(x) = \begin{cases} p - \Phi_\mu(h(x)) + I\!R^d_+ & x \in C \\ \emptyset & else \end{cases}$$

Proposition 2. *In (8), assume h to be continuous and consider some feasible point $\bar{x} \in M$. Then the two constraint qualifications*

$$\operatorname{Ker} D^*_a \Gamma_1(h(\bar{x}), 0) = \{0\} \tag{15}$$

$$\operatorname{Ker} D^*_a \Gamma_2(\bar{x}, h(\bar{x})) \cap I\!R^s_- = \{0\} \tag{16}$$

imply $\operatorname{Ker} D^*_a \Gamma(\bar{x}, 0) = \{0\}$.

Proof. All of the three multifunctions Γ_2, Γ_1 and Γ have a closed graph (due to the closedness of C, continuity of h and upper semicontinuity of Φ_μ). Let us assume for a moment that the application of Theorem 3 is justified. Then the relation $0 \in D^*_a \Gamma(\bar{x}, 0)(z^*)$ (for arbitrary z^*) along with the fact that Γ_2 is single-valued ($\Gamma_2(\bar{x}) = h(\bar{x})$) yield the existence of some $y^* \in I\!R^s$ such that

$$y^* \in D^*_a \Gamma_1(h(\bar{x}), 0)(z^*) \quad \text{and} \quad 0 \in D^*_a \Gamma_2(\bar{x}, h(\bar{x}))(y^*).$$

From Proposition 1 we know that $\operatorname{Im} D^*_a \Gamma_1(h(\bar{x}), 0) \subseteq I\!R^s_-$. This leads to

$$y^* \in \operatorname{Ker} D^*_a \Gamma_2(\bar{x}, h(\bar{x})) \cap I\!R^s_- = \{0\}$$

by (16) and to $z^* \in \operatorname{Ker} D^*_a \Gamma_1(h(\bar{x}), 0) = \{0\}$ by (15). Consequently, $\operatorname{Ker} D^*_a \Gamma(\bar{x}, 0) = \{0\}$, as desired.

To check the assumptions of Theorem 3 first note that the multifunction $M(x, z) = \Gamma_2(x) \cap \Gamma_1^{-1}(z)$ fulfills either $M(x, z) = \emptyset$ or $M(x, z) = \{h(x)\}$, so it is locally bounded by continuity of h. In particular, $M(\bar{x}, 0) = \{h(\bar{x})\}$, and again from Proposition 1 and (16) we have

$$D^*_a \Gamma_1(h(\bar{x}), 0)(0) \cap \operatorname{Ker} D^*_a \Gamma_2(\bar{x}, h(\bar{x})) \subseteq I\!R^s_- \cap \operatorname{Ker} D^*_a \Gamma_2(\bar{x}, h(\bar{x})) = \{0\}.$$

□

The result of this proposition can now be restated in terms of the ingredients of the probabilistic constraint (8) itself.

Theorem 5. *The constraint function $\Phi_\mu(h(\cdot)) - p$ in (8) is metrically regular at some feasible point $\bar{x} \in M$ w.r.t. C if the following two conditions are fulfilled:*

(i) The function $\Phi_\mu(\cdot) - p$ is metrically regular at $h(\bar{x})$ in the constraint $\Phi_\mu(z) \geq p$.
(ii) h is continuous, $N_a(\operatorname{Gph} h; (\bar{x}, h(\bar{x}))) \cap (-N_a(C; \bar{x}) \times \{0\}) = \{0\}$ and $D^*_a h(\bar{x}, h(\bar{x}))(y^*) \cap -N_a(C; \bar{x}) = \emptyset \quad \forall y^* \in I\!R^s_- \setminus \{0\}$

Proof. Obviously, condition *(i)* is equivalent to (15) by Theorem 2. Concerning *(ii)* one has $\operatorname{Gph}\Gamma_2 = \operatorname{Gph} h \cap (C \times \mathbb{R}^s)$ for the multifunction Γ_2 introduced above. The first part of *(ii)* corresponds to the assumption of Lemma 1 (with $S_1 = \operatorname{Gph} h$ and $S_2 = C \times \mathbb{R}^s$), so the lemma yields

$$N_a(\operatorname{Gph}\Gamma_2; (\bar{x}, h(\bar{x}))) \subseteq N_a(\operatorname{Gph} h; (\bar{x}, h(\bar{x}))) + N_a(C; \bar{x}) \times \{0\}$$

Choose any $y^* \in \operatorname{Ker} D_a^*\Gamma_2(\bar{x}, h(\bar{x})) \cap \mathbb{R}^s_-$. In particular, $(0, -y^*) \in N_a(\operatorname{Gph}\Gamma_2; (\bar{x}, h(\bar{x})))$ and we have $(0, -y^*) = (\xi, a) + (\pi, 0)$ according to the decomposition just stated. Then $\xi = -\pi \in -N_a(C; \bar{x})$ and $(\xi, -y^*) = (\xi, a) \in N_a(\operatorname{Gph} h; (\bar{x}, h(\bar{x})))$. It follows $\xi \in D_a^* h(\bar{x}, h(\bar{x}))(y^*) \cap -N_a(C; \bar{x})$, hence $y^* = 0$ due to the second part in *(ii)* and to $y^* \in \mathbb{R}^s_-$. However, this is (16), so Proposition 2 guarantees $\operatorname{Ker} D_a^*\Gamma(\bar{x}, 0) = \{0\}$ and, Theorem 2 implies metric regularity of $\Phi_\mu(h(\cdot)) - p$ at $\bar{x}$ w.r.t C.

□

Theorem 5 offers the possibility to check properties of the measure μ and of the function h in (8) separately. Yet the conditions imposed are rather abstract. In the following we develop criteria that are better to verify. First we turn to condition *(i)* and try to reformulate it in terms of assumptions concerning the density of the measure μ. If μ has a density, then, denoting

$$y = (y_1^1, \ldots, y_1^{s_1}, \ldots, y_d^1, \ldots, y_d^{s_d}) \quad (y \in \mathbb{R}^s;\ s = s_1 + \cdots + s_d),$$

one recognizes that the components of Φ_μ may be written as

$$\Phi_\mu^j(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{y_j^1} \cdots \int_{-\infty}^{y_j^{s_j}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_\mu(y)\, dy_d^{s_d} \cdots dy_{j+1}^1 dy_j^{s_j} \cdots dy_j^1 dy_{j-1}^{s_{j-1}} \cdots dy_1^1.$$

Next we introduce the set where this density is locally bounded below by a positive number:

$$\mathcal{D}^+ = \{y \in \mathbb{R}^s \mid \exists \varepsilon > 0 :\ f_\mu(\tilde{y}) \geq \varepsilon \text{ for almost all } \tilde{y} \in B_\varepsilon(y)\}.$$

For continuous f_μ, of course, this set reduces to $\mathcal{D}^+ = \{y \in \mathbb{R}^s \mid f_\mu(y) > 0\}$. Finally, for any subset $I \subseteq \{1, \ldots, d\}$ put

$$\Omega^I = C_1 \times \cdots \times C_d, \quad \text{where } C_i = \begin{cases} \mathbb{R}^{s_i} & i \notin I \\ \partial\mathbb{R}^{s_i}_- & i \in I \end{cases}$$

The following theorem provides a density condition guaranteeing sufficient growth of Φ_μ to arrive at the desired property of metric regularity.

Theorem 6. *For $\bar{x} \in M$ in (8), denote the set of active indices by $I(\bar{x}) = \{i \in \{1, \ldots, d\} \mid \Phi_\mu^i(h(\bar{x})) = p^i\}$. If μ has a density and $(h(\bar{x}) + \Omega^{I(\bar{x})}) \cap \mathcal{D}^+ \neq \emptyset$, then condition (i) of Theorem 5 is satisfied.*

Proof. By assumption, there exists some $\bar{y} \in \mathcal{D}^+$ such that for all $j \in I(\bar{x})$

$$\bar{y}_j^k \leq [h(\bar{x})]_j^k \; k = 1, \ldots, s_j \quad \text{and} \quad \exists k(j) \in \{1, \ldots, s_j\} : \bar{y}_j^{k(j)} = [h(\bar{x})]_j^{k(j)}$$

Here, lower and upper indices refer to the partition of vectors in $\mathbb{R}^s = \mathbb{R}^{s_1} \times \cdots \times \mathbb{R}^{s_d}$ introduced above. By definition, one has $f_\mu(y) \geq \varepsilon$ for almost all $y \in B_\varepsilon(\bar{y})$. Choose any $z \in B_{\varepsilon/2}(h(\bar{x}))$. Without loss of generality we consider the balls with respect to the maximum norm $\|\cdot\|_\infty$. As a consequence, we have for all indices $j \in I(\bar{x})$

$$z_j^k \geq \bar{y}_j^k - \varepsilon/2 \quad k = 1, \ldots, s_j.$$

Next define some vector $e \in \mathbb{R}^s$ via

$$e_j^k = \begin{cases} 1 \; j \in I(\bar{x}) \text{ and } k = k(j) \\ 0 \; else \end{cases}$$

and put $z(t) = z + te$ for $t \in (0, \varepsilon/2)$. Clearly, for all indices $j \in I(\bar{x})$ it holds

$$[z(t)]_j^{k(j)} = z_j^{k(j)} + t \quad \text{and} \quad [z(t)]_j^k = z_j^k \quad \text{if } k \neq k(j).$$

In particular, $\|z(t) - z\|_\infty = t$ and for $c \in [z_j^{k(j)}, z_j^{k(j)} + t]$ one has

$$|c - \bar{y}_j^{k(j)}| \leq |c - z_j^{k(j)}| + |z_j^{k(j)} - [h(\bar{x})]_j^{k(j)}| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Now, the following estimation can be made for the active indices $j \in I(\bar{x})$:

$$\begin{aligned}
\Phi_\mu^j(z(t)) - \Phi_\mu^j(z) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{[z(t)]_j^1} \cdots \int_{-\infty}^{[z(t)]_j^{k(j)}} \cdots \int_{-\infty}^{[z(t)]_j^{s_j}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_\mu(y)\, dy \\
&\quad - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{z_j^1} \cdots \int_{-\infty}^{z_j^{k(j)}} \cdots \int_{-\infty}^{z_j^{s_j}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_\mu(y)\, dy \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{z_j^1} \cdots \int_{z_j^{k(j)}}^{z_j^{k(j)}+t} \cdots \int_{-\infty}^{z_j^{s_j}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_\mu(y)\, dy \\
&\geq \int_{\bar{y}_1^1-\varepsilon}^{\bar{y}_1^1} \cdots \int_{\bar{y}_{j-1}^{s_{j-1}}-\varepsilon}^{\bar{y}_{j-1}^{s_{j-1}}} \int_{\bar{y}_j^1-\varepsilon}^{\bar{y}_j^1-\varepsilon/2} \cdots \int_{z_j^{k(j)}}^{z_j^{k(j)}+t} \cdots \int_{\bar{y}_j^{s_j}-\varepsilon}^{\bar{y}_j^{s_j}-\varepsilon/2} \int_{\bar{y}_{j+1}^1-\varepsilon}^{\bar{y}_{j+1}^1} \cdots \int_{\bar{y}_d^{s_d}-\varepsilon}^{\bar{y}_d^{s_d}} \varepsilon\, dy \\
&= \varepsilon^{s-s_j} \cdot (\varepsilon/2)^{s_j-1} \cdot \varepsilon \cdot \|z(t) - z\|_\infty
\end{aligned}$$

But, having in mind, that Φ_μ is continuous due to the assumption that μ possesses a density, the above estimation implies that $\Phi_\mu(\cdot) - p$ is growing at $h(\bar{x})$ (w.r.t. $\mathbb{R}^s$) in the sense of Definition 1 (put $\rho = (\varepsilon/2)^s, \eta = \varepsilon/2$ and recall that the above estimation is valid for all $t \in (0, \varepsilon/2)$). According to Lemma 2 $\Phi_\mu(\cdot) - p$ (considered with the ≥ 0 constraint) is metrically regular at $h(\bar{x})$. This is condition *(i)* of Theorem 5.

□

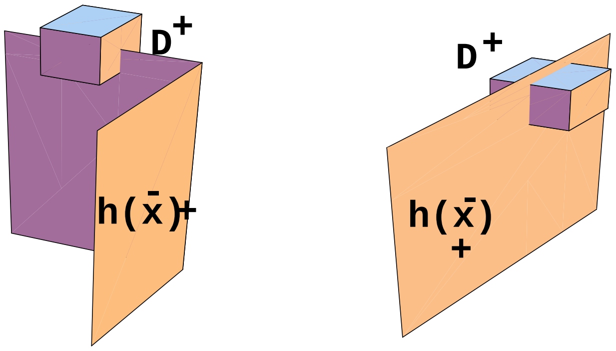

Fig. 1. Illustration of Theorem 6 for the case $d = 2, s = 3$: the left picture refers to the situation $s_1 = 2, s_2 = 1, I = \{1\}$, and the right one to $s_1 = 1, s_2 = 2, I = \{1\}$. In both cases, the two-dimensional manifolds $h(\bar{x}) + \Omega^I$ intersect the positivity region $\mathcal{D}^+$ of the density (illustrated as cuboids in the pictures). Hence, condition *(i)* of Theorem 5 is satisfied

Since, by definition, $0 \in \Omega^{I(\bar{x})}$ for an arbitrary index set $I(\bar{x})$, one concludes

Corollary 4. *If $h(\bar{x}) \in \mathcal{D}^+$, then condition (i) of Theorem 5 is satisfied.*

This density condition $h(\bar{x}) \in \mathcal{D}^+$ was used in [36] (Lemma 2.1) in order to derive a corresponding stability result for a specific probabilistic constraint ($d = 1$ and h linear in (8)). For continuous densities one simply would have to require $f_\mu(h(\bar{x})) > 0$. Note, however, that this relation is far from being necessary in order to ensure condition *(i)* of Theorem 5, as can be seen from the following example:

Example 3. In (8), we take $d = 1$, $s = m = 2$, $h(x) = x$, $p = 0.5, C = \mathbb{R}^2$. In particular, Φ_μ coincides with the distribution function of the measure μ, which we assume to be induced by the following density on $\mathbb{R}^2$:

$$f_\mu(y) = \begin{cases} a & y \in B_1(0) \\ (2 - \|y\|)a & y \in B_2(0) \setminus B_1(0) \\ 0 & y \in \mathbb{R}^2 \setminus B_2(0) \end{cases}$$

where the balls of the corresponding distances refer to the Euclidean norm and the number $a > 0$ is suitably chosen to guarantee $\int_{\mathbb{R}^2} f_\mu(y)\, dy = 1$. Obviously, f_μ is continuous and $\mathcal{D}^+ = \operatorname{int} B_2(0)$. For $\bar{x} = (0, 3)$ we deduce from the symmetry of f_μ around the origin that

$$\Phi_\mu(h(\bar{x})) = \Phi_\mu((0, 3)) = 0.5 = p,$$

hence, we have the binding case $I(\bar{x}) = \{1\}$. Of course, $f_\mu(h(\bar{x})) = f_\mu((0, 3)) = 0$, so the strong condition of Corollary 4 does not apply. Nevertheless, one may derive condition *(i)* of Theorem 5 because $[(0, 3) + \partial\mathbb{R}^2_-] \cap \mathcal{D}^+ \neq \emptyset$ (take, for instance $(0, -3) \in \partial\mathbb{R}^2_-$), hence, the weaker condition in Theorem 6 is satisfied.

Frequently, the property of metric regularity is required at points that are not given explicitly, e.g. the set of local minimizers. Therefore, it might sometimes be useful to know conditions under which metric regularity holds everywhere. For instance, as a part of this question, one could ask when condition *(i)* of Theorem 5 is satisfied everywhere, i.e., $\Phi_\mu(\cdot) - p$ is metrically regular at all $h(\bar{x})$ with $\bar{x} \in M$. Using Corollary 4 one gets an immediate criterion for such a global behaviour, namely $\mathcal{D}^+ = \mathbb{R}^s$, which is fulfilled for some of the conventional distributions (like multivariate normal). The situation becomes more interesting for densities whose support is not all of R^s. To investigate this problem in more detail we introduce the following definition:

Definition 2. *A subset $Q \subseteq \mathbb{R}^n$ is called an infinity path in $\mathbb{R}^n$ if there exists some continuous function $\pi : \mathbb{R} \to \mathbb{R}^n$ such that $\operatorname{Im} \pi = Q$ and*

$$\lim_{t\to-\infty} \max_{i=1,\dots,n} \pi_i(t) = -\infty, \qquad \lim_{t\to\infty} \max_{i=1,\dots,n} \pi_i(t) = \infty$$

Roughly speaking, one part of Q must tend to $-\infty$ with all its coordinates simultaneously, while for the other part it suffices that at least one coordinate tends to $+\infty$. Of course, any infinity path is a connected subset of $\mathbb{R}^n$. This concept allows an appropriate characterization in the case of Φ_μ having only one component, i.e., $d = 1$.

Theorem 7. *If $d = 1$, μ has a density and $\mathcal{D}^+$ contains an infinity path Q in R^s, then condition (i) in Theorem 5 holds globally, i.e., $\Phi_\mu(\cdot) - p$ is metrically regular at $h(\bar{x})$ for all $\bar{x} \in M$.*

Proof. Consider any $\bar{x} \in M$ and put $z = h(\bar{x})$. With reference to Definition 2 there exist $t_1, t_2 \in \mathbb{R}$, such that

$$\max_{i=1,\dots,s} \pi_i(t_1) < \min_{i=1,\dots,s} z_i, \qquad \max_{i=1,\dots,s} \pi_i(t_2) > \max_{i=1,\dots,s} z_i$$

Hence, for $q_1 = \pi(t_1), q_2 = \pi(t_2)$ one has $q_1 \in Q \cap \operatorname{int}(z + \mathbb{R}^s_-)$ and $q_2 \in Q \cap (\mathbb{R}^s \setminus (z + \mathbb{R}^s_-))$. Now

$$\mathbb{R}^s = [\operatorname{int}(z + \mathbb{R}^s_-)] \cup [\mathbb{R}^s \setminus (z + \mathbb{R}^s_-)] \cup [z + \partial\mathbb{R}^s_-]$$

is a disjoint decomposition of $\mathbb{R}^s$, where the first two sets are open. Therefore $Q \cap (z + \partial\mathbb{R}^s_-) \neq \emptyset$ because otherwise

$$Q = [Q \cap \operatorname{int}(z + \mathbb{R}^s_-)] \cup [Q \cap (\mathbb{R}^s \setminus (z + \mathbb{R}^s_-))]$$

would be a decomposition of Q into two open (in the relative topology of Q), disjoint and nonempty subsets in contradiction to the connectedness of Q. Taking account of $Q \subseteq \mathcal{D}^+$, we arrive at

$$\emptyset \neq \mathcal{D}^+ \cap (z + \partial\mathbb{R}^s_-) \subseteq \mathcal{D}^+ \cap (h(\bar{x}) + \Omega^{I(\bar{x})})$$

Since $\bar{x} \in M$ was arbitrary, the assertion follows from Theorem 6.

□

It is noted here, that the assertion of the theorem is not restricted to the fixed probability level p, in fact, this value does not enter the proof at any point. Consequently, under the indicated assumptions, $\Phi_\mu(\cdot) - p'$ is metrically regular at $h(\bar{x})$ not only for all $\bar{x} \in M$ but even for all $p' \in (0, 1)$. The following example shall illustrate the meaning of Theorem 7.

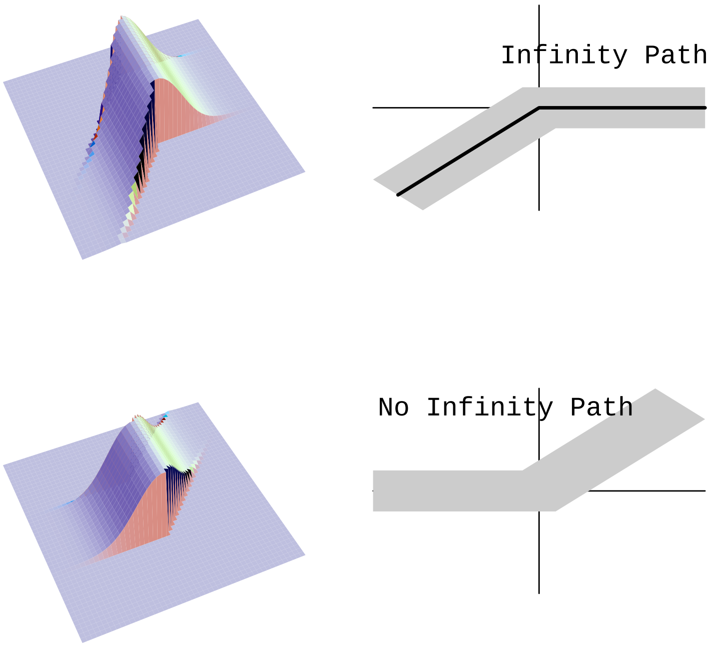

Fig. 2. Probability densities with and without infinity path contained in the positivity region of the density (shaded)

Example 4. Adopt the setting of Example 3, but with the density on $\mathbb{R}^2$ replaced by

$$f_\mu(y) = \begin{cases} ae^{-y_1^2/2} & \text{if} \quad y_1 \geq 0,\ |y_2| < 1/2 \quad \text{or} \quad y_1 < 0,\ |y_2 - y_1| < 1/2 \\ 0 & \text{else} \end{cases}$$

(a such that $\int f_\mu(y)dy = 1$). Obviously, here the set $\mathcal{D}^+$ coincides with the one which the first line in the definition of f_μ relates to, so $\mathcal{D}^+ \neq \mathbb{R}^2$. Nevertheless, condition *(i)* of Theorem 5 is satisfied globally. In fact, the continuous function $\pi : \mathbb{R} \to \mathbb{R}^2$ defined by $\pi(t) = (t, t)$ if $t \leq 0$ and $\pi(t) = (t, 0)$ if $t > 0$ generates an infinity path $Q = \operatorname{Im} \pi$ that is contained in $\mathcal{D}^+$, so Theorem 7 applies (see Fig. 2 top).

Now, reflect the density w.r.t. the origin, i.e., take $f_{\mu'}(y) = f_\mu(-y)$. Then, the set $\mathcal{D}^{+'}$ does not contain any infinity path (see Fig. 2 bottom). For instance, the canonical candidate $Q' \subseteq \mathcal{D}^{+'}$, which is defined by $Q' = \operatorname{Im} \pi'$, where $\pi'(t) = (t, 0)$ for $t \leq 0$ and $\pi'(t) = (t, t)$ for $t > 0$, fails to satisfy the first limiting condition in Definition 2 (while the second one holds true).

Now we turn to the second constraint qualification in Theorem 5. As will be seen below, this can be viewed as some kind of Mangasarian-Fromovitz Constraint Qualification for continuous inequality constraints. The first part of this condition (relating the approximate normal cones of the Graph of h and of the set C) is always fulfilled, for instance, if $C = \mathbb{R}^m$ or if h is locally Lipschitzian. In order to gain more insight, we consider the cases of locally Lipschitzian or even C^1- mappings h.

Proposition 3. *If h is locally Lipschitzian in (8), then condition (ii) of Theorem 5 reduces to*

$$\partial_a \langle y^*, h\rangle(\bar{x}) \cap -N_a(C;\bar{x}) = \emptyset \quad \forall y^* \in I\!R^s_- \setminus \{0\}. \tag{17}$$

If $h \in \mathcal{C}^1(I\!R^m, I\!R^s)$ with Jacobian Dh, then the corresponding relation reads

$$[Dh(\bar{x})]^T y^* \notin N_a(C;\bar{x}) \quad \forall y^* \in I\!R^s_+ \setminus \{0\} \tag{18}$$

Proof. For locally Lipschitzian h the first part of condition *(ii)* in Theorem 5 is automatically fulfilled. In fact, if k is a Lipschitz modulus of h near $\bar{x}$, then $\|a^*\| \le k\|b^*\|$ for all $(a^*, b^*) \in T^0(\mathrm{Gph}\, h; (x, h(x)))$ and all x near $\bar{x}$ (compare [18], Lemma 3.8). Now, the same relation must hold true for all $(a^*, b^*) \in N_a((\mathrm{Gph}\, h; (\bar{x}, h(\bar{x})))$ too. In particular, $b^* = 0$ implies $a^* = 0$.

Finally, the second part of condition *(ii)* is nothing else but (17) as a consequence of (10). Now (18) follows from the fact that the approximate subdifferential and the usual derivative coincide in the $\mathcal{C}^1$- case.

□

In case $C = I\!R^m$ (i.e., $N_a(C;\bar{x}) = \{0\}$), Gordan's theorem shows the equivalence of (18) with the condition

$$\exists \xi \in I\!R^m : \quad \nabla h_i(\bar{x}) \cdot \xi > 0 \;\; i = 1, \ldots, s,$$

where now, in contrast to the derivations above, we return to the conventional labelling of the components of h. Restricting this relation to the active indices only (which have no meaning for h in our present context) this would be the well-known Mangasarian-Fromovitz Constraint Qualification (in the absence of equations). Replacing the sets in (17) by the corresponding (bigger) concepts of Clarke's subdifferential calculus, one gets the stronger requirement

$$\partial_c \langle y^*, h\rangle(\bar{x}) \cap N_c(C;\bar{x}) = \emptyset \quad \forall y^* \in I\!R^s_+ \setminus \{0\}, \tag{19}$$

which is closely related to well-known constraint qualifications in the locally Lipschitzian setting (e.g. [34], [8], [3], [17]). However, let us emphasize once more that, in (8), the mapping h does not appear itself as a constraint, but as the inner part of a composite constraint. In particular, there is no active index set to be considered. Furthermore, the application of (17) according to Mordukhovich's calculus promises advantages over (19) for certain classes of mappings. This is confirmed by the following corollary, which illustrates the verification of condition (*iii*) in Theorem 1 by the criteria obtained so far, and where the 'production function' h is assumed to have a specific structure of nonsmoothness. In this lemma, with a compact set K we associate the set of exposed points $\mathrm{ex}\, K = \{x \in K \mid \exists z : \langle z, x\rangle < \langle z, y\rangle \; \forall y \in K \setminus \{x\}\}$ and exploit the relation (cf. [14])

$$\partial_a(\min_{y \in K} \langle \cdot, y\rangle)(x) \subseteq cl\,(\mathrm{ex}\, K) \tag{20}$$

Corollary 5. *Let C be convex, $d = 1$ and assume that $h_i(x) = \max_{y \in K_i} \langle x, y \rangle$, where $K_i \subseteq \mathbb{R}^m$ $(i = 1, \ldots, s)$ are compact (e.g. finite) subsets. Furthermore, assume that $\mu \in \mathcal{P}(\mathbb{R}^s)$ has a continuous density which is strictly positive over $h(C)$. Then, the condition*

$$\forall x^* \in T\ \exists c \in C: \quad \langle x^*, c - \bar{x} \rangle < 0, \tag{21}$$

where $T = \cup\{\mathrm{conv}\{a_1, \ldots, a_s\} \mid a_i \in \mathrm{cl}\,(\mathrm{ex}\,(-K_i))\ (i = 1, \ldots, s)\}$, is sufficient to guarantee metric regularity of the function $F_\mu(h(\cdot)) - p$ at some feasible $\bar{x} \in M_p(\mu)$.

Proof. According to Theorem 5, Theorem 6, and Proposition 3, it is sufficient to verify the following two conditions:

$$(h(\bar{x}) + \partial \mathbb{R}^s_-) \cap \mathcal{D}^+ \neq \emptyset \quad \text{and} \quad \partial_a \langle y^*, h \rangle(\bar{x}) \cap -N(C; \bar{x}) = \emptyset\ \forall y^* \in \mathbb{R}^s_- \setminus \{0\} \tag{22}$$

($N =$ normal cone to convex sets). By assumption, $h(C) \subseteq \mathcal{D}^+$, so the first relation of (22) is trivially fulfilled. Concerning the second relation, we apply (20) to obtain for $y^* \in \mathbb{R}^s_- \setminus \{0\}$:

$$\partial_a \langle y^*, h \rangle(\bar{x}) = \partial_a (\sum_{i=1}^{s} y_i^* h_i)(\bar{x}) \subseteq \sum_{i=1}^{s} (-y_i^*) \partial_a (-h_i)(\bar{x}) \subseteq \sum_{i=1}^{s} (-y_i^*)\, \mathrm{cl}\,(\mathrm{ex}\,(-K_i)), \tag{23}$$

For any $x^* \in -N(C; x)$, one has $\langle x^*, c - \bar{x} \rangle \geq 0\ \forall c \in C$. In order to prove the second relation in (22), it suffices by (23) to lead to a contradiction the existence of some $\lambda \geq 0, \lambda \neq 0$ with $x^* \in \sum_{i=1}^{s} \lambda_i\, \mathrm{cl}\,(\mathrm{ex}\,(-K_i))$. In fact, if there were such λ, then $t^{-1} x^* \in T$ with $t = \sum \lambda_i$, hence $\langle x^*, c - \bar{x} \rangle < 0$ for some $c \in C$ by (21).

□

In order to illustrate Corollary 5 as well as the difference to using the Clarke subdifferential calculus here, consider the following one-dimensional example for problem $\mathbf{P}(\mu)$:

Example 5. In (1), let

$$d = s = 1, \; H(x) = \{z \in \mathbb{R} \mid \|x\| \geq z\}, \; C = B(0, 1), \; p = 0.5, \; \mu \sim \mathcal{N}(0, 1), \; \bar{x} = 0.$$

Then, in the setting of Corollary 5, one has $h(x) = \|x\| = \max\{\langle x, y \rangle \mid y \in K\}$, where $K = B(0, 1)$. Clearly, all assumptions of the corollary are satisfied. To see this for (21), note that $\mu(H(x)) \geq 0.5\ \forall x \in \mathbb{R}^m$, hence $M_p(\mu) = C$ and $\bar{x} \in M_p(\mu)$. Furthermore, $\mathrm{ex}\,(-K)$ equals the unit sphere S^{m-1}, so $T = \cup\{\mathrm{conv}\{a\} \mid a \in S^{m-1}\} = S^{m-1}$. Then (21) is satisfied by choosing $c := -x^*$. The advantage of using (21) which relies on the application of Mordukhovich's subdifferential, over a characterization via Clarke's subdifferential ∂_c is seen in the example from the violation of the second relation in (22) when replacing ∂_a by ∂_c: $\partial_c(-1 \cdot h)(0) \cap -N(C; 0) = B(0, 1) \cap \{0\} = \{0\} \neq \emptyset$.

Similar to the considerations with respect to condition *(i)* in Theorem 5 one may ask under which circumstances condition *(ii)* of the same theorem holds globally, i.e., for all $\bar{x} \in M$. An answer may be deduced from the following corollary to Proposition 3:

Corollary 6. *In (8), let all components of h be concave and the set C be convex. If, for $\bar{x} \in M$, there exists some $x^* \in C$ such that $h(x^*) > h(\bar{x})$ (componentwise), then condition (ii) of Theorem 5 is satisfied.*

Proof. Due to concavity, h is locally Lipschitzian, so we have to check (17). If this relation does not hold, then there exist some $y^* \in I\!R^s_+ \setminus \{0\}$ and $\xi \in I\!R^m$ such that $\xi \in \partial\langle y^*, -h\rangle(\bar{x}) \cap -N(C;\bar{x})$ (note that $\langle y^*, -h\rangle$ is convex and that ∂_a and N_a coincide with the subdifferential ∂ and the normal cone N of convex analysis). Since both $\bar{x}$ and x^* belong to the convex set C, we derive $\langle \xi, x^* - \bar{x}\rangle \geq 0$. On the other hand, by the sum rule of the convex subdifferential, there are $\xi_i \in \partial(-h_i)(\bar{x})$ with $\xi = \sum_{i=1}^s y_i^* \xi_i$. In particular, by the definition of the convex subdifferential, one has $\langle \xi_i, x^* - \bar{x}\rangle \leq h_i(\bar{x}) - h_i(x^*)$. Summarizing, one obtains the contradiction

$$0 \leq \langle \xi, x^* - \bar{x}\rangle = \sum_{i=1}^s y_i^* \langle \xi_i, x^* - \bar{x}\rangle \leq \sum_{i=1}^s y_i^* (h_i(\bar{x}) - h_i(x^*)) < 0$$

from the strict inequality in the assumption.

□

The corollary corrects an error in [36] Lemma 2.1., where, in the context of linear mappings h and convex sets C, the existence of some $x^* \in C$ with $h(x^*) \geq h(\bar{x})$ was required instead of the strict inequality.

Now, the desired global property may be formulated as follows: If, in (8), h is concave (e.g. linear) and C is convex, then condition *(ii)* of Theorem 5 is fulfilled on

$$M(\mu) \cap \left[\bigcup_{x^* \in C} \bigcap_{i=1}^s h_i^{-1}(-\infty, h_i(x^*)) \right]$$

which in general may be expected to be a big subset of the chance constraint $M_p(\mu)$.

At the end of this section we reexamine Example 1 using the tools related to Theorem 5. In contrast to the previously given verification of metric regularity by means of the composite function $\Phi_\mu \circ h$, the corresponding result shall be obtained now via separate considerations of the measure and the function h.

Example 6 (Example 1 revisited). Due to $N_a(C;(0,0)) = \{(\xi_1, \xi_2) \in I\!R^2 \mid \xi_1 \leq 0\}$ one has $[Dh(0,0)]^T y^* = (y^*, y^*)^T \notin N_a(C;(0,0))\ \forall y^* > 0$. Consequently, (18) applies. On the other hand $\mathcal{D}^+ = (-0.5, 0.5)$ for the given uniform distribution over $[-0.5, 0.5]$. So $h(0,0) = 0 \in \mathcal{D}^+$ and we are in the situation of Corollary 4. Summarizing, both conditions of Theorem 5 are satisfied and the desired metric regularity result follows.

4. Quadratic growth condition and quantitative stability

In order to obtain quantitative stability results for solution sets, a certain growth condition for the objective function in a neighbourhood of the optimal set has to be verified. This is studied next for more specific (convex) stochastic programs with one joint probabilistic constraint and polyhedral deterministic constraints. More precisely, we consider the problem

$$\mathbf{P}(\mu) \qquad \min\{g(x) \mid x \in C,\, F_\mu(Ax) \ge p\}, \tag{24}$$

where $g : I\!R^m \to I\!R$ is convex quadratic, $C \subseteq I\!R^m$ is convex polyhedral, A is an (s, m)- matrix, $p \in (0, 1)$ and F_μ is the distribution function of a probability measure $\mu \in \mathcal{P}(I\!R^s)$, which is assumed to be r-concave for some $r \in (-\infty, 0)$. Due to the r-concavity of μ, $\mathbf{P}(\mu)$ represents a convex program. In the following, $\Psi(\mu)$ refers again to the set of (global) solutions to (24) and $\Psi_V(\nu)$ denotes the localized solution set to $\mathbf{P}(\nu)$, where $\nu \in \mathcal{P}(I\!R^s)$ is a perturbation of μ and $V \subseteq I\!R^m$ an open neighbourhood of $\Psi(\mu)$.

In the first step of our analysis a reduction argument is used to decompose the original problem $\mathbf{P}(\mu)$ into two auxiliary problems. The first one is a stochastic program under probabilistic constraints, again with decisions taken in R^s, whereas the second one represents a parametric quadratic program with polyhedral constraints. The reduction argument also provides insight into the structure of the solution set $\Psi(\mu)$.

Lemma 4. *In addition to the general assumptions, let $\nu \in \mathcal{P}(I\!R^s)$ and suppose the closure* $\mathrm{cl}\, V$ *of $V \subseteq I\!R^m$ to be a polytope. Then we have*

$$\varphi_V(\nu) = \inf\{\pi_V(y) \mid y \in A(C_V),\, F_\nu(y) \ge p\} \quad \textit{and} \quad \Psi_V(\nu) = \sigma_V(Y_V(\nu)),$$

where

$$\begin{aligned} Y_V(\nu) &= \operatorname{argmin}\{\pi_V(y) \mid y \in A(C_V),\, F_\nu(y) \ge p\} \\ C_V &= C \cap \mathrm{cl}\, V \\ \pi_V(y) &= \inf\{g(x) \mid Ax = y,\, x \in C_V\} \\ \sigma_V(y) &= \operatorname{argmin}\{g(x) \mid Ax = y,\, x \in C_V\} \ (y \in A(C_V)). \end{aligned}$$

Here, π_V is convex on $A(C_V)$, σ_V is Hausdorff Lipschitzian on $A(C_V)$ and there exists an $\eta > 0$ such that

$$g(x) \ge \pi_V(Ax) + \eta\, dist\,(x, \sigma_V(Ax))^2 \quad \forall x \in C_V.$$

Proof. Since the constraint set $\{x \in C_V \mid F_\nu(Ax) \ge p\}$ is compact, $\Psi_V(\nu)$ is nonempty. Let $x \in \Psi_V(\nu)$. Then $x \in C_V$, $F_\nu(Ax) \ge p$, and

$$\varphi_V(\nu) = g(x) \ge \pi_V(Ax) \ge \inf\{\pi_V(y) \mid y \in A(C_V),\, F_\nu(y) \ge p\}.$$

Conversely, let $y \in A(C_V)$ with $F_\nu(y) \ge p$. Then there exists an $x \in \sigma_V(y)$ with $\pi_V(y) = g(x) \ge \varphi_V(\nu)$. Hence

$$\varphi_V(\nu) = \inf\{\pi_V(y) \mid y \in A(C_V),\, F_\nu(y) \ge p\} \quad \text{and} \quad g(x) = \pi_V(Ax) \quad \forall x \in \Psi_V(\nu).$$

This implies $\Psi_V(\nu) = \sigma_V(Y_V(\nu))$. The convexity of π_V is immediate and the Lipschitz property of σ_V is shown in [23], Theorem 4.2.

Finally, we turn to the last statement in the lemma and assume a description $g(x) = \langle x, Hx\rangle + \langle c, x\rangle$ for the objective function with some symmetric, positive semidefinite matrix H. For each $y \in A(C_V)$ fix a corresponding $z(y) \in \sigma_V(y)$. As an elementary characterization of solution sets to quadratic programs one has

$$\sigma_V(y) = \{x \in C_V \cap A^{-1}(y) \mid Hx = Hz(y),\ \langle c, x\rangle = \langle c, z(y)\rangle\}$$

By Hoffman's theorem [15] there exists some $L > 0$ such that

$$dist\,(x, \sigma_V(y)) \le L(\|Hx - Hz(y)\| + |\langle c, x\rangle - \langle c, z(y)\rangle|) \quad \forall y \in A(C_V)\ \forall x \in C_V \cap A^{-1}(y)$$

With the decomposition $H = H^{1/2}H^{1/2}$ one arrives at $\langle c, x\rangle - \langle c, z(y)\rangle = g(x) - \pi_V(y) - \|H^{1/2}x\|^2 + \|H^{1/2}z(y)\|^2$, hence

$$\begin{aligned} dist\,(x, \sigma_V(y)) &\le L(\|H^{1/2}\|\|H^{1/2}(x - z(y))\| + (\|H^{1/2}z(y)\| \\ &\quad -\|H^{1/2}x\|)(\|H^{1/2}z(y)\| + \|H^{1/2}x\|) + g(x) - \pi_V(y)) \\ &\le L([\|H^{1/2}\| + \|H^{1/2}z(y)\| + \|H^{1/2}x\|]\|H^{1/2}(x - z(y))\| \\ &\quad +g(x) - \pi_V(y)) \\ &\le L((2\kappa + 1)\|H^{1/2}\|\|H^{1/2}(x - z(y))\| + g(x) - \pi_V(y)) \end{aligned}$$

for all $y \in A(C_V)$ and all $x \in C_V \cap A^{-1}(y)$, where $\kappa = \max\{\|\tilde{x}\| \mid \tilde{x} \in C_V\}$. Consequently,

$$dist\,(x, \sigma_V(y))^2 \le \tilde{L}(\|H^{1/2}(x - z(y))\|^2 + (g(x) - \pi_V(y))^2) \quad \forall y \in A(C_V)\ \forall x \in C_V \cap A^{-1}(y)$$

for some $\tilde{L} > 0$. Furthermore, the equality

$$g((x + z(y))/2) = g(x)/2 + g(z(y))/2 - \|H^{1/2}(x - z(y))\|^2/4$$

implies

$$\|H^{1/2}(x - z(y))\|^2/2 \le g(x) - \pi_V(y) \quad \forall y \in A(C_V)\ \forall x \in C_V \cap A^{-1}(y)$$

Summarizing, one gets

$$\begin{aligned} dist\,(x, \sigma_V(y))^2 &\le \tilde{L}[2(g(x) - \pi_V(y)) + \tilde{\pi}_V(g(x) - \pi_V(y))] \\ &\le \tilde{L}(2 + \tilde{\pi}_V)(g(x) - \pi_V(y)) \end{aligned}$$

for all $y \in A(C_V)$ and all $x \in C_V \cap A^{-1}(y)$, where

$$\tilde{\pi}_V = \max\{g(\tilde{x}) \mid \tilde{x} \in C_V\} - \min\{g(\tilde{x}) \mid \tilde{x} \in C_V\}$$

Inserting any $x \in C_V$ along with $y = Ax \in A(C_V)$ yields

$$g(x) \ge \pi_V(Ax) + \eta\, dist\,(x, \sigma_V(Ax))^2 \quad \forall x \in C_V.$$

□

The preceding result enables us first to study the growth behaviour of the objective function in the auxiliary problem

$$\min\{\pi_V(y) \mid y \in A(C_V), F_\nu(y) \geq p\},$$

where V is some suitably chosen subset of $\mathbb{R}^m$. In a second step, the formula for Ψ_V in the above proposition and the properties of σ_V may be exploited. This two-stage procedure forms the basis of the proof of the following results.

Theorem 8. *In addition to the general assumptions in this section, suppose that*

(i) $\Psi(\mu)$ *is nonempty and bounded;*
(ii) $\Psi(\mu) \cap \operatorname{argmin}\{g(x) \mid x \in C\} = \emptyset$*;*
(iii) $\exists \bar{x} \in C : F_\mu(A\bar{x}) > p$ *(Slater condition);*
(iv) F_μ^r *is strongly convex on some open convex neighbourhood* U *of* $A(\Psi(\mu))$.

Then the following quadratic growth condition is satisfied:

$$\exists c > 0 \, \exists V \supseteq \Psi(\mu) \ (V \text{ open}) : \quad g(x) \geq \varphi(\mu) + c d(x, \Psi(\mu))^2 \quad \forall x \in C \cap V, F_\mu(Ax) \geq p.$$

Proof. Let $V_0 \subseteq \mathbb{R}^m$ be an open convex set such that $\Psi(\mu) \subseteq V_0$ and $A(V_0) \subseteq U$. For each $x \in \Psi(\mu)$ select $\varepsilon(x) > 0$ such that the closed ball (w.r.t. the norm $\|\cdot\|_\infty$) $B_\infty(x, \varepsilon(x))$ around x with radius $\varepsilon(x)$ is contained in V_0. Since $\Psi(\mu)$ is compact, a finite number of these balls cover $\Psi(\mu)$. The closed convex hull $\bar{V}$ of their union is a polyhedron with $\Psi(\mu) \subseteq V \subset \bar{V} \subseteq V_0$, where $V = \operatorname{int}\bar{V}$. With the notations from Lemma 4 consider now the problem

$$\min\{\pi_V(y) \mid y \in S_V, F_\mu(y) \geq p\}, \quad \text{with} \quad S_V = A(C_V)$$

or, equivalently,

$$\min\{\pi_V(y) \mid y \in S_V, h(y) \leq 0\} \quad \text{where} \quad h(y) = F_\mu^r(y) - p^r.$$

According to Lemma 4 the solution set $Y_V(\mu)$ of this problem fulfills $\Psi(\mu) = \Psi_V(\mu) = \sigma_V(Y_V(\mu))$. Let $y_* \in Y_V(\mu)$ and $\bar{y} = A\bar{x}$ with $\bar{x} \in C$ from *(iii)*. Then r-concavity of μ implies for any $\lambda \in (0, 1]$:

$$\begin{aligned} h(\lambda\bar{y} + (1-\lambda)y_*) &= F_\mu^r(\lambda\bar{y} + (1-\lambda)y_*) - p^r \leq \lambda F_\mu^r(\bar{y}) + (1-\lambda)F_\mu^r(y_*) - p^r \\ &\leq \lambda(F_\mu^r(\bar{y}) - p^r) < 0. \end{aligned}$$

Thus, we may select $\hat{\lambda} \in (0, 1]$ such that $\hat{y} = \hat{\lambda}\bar{y} + (1-\hat{\lambda})y_*$ belongs to S_V and has the property $h(\hat{y}) < 0$. This constraint qualification implies the existence of a Kuhn-Tucker coefficient $\lambda_* \geq 0$ such that

$$\pi_V(y_*) = \min\{\pi_V(y) + \lambda_* h(y) \mid y \in S_V\} \quad \text{and} \quad \lambda_* h(y_*) = 0$$

In case $\lambda_* = 0$, this would imply $y_* \in \operatorname{argmin}\{\pi_V(y) \mid y \in S_V\}$ and, hence, the existence of some $x_* \in \Psi(\mu)$ with $g(x_*) = \pi_V(Ax_*) = \min\{g(x) \mid Ax = y_*, x \in C_V\}$. Then, in contradiction to condition *(ii)*, x^* would minimize g w.r.t. C due to

$x^* \in \operatorname{int} V$. Thus $\lambda_* > 0$ and $\pi_V + \lambda_* h$ is strongly convex on S_V. Hence, y_* is the unique minimizer of $\pi_V + \lambda_* h$ and the growth property

$$\exists \rho > 0 \quad \rho \|y - y_*\|^2 \le \pi_V(y) + \lambda_* h(y) - \pi_V(y^*) \quad \forall y \in S_V \tag{25}$$

is valid (recall that $S_V \subseteq U$). From Lemma 4 we conclude $\Psi(\mu) = \Psi_V(\mu) = \sigma_V(y_*)$ and

$$\|Ax - y_*\|^2 \le \rho^{-1}(\pi_V(Ax) - \varphi(\mu)) \quad \forall x \in C_V,\ F_\mu(Ax) \ge p. \tag{26}$$

Now, choose any $x \in C \cap V$ such that $F_\mu(Ax) \ge p$. Obviously

$$\mathit{dist}\,(x, \Psi(\mu)) = \mathit{dist}\,(x, \sigma_V(y^*)) \le \mathit{dist}\,(x, \sigma_V(Ax)) + d_H(\sigma_V(Ax), \sigma_V(y^*)),$$

where d_H refers to the Hausdorff distance on bounded subsets of $I\!R^m$. Using the last two statements of Lemma 4 (with some Hausdorff Lipschitz modulus $L > 0$) along with (26) we continue by

$$\begin{aligned}
\mathit{dist}\,(x, \Psi(\mu))^2 &\le 2(\mathit{dist}\,(x, \sigma_V(Ax))^2 + d_H(\sigma_V(Ax), \sigma_V(y^*))^2) \\
&\le 2(\eta^{-1}(g(x) - \pi_V(Ax)) + L^2\|Ax - y_*\|^2) \\
&\le 2(\eta^{-1}(g(x) - \pi_V(Ax)) + L^2\rho^{-1}(\pi_V(Ax) - \varphi(\mu))) \\
&\le 2\max\{\eta^{-1}, L^2\rho^{-1}\}(g(x) - \varphi(\mu))
\end{aligned}$$

□

Together with Theorem 1 the preceding result leads to upper Hölder continuity of the localized solution set mapping Ψ_V at μ (with rate 1/2) immediately. Using the special structure of problem $\mathbf{P}(\mu)$ we are able to show even the Hausdorff Hölder continuity of Ψ_V at μ.

Theorem 9. *Adopt the setting of Theorem 8. Then there exist* $L > 0, \delta > 0$ *and a neighbourhood* V *of* $\Psi(\mu)$ *with*

$$d_H(\Psi(\mu), \Psi_V(\nu)) \le L\|F_\mu - F_\nu\|_\infty^{1/2} \quad \textit{whenever} \quad \nu \in \mathcal{P}(I\!R^s),\ \|F_\mu - F_\nu\|_\infty < \delta.$$

Here, again, d_H *denotes the Hausdorff distance and* $\|F_\mu - F_\nu\|_\infty = \sup_{z \in I\!R^s} |F_\mu(z) - F_\nu(z)|$.

Proof. As in the proof of Theorem 8 we construct a polyhedron $\bar{V} \subseteq I\!R^m$ such that $\Psi(\mu)$ is contained in the interior V of $\bar{V}$. Since the assumptions of Corollary 1 are satisfied, the localized solution-set mapping Ψ_V is upper semicontinuous at μ and $\Psi_V(\nu) \ne \emptyset$ is a complete local minimizing set for $\mathbf{P}(\nu)$ if $\alpha_K(\mu, \nu)$ is sufficiently small. Hence, there exists a $\delta > 0$ such that $\emptyset \ne \Psi_V(\nu) \subseteq V$ for all $\nu \in \mathcal{P}(I\!R^s)$ with $\|F_\mu - F_\nu\|_\infty < \delta$. With the notations from Lemma 4 and using the fact that $Y_V(\mu) = \{y_*\}$ and $\Psi(\mu) = \Psi_V(\mu) = \sigma_V(y_*)$ we obtain

$$d_H(\Psi(\mu), \Psi_V(\nu)) = d_H(\sigma_V(y_*), \sigma_V(Y_V(\nu))) \le \hat{L} \sup_{y \in Y_V(\nu)} \|y - y_*\|,$$

where $\hat{L} > 0$ is the Hausdorff Lipschitz constant of σ_V (cf. Lemma 4). Using (25), the above chain of inequalities extends to (due to $Y_V(\nu) \subseteq S_V$)

$$\begin{aligned} d_H(\Psi(\mu), \Psi_V(\nu)) &\leq \hat{L}\rho^{-1/2} \sup_{y \in Y_V(\nu)} [\pi_V(y) + \lambda_* h(y) - \pi_V(y_*)]^{1/2} \\ &= \hat{L}\rho^{-1/2}[\varphi_V(\nu) - \varphi(\mu) + \lambda_*(F_\mu^r(y) - p^r)]^{1/2} \\ &\leq \hat{L}\rho^{-1/2}[\varphi_V(\nu) - \varphi(\mu) + \lambda_*(F_\mu^r(y) - F_\nu^r(y))]^{1/2} \\ &\leq \hat{L}\rho^{-1/2}[|\varphi_V(\nu) - \varphi(\mu)| + \lambda_*|r|(p-\delta)^{r-1}|F_\mu^r(y) - F_\nu^r(y)|]^{1/2} \\ &\leq \hat{L}\rho^{-1/2}[(L + \lambda_*|r|(p-\delta)^{r-1})d_K(\mu, \nu)]^{1/2}, \end{aligned}$$

where $L > 0$ is the constant from Theorem 1 and we used that $F_\nu^r(y) \leq p^r$ for any $y \in Y_V(\nu)$ and that the inequality

$$|u^r - v^r| \leq |r| \max\{u^{r-1}, v^{r-1}\}|u - v|$$

holds for any $u, v \in (0, 1]$. This completes the proof.

□

The assumptions *(i)-(iv)* imposed in the Theorems 8 and 9 all concern the original problem $\mathbf{P}(\mu)$. Condition *(i)* is basic for our stability analysis and is satisfied, for example, if C is a polytope. The conditions *(ii)* and *(iii)* mean that the probability level p is not chosen too low and too high, respectively. *(ii)* expresses the fact that the presence of the probabilistic constraint $F_\mu(Ax) \geq p$ moves the solution set $\Psi(\mu)$ away from that obtained without imposing the reliability constraint for '$Ax \geq \xi$'. From a modelling point of view, both conditions show the significance of the choice of the reliability level p. Assumption *(iv)* is decisive for the desired growth condition of the objective function around $\Psi(\mu)$. In contrast to the (global) r-concavity of μ, *(iv)* requires strong convexity of F_μ^r as a local property around $A(\Psi(\mu))$ (in addition to the convexity of F_μ^r on $\mathbb{R}^s$ with values in the extended real numbers). Although no general sufficient criterion for *(iv)* is available so far, *(iv)* seems to be satisfied in many cases when $A(\Psi(\mu))$ belongs to the interior of the support of μ.

Proposition 4. *Let $\mu \in \mathcal{P}(\mathbb{R}^s)$ be logarithmic concave and, hence, F_μ have the form $F_\mu(z) = \exp(-f(z))$, $z \in \mathbb{R}^s$, where $f : \mathbb{R}^s \to \mathbb{R} \cup \{\infty\}$ is convex. Assume that f is continuous and strongly convex on some convex compact set $U \subseteq \operatorname{int}\operatorname{dom} f$. Then F_μ^r is strongly convex on U for each $r < 0$.*

Proof. Let $r < 0$ and $z, \tilde{z} \in U$, $\lambda \in [0, 1]$. Denoting by $c > 0$ the strong convexity constant of f on U, we obtain

$$\begin{aligned} F_\mu^r(\lambda z + (1-\lambda)\tilde{z}) &= \exp(-r f(\lambda z + (1-\lambda)\tilde{z})) \\ &\leq \exp(-r[\lambda f(z) + (1-\lambda) f(\tilde{z}) - c\lambda(1-\lambda)\|z - \tilde{z}\|^2]) \\ &\leq (\lambda F_\mu^r(z) + (1-\lambda)F_\mu^r(\tilde{z}))\exp(rc\lambda(1-\lambda)\|z - \tilde{z}\|^2), \end{aligned}$$

where we used the monotonicity and convexity of exp. Let

$$K := \max_{z \in U} F_\mu^r(z), \quad k := \min_{z \in U} F_\mu^r(z), \quad \bar{a} := \max_{z, \tilde{z} \in U}\{-\frac{rc}{4}\|z - \tilde{z}\|^2\}.$$

Then there exists a constant $\kappa > 0$ such that $t \exp(-a) \leq t - \kappa a$ for all $a \in [0, \bar{a}]$ and $t \in [k, K]$, and we can continue to

$$F_\mu^r(\lambda z + (1-\lambda)\tilde{z}) \leq \lambda F_\mu^r(z) + (1-\lambda) F_\mu^r(\tilde{z}) - \kappa(-r)c\lambda(1-\lambda)\|z - \tilde{z}\|^2,$$

which is the desired inequality.

□

Note that for the uniform distribution μ on some rectangle $D = \times_{i=1}^s [a_i, b_i]$ the function f has the form $f(z) = -\sum_{i=1}^s \log(z_i - a_i)$, $z \in D$, and the proposition applies to each convex compact subset U of $\text{int}\, D$.

5. Exponential bounds for empirical solution estimates

Finally, we show how our quantitative stability results can be employed to derive (asymptotic) properties of solutions to $\mathbf{P}(\mu)$ when estimating the (unknown) probability distribution μ by empirical measures. Consider independent $\mathbb{R}^s$- valued random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ on some probability space $(\Omega, \mathcal{A}, P)$ having common law μ. The empirical measure μ_n is a discrete random measure putting mass n^{-1} at each of the points $\xi_1(\omega), \ldots, \xi_n(\omega)$, i.e., $\mu_n = n^{-1} \sum_{i=1}^n \delta_{\xi_i}$ $(n \in \mathbb{N})$, where δ_z is the dirac measure placing mass one at $z \in \mathbb{R}^s$.

The relevant term in our stability analysis is the $\mathcal{B}$- discrepancy evaluated at μ and μ_n,

$$\alpha_{\mathcal{B}}(\mu_n, \mu) = \sup\{|n^{-1} \sum_{i=1}^n 1_B(\xi_i) - E[1_B]| \mid B \in \mathcal{B}\},$$

where $\mathcal{B}$ is some collection of closed sets in $\mathbb{R}^s$, 1_B denotes the characteristic function of B and E denotes expectation. Thus, the empirical process $\{n^{-1} \sum_{i=1}^n 1_B(\xi_i) - E[1_B]\}_{B \in \mathcal{B}}$ indexed by sets and its uniform convergence properties are of interest. We refer to [44] for a recent exposition of the modern empirical process theory. When studying empirical measures, measurability complications arise. Here, we have to take care of possibly nonmeasurable suprema over uncountable sets of measurable functions . To simplify matters, we call a collection $\mathcal{B}$ of closed subsets of $\mathbb{R}^s$ *permissible* if there exists a countable subclass $\mathcal{B}_0$ such that each characteristic function 1_B with $B \in \mathcal{B}$ is the pointwise limit of a sequence (1_{B_k}) with B_k belonging to $\mathcal{B}_0$. Clearly, if $\mathcal{B}$ is permissible we have $\alpha_{\mathcal{B}}(\mu_n, \mu) = \alpha_{\mathcal{B}_0}(\mu_n, \mu)$, i.e., the further analysis is reduced to countable classes and, in particular, $\alpha_{\mathcal{B}}(\mu_n, \mu)$ is measurable.

An important family of classes of (Borel) measurable sets are the Vapnik-ervonenkis (VC) classes. Recall that $\mathcal{B}$ is called a *VC class* of index $v \in \mathbb{N}$ if it does not shatter any subset of $\mathbb{R}^s$ of cardinality $v + 1$, but does shatter at least a subset of cardinality v. $\mathcal{B}$ is said to shatter $\{x_1, \ldots, x_k\}$ if each of its 2^k subsets is of the form $B \cap \{x_1, \ldots, x_k\}$ for some $B \in \mathcal{B}$. The role of VC classes for empirical processes indexed by sets is enlightened by the following result which is proved in the recent paper [43].

Lemma 5. *Let $\mathcal{B}$ be a permissible VC class of index v. Then there exists a constant $K > 0$ (not depending on v) such that we have for all $n \in \mathbb{N}$ and $\lambda > 0$,*

$$P(\alpha_{\mathcal{B}}(\mu_n, \mu) \geq \lambda) \leq \frac{K}{\lambda\sqrt{n}} \left(\frac{K\lambda^2 n}{v}\right)^v exp(-2\lambda^2 n).$$

Examples of permissible VC classes are the collection of cells $\{z + \mathbb{R}^s_- \mid z \in \mathbb{R}^s\}$ (with $v = s$), the collections of all closed balls in $\mathbb{R}^s$ (with $v = s + 1$), all half-spaces in $\mathbb{R}^s$ (with $v = s + 1$) and all polyhedra with at most k faces. Note that the collection of all closed convex subsets of $\mathbb{R}^s$ is permissible, but too large for being a VC class (cf. [42]).

We return to the setting of Section 2 and show next that the bound in Lemma 5 leads in a straightforward way to exponential bounds for the deviation of the sets of local solutions to $\mathbf{P}(\mu_n)$ and $\mathbf{P}(\mu)$, respectively, if the collection $\{H_j(x) \mid j = 1, \ldots, d;\ x \in C\}$ is contained in a permissible VC class.

Proposition 5. *Adopt the setting of Section 2 and assume the conditions (i)-(iv) of Theorem 1 to be satisfied and that the collection $\{H_j(x) \mid j = 1, \ldots, d;\ x \in C\}$ is contained in a permissible VC class $\hat{\mathcal{B}}$. Then there exist constants $K > 0$, $v \in \mathbb{N}$ such that we have for all $n \in \mathbb{N}$ and $\varepsilon > 0$,*

$$P(\sup_{x \in \Psi_V(\mu_n)} dist\,(x, \Psi_V(\mu)) \geq \varepsilon) \leq \frac{K}{\lambda\sqrt{n}} \left(\frac{K\lambda^2 n}{v}\right)^v exp(-2\lambda^2 n),$$

where $\lambda = \min\{\delta, \varepsilon^2 L^{-2}\}$, L and δ denote the constants and V the bounded open set arising in Theorem 1.

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$, and let L, δ and V be as in Theorem 1. First we notice that $\sup\{dist\,(x, \Psi_V(\mu)) \mid x \in \Psi_V(\mu_n)\}$ is a (possibly extended real-valued) measurable mapping (Theorem 2.K in [33]). Next we define the class $\mathcal{B}$ as the union of $\hat{\mathcal{B}}$ and of the collection $\{z + \mathbb{R}^s_- \mid z \in \mathbb{R}^s\}$. Then $\mathcal{B}$ is a determining class as well as a permissible VC class. Let v be its (VC) index. Now we set $A_\delta := \{\omega \mid \alpha_{\mathcal{B}}(\mu_n, \mu) < \delta\} \in \mathcal{A}$ and $\bar{A}_\delta = \Omega \setminus A_\delta$, and obtain the following inclusion from Theorem 1:

$$\begin{aligned}\{\omega \mid \sup_{x \in \Psi_V(\mu_n)} dist\,(x, \Psi_V(\mu)) \geq \varepsilon\} &\subseteq \bar{A}_\delta \cup \{\omega \mid \varepsilon \leq L\alpha_{\mathcal{B}}(\mu_n, \mu)^{1/2}\} \\ &\subseteq \{\omega \mid \alpha_{\mathcal{B}}(\mu_n, \mu) \geq \min\{\delta, \varepsilon^2 L^{-2}\}\}.\end{aligned}$$

Setting $\lambda = \min\{\delta, \varepsilon^2 L^{-2}\}$, the result follows from Lemma 5.

□

An immediate consequence of the preceding bound is the following large deviation result:

$$\limsup_{n \to \infty} n^{-1} \log P(\sup_{x \in \Psi_V(\mu_n)} dist\,(x, \Psi_V(\mu)) \geq \varepsilon) \leq -2 \min\{\delta^2, \varepsilon^4 L^{-4}\}.$$

All of this applies to the particular case $H_j(x) = \{z \in \mathbb{R}^s \mid h_j(x) \geq z_j\}$ with $h_j : \mathbb{R}^m \to \mathbb{R}^{s_j}$, $j = 1, \ldots, d$, which is considered in Section 3 and for which various verifiable sufficient conditions for the metric regularity condition (iii) are established there (e.g. Theorems 4, 5 and 6). In this case we choose $\hat{\mathcal{B}} = \mathcal{B}_K$. For the special case $d = 1, s_1 = s$ in Section 4 we have $\upsilon = s$ and a slightly modified bound in Lemma 5 (cf. the discussion in [43] after Theorem 1). The corresponding conclusion takes the form:

Proposition 6. *Adopt the setting of Theorem 9. Then there exists a constant* $K > 0$ *such that it holds for all* $n \in \mathbb{N}$ *and* $\varepsilon > 0$,

$$P(d_H(\Psi(\mu), \Psi_V(\mu_n)) \geq \varepsilon) \leq K(\lambda^2 n)^{s-\frac{1}{2}} exp(-2\lambda^2 n),$$

where $\lambda = \min\{\delta, \varepsilon^2 L^{-2}\}$, L *and* δ *denote the constants and* V *the bounded open set arising in Theorem 9.*

6. Appendix

In this appendix, a few examples shall illustrate how nonsmoothness may enter the model (2) of stochastic programming with chance constraints in a natural way and thus requires more general tools for the characterization of stability than the classical ones from differentiable or convex analysis. The impact of a nonsmooth distribution function on the characterization of stability in (2) is easily seen from the following example:

Example 7. In (2), let $m = s = 2$, $g(x_1, x_2) = x_1$, $C = [0, 2] \times [0, 2]$, $A := I$ (=identity matrix), $p = 1/4$ and $\mu =$ uniform distribution over $[0, 1] \times [0, 1]$. Then, the solution set becomes the line segment joining the points $(1/4, 1)$ and $(1/4, 2)$. According to Theorem 1, one has to check metric regularity w.r.t. C of the constraint function $F_\mu(x_1, x_2) - p$ at all these points. Around $(1/4, 1) \in \operatorname{int} C$, this function equals $\min\{x_1 x_2, x_1\} - p$, hence no criterion based on differentiablity applies.

Of course, in this example, one may compensate the lacking differentiability by a convexity argument: the measure μ is logarithmic concave and $(1, 1)$ is the kind of Slater point required in Theorem 8. Also, one might object that the point discussed is located on the boundary of the support of the underlying density, where non-differentiabilities are expected to occur. A modification of the first example towards a uniform distribution over a nonconvex but still connected and even polyhedral set along with a (convex) quadratic objective answers these objections:

Example 8. In (2), let $m = s = 2$, $g(x_1, x_2) = (x_1 - 3/4)^2 + (x_2 - 1/2)^2$, $C = [0, 2] \times [0, 2]$, $A := I$, $p = 1/6$ and $\mu =$ uniform distribution over $([0, 1] \times [0, 1]) \setminus ([0, 1/2] \times [0, 1/2])$. Then, the point $x^0 = (3/4, 1/2)$ is feasible (the probability level is binding at x^0), hence the solution set reduces exactly to $\{x^0\}$. Around x^0, the constraint function equals $F_\mu(Ax) - p = F_\mu(x) - p = 4/3 \max\{x_2(x_1 - 1/2), x_1(x_2 - 1/2), x_1 x_2 - 1/4\} - p$, and it is non-differentiable at x^0, although x^0 lies in the interior of the support of the underlying constant density (see left part of Fig. 3). Also, the measure is not quasi-concave since the support of μ is non-convex. Consequently, neither differentiable nor convex criteria apply in this case.

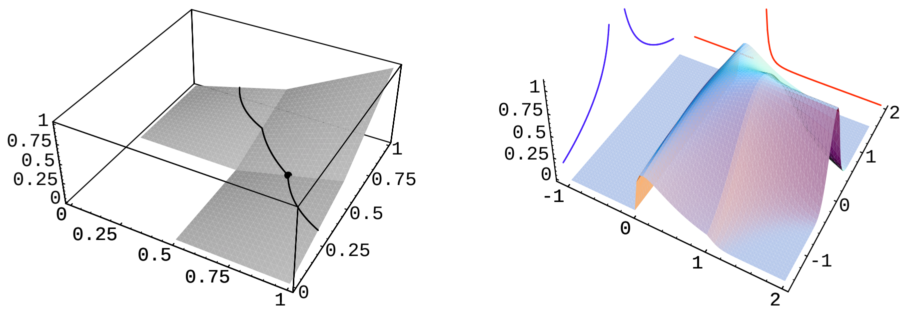

Fig. 3. Left: Illustration of the distribution function F_μ for the uniform distribution on three quarters of a square. The (nonsmooth) level line $F_\mu(x) = p$ as well as the solution point (lifted to the graph of F_μ) are indicated. Right: Plot of the density defined in Example 9 and of the corresponding marginal densities

Starting from dimension two, there may occur unexpected relations between the qualities of densities and corresponding distribution functions. For instance, the last example has shown, that the distribution function may become non-differentiable even at points in a neighborhood of which the underlying density is the nicest possible (constant). The next example (communicated to us bei A. Wakolbinger) highlights another aspect of this dimensionality phenomenon but now focusing on the Lipschitzian property of the distribution function.

Example 9. Consider the following probability density in two variables:

$$f(x_1, x_2) = \begin{cases} 0 & x_1 < 0 \\ c x_1^{1/4} e^{-x_1 x_2^2} & x_1 \in [0, 1] \\ c e^{-x_1^4 x_2^2} & x_1 > 1 \end{cases} \quad \left(c \text{ such that } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2) dx_1 dx_2 = 1\right)$$

This density is bounded and continuous. Yet, the distribution function is not locally Lipschitzian, since the marginal densities are not locally bounded (see right part of Fig. 3).

Consequently, even in the class of random variables with bounded and continuous density one may be led to renounce tools relying on Lipschitzian properties (like Clarke's subdifferential in its original definition) in the study of problem (2). Then, Theorem 4 still provides a tool for checking stability.

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